# Amplification Factors for Beam-Columns

BENJAMIN KOO

BEAM-COLUMNS are structural members subject to axial compression and simultaneous bending moments. In practice, the bending moments in such members are often induced by lateral loads. The axial force influences some functions of the beam, such as deflection, slope, shear force and bending moment. These functions are increased by an amplification factor which depends upon the magnitude of the axial compressive force. Since the member used for structural purposes has a variety of applications, it is desirable to have some information readily available for such beam-columns. Formulas for the determination of magnification factors under the most generalized conditions under symmetrical loading are presented herein. In particular, the end slope of a member equal to the small angle of rotation of the end is given, because many statically indeterminate structures can be solved by a proper consideration of angle rotation. Beam-columns which are symmetrically loaded with concentrated, uniformly distributed and triangularly distributed loads are considered.

In elementary theory of bending, the principal of superposition is valid provided that Hook's Law holds for the material. In beam-columns, the presence of the axial force shows that the deflection of the member is not directly proportional to the lateral loads. However, the superposition method can be applied in a slightly modified form. Thus results shown in this paper can cover a large area of symmetrical loading.

### EQUATIONS FOR SYMMETRICAL LOADS

There are various methods and techniques available for determining theoretical magnification factors. The development shown herein is based on the classical version and is presented in a form that will be most suitable for engineering design use. Three types of symmetrical loading for a pin-end member with uniform cross-section under an axial compressive force are shown in Fig. 1.

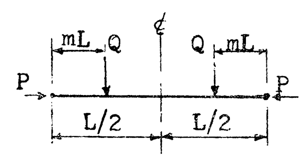
The differential equation for the deflection curve of an ideal beam-column subjected to a concentric axial load $P$ and a lateral load $Q$ as shown in Fig. 2 is:

$$EI\frac{d^2y}{dx^2} + Py = -\frac{Qc}{L}x \tag{1}$$

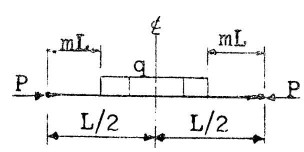
where $E$ is the modulus of elasticity, $I$ is the moment of inertia, and $y$ represents lateral deflections of the member. The general solution of Equation (1) is

$$y = \frac{Q}{P}\frac{\sin kx \sin kc}{\sin kL} - \frac{Qc}{PL}x \tag{2}$$

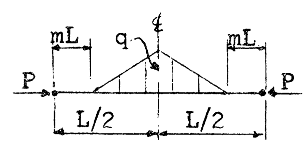
where $k^2 = P/EI$.




Case 1: Concentrated Load




Case 2: Uniform Load




Case 3: Triangular Load

*Figure 1*

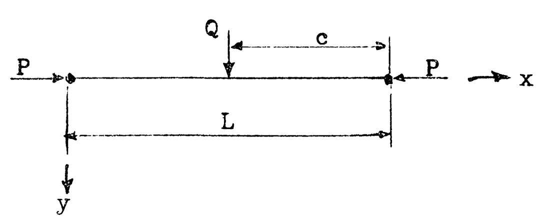



*Figure 2*

*Benjamin Koo is Associate Professor in Structural Engineering, The University of Toledo, Toledo, Ohio.*

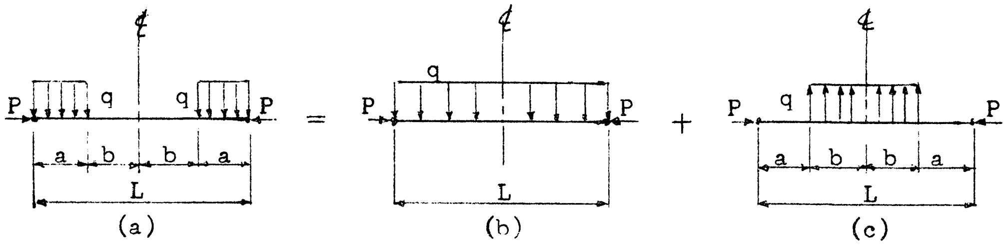



*Figure 3*

The slope of the deflection curve is obtained by differentiating Equation (2) as follows:

$$y' = \frac{Qk}{P} \frac{\cos kx \sin kc}{\sin kL} - \frac{Qc}{PL} \qquad (3)$$

By substituting $qdc$ for $Q$ in Equations (2) and (3) and integrating from limits $mL$ to $(L - mL)$, both the deflection and slope equations of Case 2 and 3 are obtained.

The moment equations are obtained by adding the moment produced by $Py$ to the static moment produced by a lateral load acting only on a simply supported span. Thus,

$$M = M_{\text{static}} + Py \qquad (4)$$

Table 1 gives expressions for the lateral load functions and the limits of integration. Table 1(a) shows the location at which the maximum functions occur.

**RESULTS**

The equations for maximum deflection, slope and moment obtained by solving Equation (1) are listed in Table 2. The equations are expressed in static functions multiplied by an amplification factor. Static functions represent simple beam functions which are obtained if there exists a uniformly or triangularly distributed load over the entire span, of if there is a concentrated load acting alone at the center of the span. The magnification factor gives the influence of the longitudinal force $p$ on the deflection, slope and moment due to actual loading conditions.

These factors are expressed as trigonometrical functions with parameters $m$ and $U$ which are as follows:

$m$ = ratio of the unloaded span to the overall span
$U = \frac{1}{2}kL.$

Table 3 lists the simple beam functions (i.e., axial load $P = 0$) and Table 3(a) lists beam-column functions of corresponding lateral loadings.

**Table 1. Loading Functions and Integration Limits**

| | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| Lateral Loading Function | Q | q dc | a) $q \frac{c - mL}{L/2 - mL} dc$ <br> b) $q \frac{(L - mL) - c}{(L - mL) - L/2} dc$ |
| Limits of Integration | - | from mL to L - mL | a) from mL to L/2 <br> b) from L/2 to L - mL |

**Table 1(a). Location of Maximum Function**

| Location | Deflection (Δ) | Slope (θ) | Moment (M) |
|---|---|---|---|
| X | L/2 | 0 | L/2 |

## APPLICATIONS

By using the method of superposition in its modified form, the deflections due to various arrangements of lateral loadings acting on a beam-column may be obtained by superimposing deflections produced separately by each lateral load acting in combinations with the identical axial force $P$. The deflection due to the lateral loading in Fig. 3(a) is equal to the sum of deflections produced by loadings in Figs. 3(b) and 3(c).

Similarly the end moments of a fixed-end member can be obtained. With reference to Fig. 4, the statically indeterminate end moment $M$ is obtained from the condition that the slope at the built-in end is equal to zero. Therefore, the rotation of the ends produced by the two symmetrical concentrated loads plus the rotation from the action of the end moments $M$ as shown in Table 2, Case 1(C) and Table 3(a), Case 4(F) must be zero.

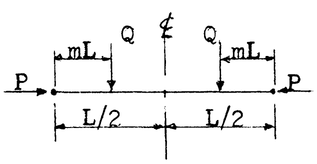
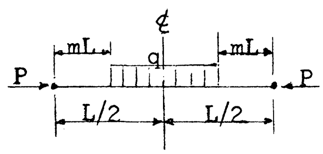
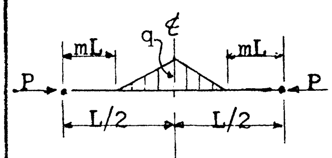
$$\theta_1 + \theta_2 = 0$$

$$\frac{ML}{2EI}\left(\frac{\tan U}{U}\right) + \frac{QL^2}{16EI} \times \left[2\,\frac{2\sin 2mU + \sin 2U(1-m)}{U^2 \sin 2U} - \frac{2}{U^2}\right] = 0$$

$$M = -\frac{QL}{8}\left[\frac{U}{\tan U}\right] \times \left[2\,\frac{2\sin 2mU + \sin 2U(1-m)}{U^2\sin 2U} - \frac{2}{U^2}\right] \tag{5}$$

The $(-)$ sign on the right-hand side of Equation (5) indicates that the end moment $M$ acts opposite to the direction assumed in Fig. 4(a).

**Table 2. Beam-Column Amplification Factors for Symmetrical Loading**

| | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| |  |  |  |
| A | $Y_{℄} = \Delta_{st}\left[\frac{3(\sin 2mU - 2mU\cos U)}{U^3 \cos U}\right]$ | $Y_{℄} = \Delta_{st}\left\{\frac{12}{5}\left[\frac{2}{U^4}(\cos 2mU \sec U - 1) - \frac{1}{U^2} + \frac{4}{U^2}m^2\right]\right\}$ | $Y_{℄} = \Delta_{st}\left\{\frac{15}{4(\frac{1}{2}-m)U^5}\left[\tan U - \frac{\sin 2mU}{\cos U}\right] - \frac{15}{2U^4} - \frac{5}{U^2}\left(\frac{1}{2} - \frac{m}{2} - m^2\right)\right\}$ |
| B | $M_{℄} = M_{st}\left[2m + \frac{\sin 2mU - 2mU\cos U}{U\cos U}\right]$ | $M_{℄} = M_{st}\left[\frac{2(\cos 2mU - \cos U)}{U^2 \cos U}\right]$ | $M_{℄} = M_{st}\left\{\frac{3}{U^3(1-2m)}\left[\tan U - \frac{\sin 2mU}{\cos U}\right] - \frac{3}{U^2}\right\}$ |
| C | $\theta = \theta_{st}\left[2\,\frac{\sin 2mU + \sin 2U(1-m)}{U^2 \sin 2U} - \frac{2}{U^2}\right]$ | $\theta = \theta_{st}\left\{\left[\frac{3(\cos 2mU - \cos 2U(1-m))}{U^3 \sin 2U} - \frac{3}{U^2} + \frac{6}{U^2}(m)\right]\right\}$ | $\theta = \theta_{st}\left[\frac{12}{5U^4}\frac{1}{(\frac{1}{2}-m)}\frac{2\sin U - \sin 2U(1-m)}{\sin 2U} - \frac{\sin 2mU}{\sin 2U} - \frac{12}{5U^2}(1-2m)\right]$ |

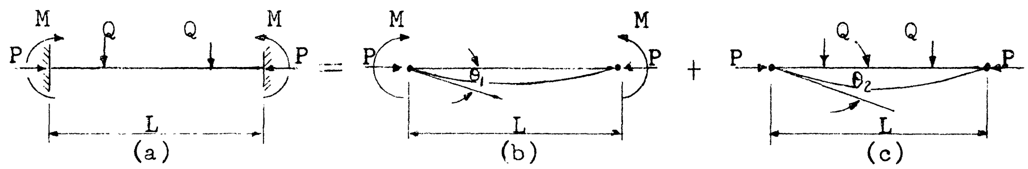
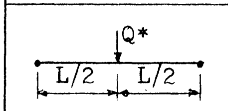
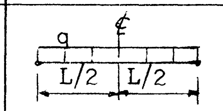
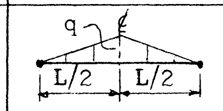
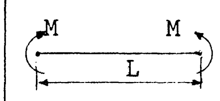


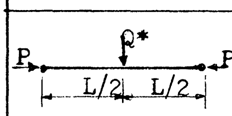
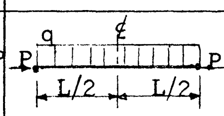
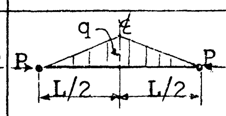
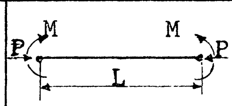

*Figure 4*

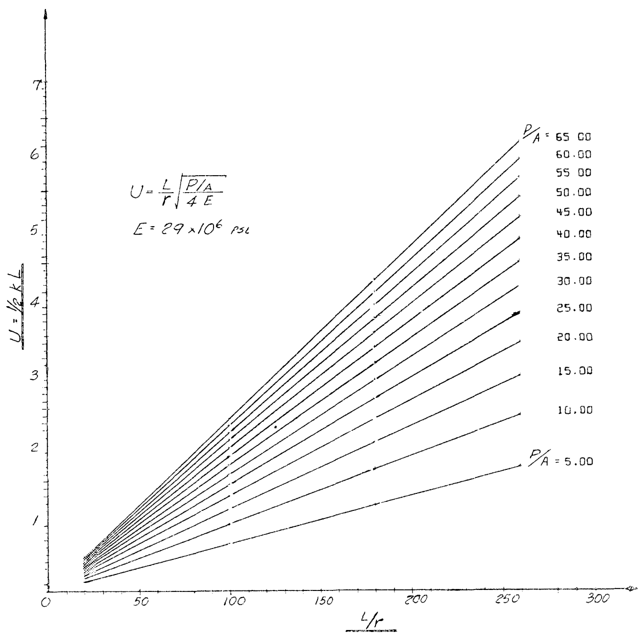
**Table 3**

| | | | Case 1 | Case 2 | Case 3 | Case 4 |
|---|---|---|---|---|---|---|
| | | |  |  |  |  |
| Simple Beam (P = 0) | A | $Y_{\text{¢}}$ | $\Delta_{st} = \frac{Q^* L^3}{48\ EI}$ | $\Delta_{st} = \frac{5\ q L^4}{384\ EI}$ | $\Delta_{st} = \frac{q L^4}{120\ EI}$ | $\Delta_{st} = \frac{M L^2}{8\ EI}$ |
| | B | $M_{\text{¢}}$ | $M_{st} = \frac{Q^* L}{4}$ | $M_{st} = \frac{q L^2}{8}$ | $M_{st} = \frac{q L^2}{12}$ | $M_{st} = M$ |
| | C | $\theta_{End}$ | $\theta_{st} = \frac{Q^* L^2}{16\ EI}$ | $\theta_{st} = \frac{q L^3}{24\ EI}$ | $\theta_{st} = \frac{5\ q L^3}{192\ EI}$ | $\theta_{st} = \frac{M L}{2\ EI}$ |

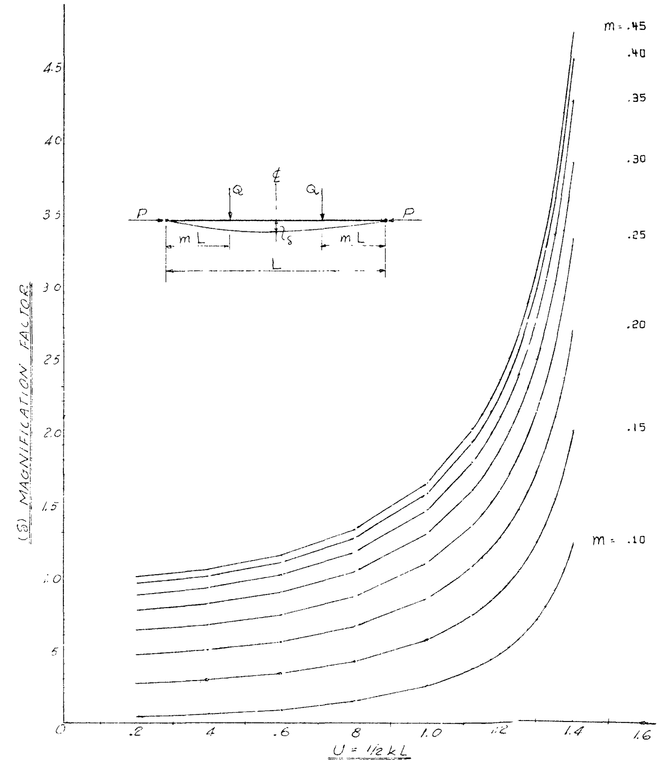
Q* = 2Q shown in Table 2 Case 1.

**Table 3 (a).**

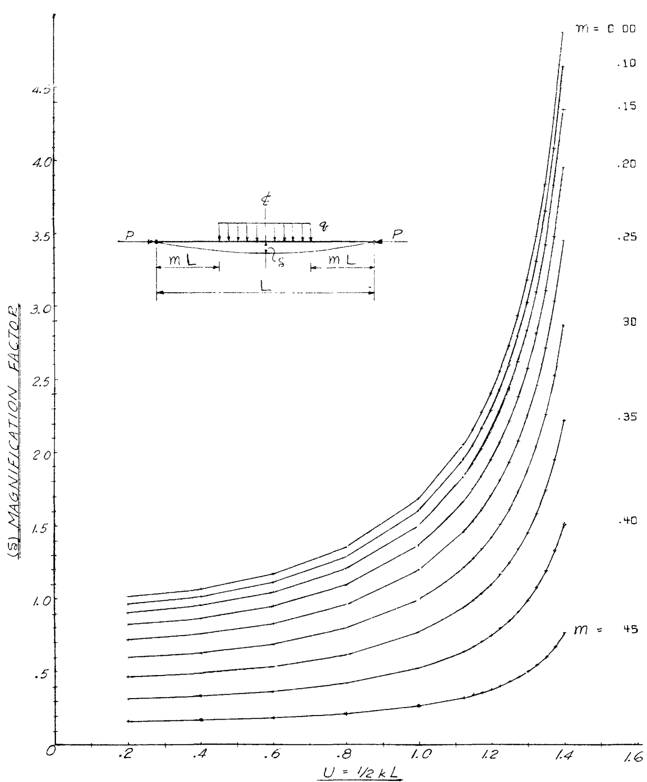
| | | | Case 1 | Case 2 | Case 3 | Case 4 |
|---|---|---|---|---|---|---|
| | | |  |  |  |  |
| Beam-Column | D | $Y_{\text{¢}}$ | $\delta = \Delta_{st}\left(3\ \frac{\tan U - U}{U^3}\right)$ | $\delta = \Delta_{st} \frac{12(\sec U - 2 - U^2)}{5 U^4}$ | $\delta = \Delta_{st} \frac{15(\tan U - U) - 5U^3}{2U^5}$ | $\delta = \Delta_{st} \frac{2(1 - \cos U)}{U^2 \cos U}$ |
| | E | $M_{\text{¢}}$ | $M = M_{st}\left(\frac{\tan U}{U}\right)$ | $M = M_{st}\left(\frac{2 - 2\cos U}{U^2 \cos U}\right)$ | $M = M_{st}\left(\frac{3\tan U - 3U}{U^3}\right)$ | $M = M_{st}(\sec U)$ |
| | F | $\theta_{End}$ | $\theta = \theta_{st}\left(\frac{2 - 2\cos U}{U^2 \cos U}\right)$ | $\theta = \theta_{st}\left(\frac{3\tan U - 3U}{U^3}\right)$ | $\theta = \theta_{st}\left(\frac{12}{5}\ \frac{2 - 2\cos U - U^2\cos U}{U^4 \cos U}\right)$ | $\theta = \theta_{st}\left(\frac{\tan U}{U}\right)$ |



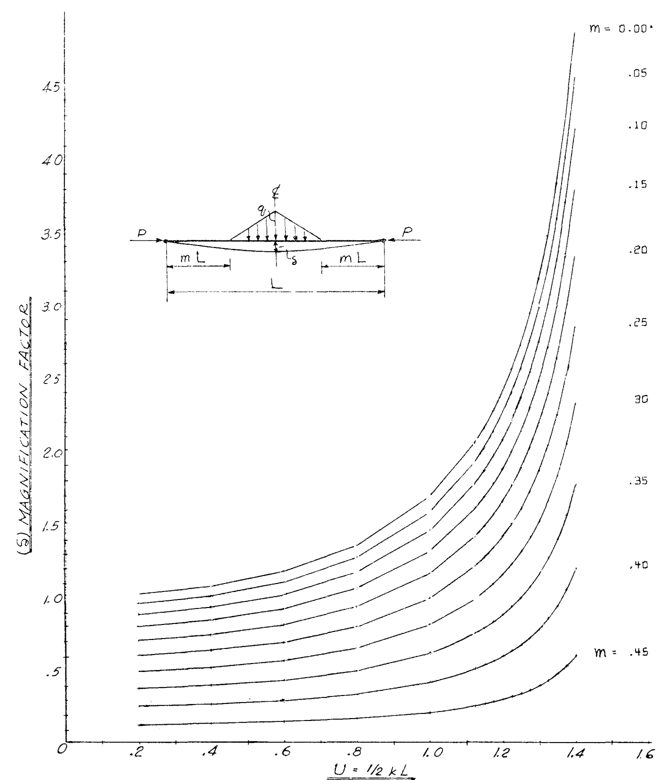

Graph 1




Graph 2




Graph 3




Graph 4

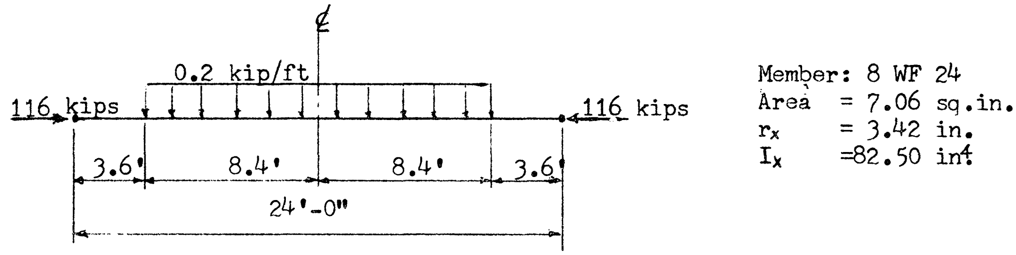



*Figure 5*

**Example**—The use of graphs as aids to engineering design and analysis is illustrated in a numerical example. Figure 5 shows a structural member (8 WF 24) subject to axial and lateral loads as shown. It is required to determine the maximum deflection and moment at the center line of the member,

1. From Table 3, Case 2(A): $\Delta_{st} = 0.624$ in.
2. Calculate $L/r$, which is equal to 84.
   Stress $= P/A = 16.4$ ksi
   $m = 3.6/24 = 0.15$
3. In Graph 1 with the above $L/r$ and $P/A$ values, find the $U$ value on the vertical axis, which is equal to 1.0.
4. In Graph 3, with the $m$ and $U$ values as found above, read across horizontally the magnification value on the vertical axis, which is 1.503.
5. Maximum deflection $= 0.624 \times 1.503 = 0.9378$ in.

For maximum bending moment, use the formula for Case 2(B) in Table 3 and obtain $M = 172.8$ kip-in. From the formula for Case 2(B) in Table 2,

$$M = (172.8)\left[2\frac{\cos(2)(0.15)(1) - \cos(1)}{(1)^2(\cos(1))}\right]$$

$$= (172.8)(1.545) = 267 \text{ kip-in.}$$

Alternately, the maximum moment is equal to the static moment at the center line plus the axial load times the deflection at the center line, i.e.,

$$M = (0.2)(16.8/2)(12)(12) - (0.200)(8.4)(8.4)(12)/2 + (116)(0.9378) = 266 \text{ kip-in.}$$

## CONCLUSION

The results presented herein can form the basis for a useful beam-column analysis subject to symmetrical loadings. The magnification factors can be calculated, if desired, from the equations given in Table 2. The deflection magnification factor may be obtained from Graphs 2 through 4.

For the problem illustrated, the technique has been simple and fast. It appears that the procedures may be applied to more complicated situations such as members with built-in ends, rigid frames and three dimensional structures. Several other topics presently under development may add to the application.

## ACKNOWLEDGMENT

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