A Case for a Single Stiffness Reduction Factor in the 2010 AISC Specification

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ABSTRACT

The 2005 AISC *Specification for Structural Steel Buildings* includes a stiffness reduction factor, τ_b, in Appendix 7 to be used in the direct analysis method to account for the presence of residual stresses and their influence on the second-order effects of frame behavior. The 2005 Commentary includes a stiffness reduction factor, τ_a, to be used along with the effective length nomograph to account for the influence of column inelasticity due to residual stresses on effective length. These two stiffness reduction factors are intended to account for the same effect yet they are different. This paper provides the background for these two factors, and it will demonstrate that τ_b is the more correct stiffness reduction factor. The 2010 AISC *Specification* will recommend its use with both the direct analysis method and the effective length nomograph.

Keywords: stiffness reduction factor, direct analysis method, effective length nomograph, column inelasticity, residual stresses.

INTRODUCTION

Appendix 7 of the *Specification for Structural Buildings* (AISC, 2005) includes a stiffness reduction factor, τ_b, to be used in the direct analysis method to account for the presence of residual stresses and their influence on the second-order effects of frame behavior. The 2005 Commentary includes a stiffness reduction factor, τ_a, to be used along with the effective length nomograph to account for the influence of column inelasticity due to residual stresses on effective length. These two stiffness reduction factors are intended to account for the same effect, yet they are different. This paper provides the background for these two factors and the first stiffness reduction factor introduced in the literature in the early 1970s. It will demonstrate that τ_b is the more correct stiffness reduction factor and it should be used with both the direct analysis method and the effective length nomograph.

STIFFNESS REDUCTION FACTOR FOR DETERMINATION OF INELASTIC *K*-FACTOR

The requirement to use an effective length factor, *K*, in the determination of column strength was introduced with the 1963 AISC *Specification*. At about the same time, two nomographs were published to assist in determining these effective length factors. One of the assumptions used in the derivation of the equations upon which the nomographs are based was that all members in the frame behave elastically. Since it was known at the time that columns behaved inelastically if they had sufficiently low slenderness ratios, there was a need to address the influence of inelastic column behavior on *K*. This was accomplished by using a stiffness reduction factor.

The first introduction of a stiffness reduction factor to be used in conjunction with the nomograph is found in the 1971 paper by Yura (1971). He shows that the joint stiffness ratio, *G*, used with the nomograph included the modulus of elasticity, *E*, in the numerator and denominator so that

$$G = \frac{\text{Column stiffness}}{\text{Beam stiffness}} = \frac{\Sigma\left(EI/L\right)_{col}}{\Sigma\left(EI/L\right)_{beam}} = \frac{\Sigma\left(I/L\right)_{col}}{\Sigma\left(I/L\right)_{beam}} = G_{elastic} \tag{1}$$

However, if the columns behaved inelastically, *E* for the columns should be replaced by the tangent modulus, E_T, and the joint stiffness ratio became

$$G_{inelastic} = \frac{\Sigma\left(E_T I/L\right)_{col}}{\Sigma\left(EI/L\right)_{beam}} = \frac{E_T}{E} G_{elastic} \tag{2}$$

Yura pointed out that the difficulty would be in determining E_T/E. He went on to say that this could be accomplished in a reasonably accurate manner by noting that for a given *Kl/r*,

$$\frac{E_T}{E} = \frac{F_{cr(inelastic)}}{F_e} \tag{3}$$

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where $F_{cr(inelastic)}$ is the critical stress in the inelastic region and

$$F_e = \frac{\pi^2 E}{\left(\frac{Kl}{r}\right)^2}$$

By using the stress equations from the AISC *Specification*, an approximate relationship could be established as

$$\frac{E_T}{E} = \frac{F_{cr(inelastic)}}{F_e} \approx \frac{F_a}{F'_e} \tag{4}$$

where F_a is the allowable stress given by the *Specification* and

$$F'_e = \frac{12\pi^2 E}{23\left(\frac{Kl}{r}\right)^2}$$

Yura also noted that F_a and F'_e use different factors of safety, but that this could be ignored as a minor factor in the development.

Equation 4 is the basis for the stiffness reduction factors used with the Allowable Stress Design (ASD) specifications from 1969 through 1978, while Equation 3 is the basis for τ_a presented in the 2005 AISC *Specification*. Although these stiffness reduction factors are all based on the same relationship, Equation 3, the actual value of the stiffness reduction factor has gone through some variation as the column strength equations have changed through subsequent editions of the specification.

USE OF ACTUAL STRESS RATHER THAN ALLOWABLE STRESS

The use of the stiffness reduction factor as originally presented by Yura (1971) was an iterative process. First, the stiffness reduction factor was assumed to be 1.0 and K was determined. With this K, the allowable stress was determined and a new stiffness reduction factor was found. This stiffness reduction factor lead to a new, reduced K, which led to a new allowable stress and another new reduced K. This process eventually converged and, in the example presented by Yura, the column actually ended up having $K = 1.0$.

In order to eliminate the need to iterate in the determination of the stiffness reduction factor, Disque (1973) recommended using the actual stress rather than the allowable stress. He also suggested that an even more conservative approach would be to use the maximum possible allowable stress, $0.6F_y$, and he provided tables in his paper for that approach. The 8th edition *Steel Construction Manual* included a stiffness reduction factor table based on the actual stress.

1969 AISC SPECIFICATION

From the 1969 AISC *Specification*, column allowable stresses are given:

for $\frac{Kl}{r} \le C_c = \sqrt{\frac{2\pi^2 E}{F_y}}$ (inelastic behavior)

$$F_a = \frac{\left[1 - \frac{(Kl/r)^2}{2C_c^2}\right] F_y}{\frac{5}{3} + \frac{3(Kl/r)}{8C_c} - \frac{(Kl/r)^3}{8C_c^3}} \tag{5}$$

and for $\frac{Kl}{r} > C_c = \sqrt{\frac{2\pi^2 E}{F_y}}$ (elastic behavior)

$$F_a = \frac{12\pi^2 E}{23(Kl/r)^2} \tag{6}$$

while

$$F'_e = \frac{12\pi^2 E}{23(Kl/r)^2} \tag{7}$$

Equations 5, 6, and 7 include the effects of both residual stresses and initial out of straightness (Johnson, 1966). Thus, the ratio of F_a/F'_e is not the ratio of E_T/E but an approximation, as already indicated by Yura.

Substituting Equations 5 and 7 into Equation 4 and defining the stiffness reduction factor as SRF_{1969} yields

$$SRF_{1969} = \frac{F_a}{F'_e} = \frac{\frac{23}{12}\left(1 - \frac{F_y}{4F_e}\right)\frac{F_y}{F_e}}{\left(\frac{5}{3} + \frac{3(Kl/r)}{8C_c} - \frac{(Kl/r)^3}{8C_c^3}\right)} \tag{8}$$

The factor 23/12 in the numerator is the safety factor for elastic buckling while the three-term denominator is the safety factor for inelastic buckling. In the inelastic region, for $Kl/r = 0$ the safety factor is 5/3 while for $Kl/r = C_c$, the safety factor is 23/12. For design according to the 1969 AISC *Specification*, the safety factors account for the influence of initial out of straightness while inelasticity is accounted for through the basic equations.

It should be noted that in the elastic buckling region, when $Kl/r > C_c$, the ratio of F_a/F'_e is 1.0 and there is no inelastic stiffness reduction factor, as would be expected.

1986 AISC LRFD SPECIFICATION

With the introduction of the 1986 AISC LRFD (Load and Resistance Factor Design) *Specification*, column behavior was defined at the critical stress level, without the use of factors of safety. In addition, the column strength equation in the inelastic region was changed. Thus,

for $\lambda_c = \frac{Kl}{r\pi}\sqrt{\frac{F_y}{E}} \leq 1.5$ (inelastic behavior)

$$F_{cr} = \left(0.658^{\lambda_c^2}\right)F_y \tag{9}$$

and for $\lambda_c > 1.5$ (elastic behavior)

$$F_{cr} = \left(\frac{0.877}{\lambda_c^2}\right)F_y \tag{10}$$

These equations also include the influence of initial out of straightness and residual stresses. In the elastic buckling region, Equation 10, the primary factor influencing column strength is the initial out of straightness. Thus, the 0.877 factor can be thought of as accounting for initial out of straightness. In the inelastic region, Equation 9, the separate influences of residual stresses and initial out of straightness can not be clearly distinguished. This is the same situation that was seen when using the 1969 ASD equations.

The stiffness reduction factor based on Eq. 3 and the LRFD column strength equations then becomes, for columns with $\lambda_c \leq 1.5$

$$SRF_{1986} = \frac{E_T}{E} = \frac{F_{cr(inelastic)}}{F_{cr(elastic)}} = \frac{\left(0.658^{\lambda_c^2}\right)F_y}{\left(\frac{0.877}{\lambda_c^2}\right)F_y} \tag{11}$$

2005 AISC SPECIFICATION

The column strength equations found in the 2005 AISC *Specification* are essentially the same as those from the 1986 AISC LRFD *Specification*. The only change is how the division between elastic and inelastic behavior is defined and the format of the actual terms in the equations. For the 2005 *Specification*, Eq. 11 becomes

$$\tau_a = \frac{E_T}{E} = \frac{\left(0.658^{\frac{F_y}{F_e}}\right)F_y}{0.877F_e} \tag{12}$$

The 2005 Commentary gives τ_a in the format developed by the ASCE Task Committee on Effective Length (ASCE, 1997) as

$$\tau_a = -2.724\left(\frac{P_n}{P_y}\right)\ln\left(\frac{P_n}{P_y}\right) \tag{13}$$

Where $P_n = F_{cr}A_s$ and F_{cr} is determined through Equation 9. Equations 11, 12 and 13 provide the same results.

A new stiffness reduction factor, τ_b, to be used in the Direct Analysis Method was introduced in the 2005 AISC *Specification*. It is based on column strength curves proposed by the Column Research Council in 1960 (Johnson, 1960; Galambos, 1998), where

for $\lambda = \frac{KL}{r\pi}\sqrt{\frac{F_y}{E}} \leq \sqrt{2}$ (inelastic behavior)

$$\frac{F_{cr}}{F_y} = \left(1 - \frac{\lambda^2}{4}\right) \tag{14}$$

and for $\lambda > \sqrt{2}$ (elastic behavior)

$$\frac{F_{cr}}{F_y} = \frac{1}{\lambda^2} \tag{15}$$

The slenderness parameter, λ, is the same as λ_c introduced in the 1986 AISC LRFD *Specification*. Equations 14 and 15 include the effects of residual stresses but do not include the effects of out of straightness (Johnson, 1960). With the substitution of λ, Equation 15 reduces to the Euler buckling stress equation,

$$F_{cr} = F_e = \frac{\pi^2 E}{\left(\frac{Kl}{r}\right)^2} \tag{16}$$

and Equation 14 reduces to

$$F_{cr} = \left(1 - \frac{F_y}{4F_e}\right)F_y \tag{17}$$

The 2005 Commentary says that τ_b is similar to the inelastic stiffness reduction factor. For use with the nomograph, the stiffness reduction factor defined by Equation 3, using Equations 16 and 17, yields

$$\tau_b = \frac{E_T}{E} = \left(1 - \frac{F_y}{4F_e}\right)\frac{F_y}{F_e} \tag{18}$$

COMPARISON OF STIFFNESS REDUCTION FACTORS BASED ON SLENDERNESS RATIOS

The three stiffness reduction factors given in Equations 8, 12 and 18 are each a function of *Kl*/*r*, directly and through F_e. Figure 1 presents these three equations as a function of λ, which itself is a function of *Kl*/*r*, for F_y = 50 ksi. Table 1 gives values for Equations 8, 12, and 18 for λ = 0 to 1.5 (*Kl*/*r* = 0 to 113.5). It is seen from these results that the three equations for determining the stiffness reduction factor provide essentially the same results when presented as a function of slenderness. This confirms the claim of Yura

(1971) that the variation in safety factor was a minor factor. It also shows that the use of strength equations that include initial out of straightness and residual stresses (Equations 8 and 12), rather than just residual stresses (Equation 18) is also a minor factor in determining the stiffness reduction factor. Since the stiffness reduction factor was initially developed to assist in the proper use of the nomographs, which are of course based on *Kl*/*r*, it is clear that any of these three approaches would be reasonable for this purpose.

STIFFNESS REDUCTION FACTOR FOR USE IN THE DIRECT ANALYSIS METHOD

The stiffness reduction factor to be used with the direct analysis method is defined as τ_b as discussed above and is given directly as an unnumbered equation in Appendix 7 of the 2005 AISC *Specification*.

For $P_r/P_y \le 0.5$

$$\tau_b = 1.0 \tag{19}$$

and for $P_r/P_y > 0.5$

$$\tau_b = 4\left[\frac{\alpha P_r}{P_y}\left(1 - \frac{\alpha P_r}{P_y}\right)\right] \tag{20}$$

In Equation 20, $\alpha = 1.0$ for LRFD and $\alpha = 1.6$ for ASD. This factor is used to insure that the calculation is performed at an ultimate strength level.

This relationship was developed from the CRC column strength equations presented above as Equations 14 and 15.

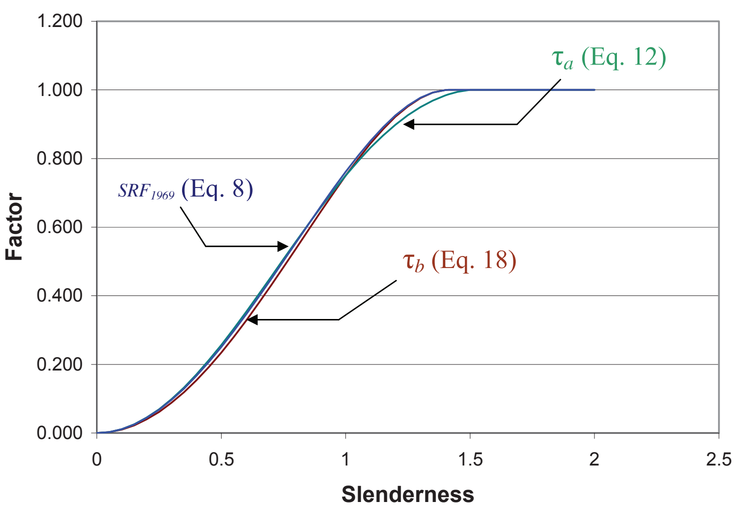

Fig. 1. Comparison of the three approaches to determining the stiffness reduction factor as a function of slenderness parameter, λ, for use with the nomograph.

Table 1. Stiffness Reduction Factors Based on Slenderness

λ	SRF_{1969}	τ_a	τ_b
	Eq. 8	Eq. 12	Eq. 18
0	0.000	0.000	0.000
0.05	0.003	0.003	0.002
0.10	0.011	0.011	0.010
0.15	0.025	0.025	0.022
0.20	0.044	0.045	0.040
0.25	0.068	0.069	0.062
0.30	0.097	0.099	0.088
0.35	0.129	0.133	0.119
0.40	0.166	0.171	0.154
0.45	0.207	0.212	0.192
0.50	0.250	0.257	0.234
0.55	0.297	0.304	0.280
0.60	0.346	0.353	0.328
0.65	0.396	0.404	0.378
0.70	0.449	0.455	0.430
0.75	0.502	0.507	0.483
0.80	0.555	0.558	0.538
0.85	0.608	0.609	0.592
0.90	0.661	0.658	0.646
0.95	0.712	0.705	0.699
1.00	0.762	0.750	0.750
1.05	0.808	0.792	0.799
1.10	0.852	0.831	0.844
1.15	0.891	0.867	0.885
1.20	0.926	0.899	0.922
1.25	0.955	0.926	0.952
1.30	0.977	0.950	0.976
1.35	0.993	0.969	0.992
1.40	1.000	0.984	1.000
1.45	1.000	0.994	1.000
1.50	1.000	1.000	1.000

If the inelastic critical stress is taken as τ_b times the elastic critical stress, then from Equation 15

$$F_{cr} = \tau_b\left(\frac{1}{\lambda^2}\right)F_y \tag{21}$$

This is simply a restatement of the Equation 3 and applies for both elastic and inelastic behavior when using Equations

19 and 20 for τ_b. If Equation 21 is solved for λ^2 and the result substituted into Equation 14,

$$\frac{F_{cr}}{F_y} = \left[1 - \frac{1}{4}\left(\frac{\tau_b F_y}{F_{cr}}\right)\right] \tag{22}$$

Equation 22 can then be solved for τ_b, which yields

$$\tau_b = 4\left[\frac{F_{cr}}{F_y}\left(1 - \frac{F_{cr}}{F_y}\right)\right] \tag{23}$$

which can be written in terms of strength as

$$\tau_b = 4\left[\frac{P_n}{P_y}\left(1 - \frac{P_n}{P_y}\right)\right] \tag{24}$$

Equations 20 and 24 are the same except that one is written in terms of αP_r, the amplified required strength, and one in terms of P_n, the nominal strength. This difference is similar to the two approaches presented by Disque (1973) and Yura (1971).

To allow for additional comparisons, the stiffness reduction factor using the 1969 AISC *Specification*, SRF_{1969}, can be linked to nominal strength. To accomplish this, F_a/F_y is increased by 5/3, removing a uniform factor of safety and increasing the allowable stress at $Kl/r = 0$ to F_y and the stresses at other values of Kl/r to a comparable critical stress level. This links Equation 8 to F_{cr}/F_y or P_n/P_y.

Figure 2 shows the three stiffness reduction factors as a function of P_n/P_y and Table 2 gives values for P_n/P_y from 1 to 0.39. It is clear from these results that the three approaches no longer can be viewed as giving essentially the same answer. The differences between the factors can be attributed to the inelastic column strength equations used. A detailed comparison of Equations 5 and 14 would show that the 1969 ASD equation is the CRC equation with a variable safety factor. The LRFD strength equation, Equation 9, is quite different when compared to both of the others.

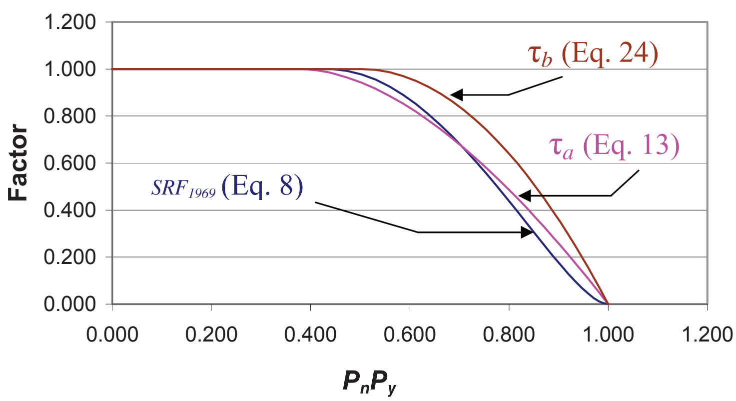

Fig. 2. Comparison of three stiffness reduction factors as a function of the nominal strength to yield strength ratio.

Table 2. Stiffness Reduction Factors Based on P_n/P_y

P_n/P_y	SRF_{1969}	τ_a	τ_b
	Eq.8	Eq. 13	Eq. 24
1.000	0.000	0.000	0.000
0.991	0.003	0.023	0.034
0.982	0.011	0.049	0.071
0.971	0.025	0.077	0.112
0.960	0.044	0.108	0.155
0.947	0.068	0.140	0.200
0.934	0.097	0.175	0.248
0.919	0.129	0.211	0.297
0.904	0.166	0.249	0.347
0.888	0.207	0.287	0.398
0.871	0.250	0.327	0.449
0.853	0.297	0.368	0.500
0.835	0.346	0.410	0.551
0.816	0.396	0.452	0.601
0.796	0.449	0.495	0.649
0.776	0.502	0.537	0.696
0.754	0.555	0.579	0.741
0.732	0.608	0.622	0.784
0.710	0.661	0.663	0.824
0.686	0.712	0.704	0.861
0.662	0.762	0.744	0.895
0.637	0.808	0.782	0.924
0.612	0.852	0.819	0.950
0.586	0.891	0.853	0.971
0.559	0.926	0.886	0.986
0.531	0.955	0.915	0.996
0.503	0.977	0.942	1.000
0.474	0.993	0.964	1.000
0.443	1.000	0.982	1.000
0.435	1.000	0.986	1.000
0.414	1.000	0.995	1.000
0.390	1.000	1.000	1.000

A SINGLE STIFFNESS REDUCTION FACTOR

There are three stiffness reduction factors available for use with the nomograph as discussed earlier: Equations 8, 12 and 18. If the intent is only to use the stiffness reduction factor to modify the column end stiffness ratios, G, for use with the nomograph, it has been shown that any of these approaches will provide satisfactory results.

The SRF_{1969} and τ_a are based on strength equations that include initial out of straightness and residual stresses while τ_b is based on column strength equations that include only the effects of residual stresses. A comparison of these three approaches, Equations 8, 13 and 24, as a function of P_n/P_y, shows that they do not give similar results. Since the intent of the stiffness reduction factor, in all cases, is to include only the influence of inelastic behavior due to residual stresses, clearly τ_b should be used. Thus, the stiffness reduction factor for both the nomograph and the direct analysis method should be taken as τ_b.

Equations 19 and 20 for τ_b will be included in the 2010 AISC *Specification* as Equations C2-2a and C2-2b for use with the direct analysis method. The effective length method is covered in Appendix 7 of the 2010 *Specification,* and Commentary Section A7.2 indicates that the same equations used for stiffness reduction with the direct analysis method should be used with the effective length method.

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