Plastic Design by Moment Balancing

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TWO THEOREMS ARE important in the plastic analysis of rigid-frame structures. These are usually called the upper-bound theorem and the lower-bound theorem. According to the upper-bound theorem, the load corresponding to an assumed mechanism of collapse will always be greater than, or at best equal to, the true ultimate load. According to the lower-bound theorem, the load corresponding to a given moment diagram for which the moment nowhere exceeds the plastic-moment capacity of the frame will always be less than, or at most equal to, the true ultimate load.

If one constructs a moment diagram for a structure—any moment diagram, so long as it satisfies statics—and proportions the structure to this moment diagram so that there are enough plastic hinges to form a mechanism, the upper-bound and lower-bound theorems are satisfied simultaneously. This conclusion is the basis for a scheme proposed by Horne[1] that is particularly suited for purposes of design. It can be used to construct a distribution of moments about which the structure can then be developed in such a way as to result in a sufficient number of plastic hinges to produce a mechanism. The procedure is very efficient in application to frames of several stories, and offers the designer liberty he does not enjoy in design based on elastic methods of analysis. The method is also useful for preliminary sizing of frames which are to be designed elastically.

A pin-based rectangular frame is shown supporting a concentrated gravity load in Fig. 1(a), a concentrated lateral force in Fig. 1(b), and a combination of the two in Fig. 1(c). The corresponding deflected shapes are shown in Figs. 1(d), 1(e), and 1(f). Points where plastic hinges develop are shown in Figs. 1(g), 1(h), and 1(i). In each case, the loss in rotational stiffness resulting from the formation of the plastic hinges reduces the frame to a mechanism. Thus, the determination of the ultimate load a frame can support requires the identification of points of peak moment where plastic hinges can develop in sufficient number to result in the formation of a mechanism.

Figure 2 shows simple, basic adjustments which can be made in an initial or trial moment diagram for the purpose of developing a moment diagram which yields a collapse mechanism. Interior moments are taken positive in the usual sense, that is, they produce tension on the bottom fibers of a horizontal beam. End moments are positive when they are clockwise on the member. One may think of each of these distributions as involving a shifting of an initial baseline. In Fig. 2(a), base **a-a** of the beam-mechanism moment diagram for a concentrated load is shifted to position **b-b** by adding uniform moment, i.e., moments in the ratio +1, +1, −1. If base **a-a** is rotated about its midpoint to **b-b** (Fig. 2(b)), the distribution is in the ratio +1, 0, +1. The base may also be rotated about either end, as in Figs. 2(c) and 2(d).

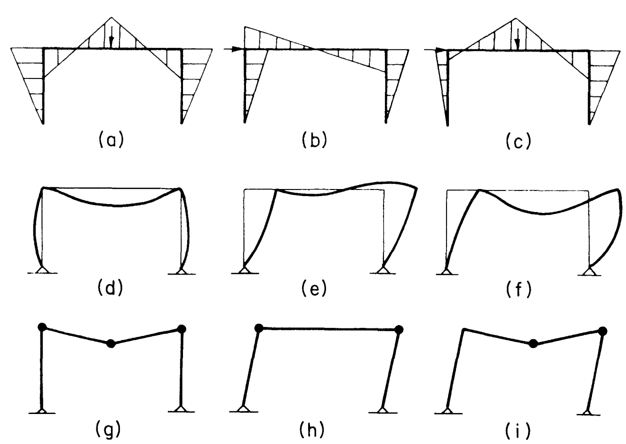

Figure 1

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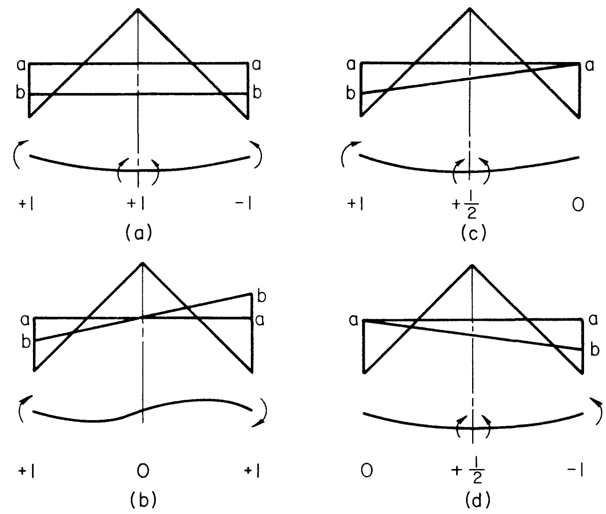


Figure 2

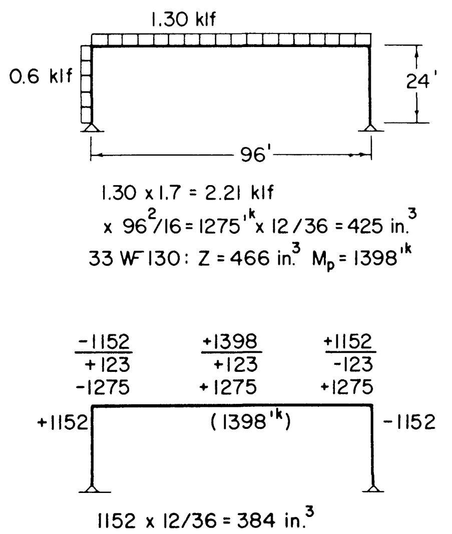


Figure 3

SINGLE-SPAN RECTANGULAR FRAME

A simple example is given in Fig. 3, where the frame shown is designed for the gravity load of 1.3 kips/lin ft at a load factor of 1.7. The beam-mechanism moment is computed first. This implies that the beam and columns will have equal resisting moments, so that the relative size of the members has in fact been assumed. A 33WF130 (A36 steel) is required. However, since it furnishes M_p = 1398 ft-kips, a frame consisting of this shape for beam and columns will not form a mechanism at the specified ultimate load. Instead, it will support about 9.5 percent more load. But if the distribution of moments is revised by adding +123, +123, −123, corresponding to Fig. 2(a), a hinge will develop at midspan and a mechanism results if the columns can be sized to develop only 1152 ft-kips. However, a 33WF118 is the best selection for the columns, and, although no mechanism results, the revised frame has only 3.5 percent excess capacity and saves more than 500 lbs of steel.

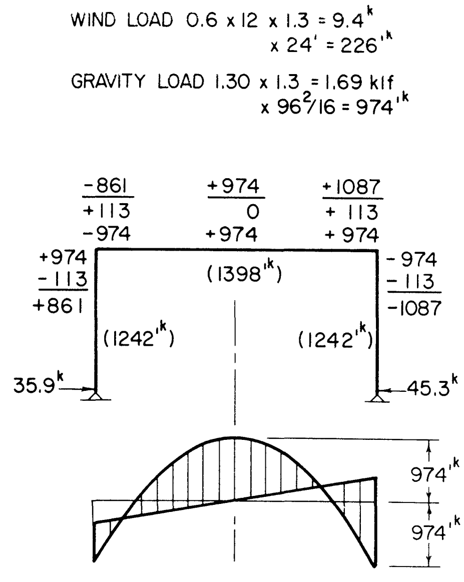


Figure 4

The resulting frame is checked for adequacy against wind and gravity load in Fig. 4. The load factor for this combination is 1.3. The sway moment due to the wind load is 226 ft-kips. The analysis begins with the gravity-load beam-mechanism moments (974 ft-kips) for the reduced load factor. Figure 1(b) suggests that the required sway moment may be developed with the correction shown in Fig. 2(b), which gives +113, 0, +113. This yields the necessary sway moment, −226 ft-kips, with a moment distribution which nowhere exceeds the plastic moment capacity of the frame. Therefore, the frame is adequate for wind and gravity loads. Although not necessary to the solution, the starting moment diagram and the subsequent adjustment are shown in Fig. 4.

If the adjustment in Fig. 4 is made +268, 0, +268, a hinge (1242 ft-kips) develops in the right-hand column. This corresponds to a wind force of 2 × 268/24 = 22.3 kips, instead of the 9.4 kips required. However, one hinge does not transform the frame to a mechanism. Figures 1(c) and 1(i) show that, at ultimate load, there should also be a hinge near midspan. This can be accomplished by superimposing the moment patterns of Figs. 2(b) and 2(c) as shown in Fig. 5. The off-center location of the interior hinge must be anticipated by raising the moment at midspan to something less than the plastic moment capacity of the beam. The location of the interior hinge is easily found, as is shown in the shear diagram, and the hinge moment determined by computing the increment ΔM. Mechanisms of this type tend to develop in the lower stories of multi-story frames.

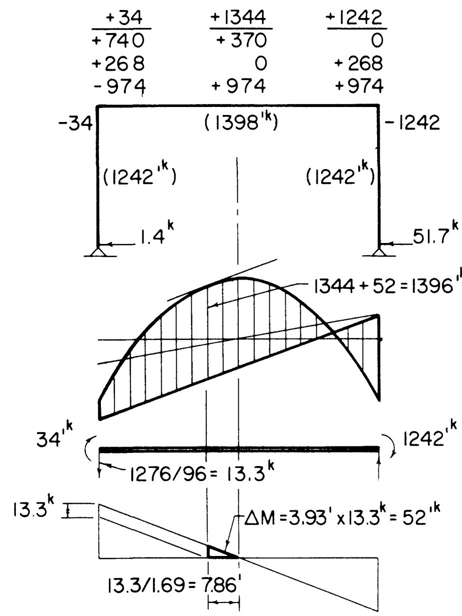
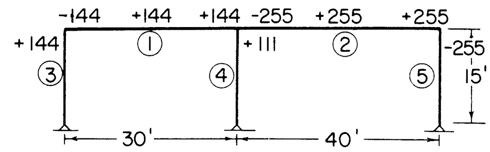


Figure 5



FRAMES 25' o.c. ROOF LOAD 60 psf. WIND 40 psf.
COLUMNS NOT TO EXCEED 12".
$60 \times 25 = 1.5$ klf $\times 1.7 = 2.55$ klf
$2.55 \times 30^2/16 = 144'^k$; $2.55 \times 40^2/16 = 255'^k$

	M	SECTION	M_p	M_o
1	144	$16 \times 5\frac{1}{2}$ B31	161	
2	255	$18 \times 7\frac{1}{2}$ WF45	269	
3	144	$12 \times 6\frac{1}{2}$ WF36	154	154
4	111	10×8 WF39*	141	125+
5	255	12×10 WF58	260	260

* Lightest section with flange at least $7\frac{1}{2}$" (to fit beam 2)

+ Formula 22 (Case II Column)

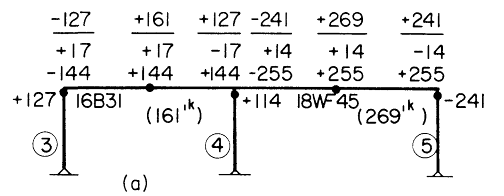
Figure 6

TWO-BAY FRAME

Design of a two-bay frame for gravity load is shown in Fig. 6. The distribution of moments, assuming beam mechanisms in each span, is shown. The required section moduli are shown in the table. With columns restricted to not more than 12 in. wide in the plane of the frame, the required minimum sections are those shown. The 10WF39 (interior column) is the lightest shape which has a flange wide enough to accommodate a welded connection of the 18WF45. Each column is checked against Formula 22 of the AISC Specification, but only the interior column has an axial force large enough to reduce the plastic moment resistance.

It will be noted that none of these shapes need be checked for local buckling. The AISC Manual's table of plastic section moduli excludes shapes for which the flange slenderness exceeds 8.5, and denotes with the symbol § those for which web buckling must be checked. Of course, spacing of supports against lateral buckling must be checked for compliance with Sect. 2.8 of the AISC Specification.

Figure 7(a) shows how the distribution of moments in Fig. 6 can be improved to take advantage of the excess capacities of the members of the frame. A midspan hinge is developed in each beam by adding the distributions +17, +17, −17, and +14, +14, −14. These do not upset the sway balance, which was +144 + 111 − 255 = 0 in Fig. 6 and is +127 + 114 − 241 = 0 in Fig. 7. The resulting reduction in moment in the left column allows the section to be reduced by 5 lbs/lin ft. However, the web of the 12WF31 must be checked for local buckling (Formula 25 of the AISC Specification). It is found to be adequate.



	M	Z	SECTION	M_p	M_o	
3	127	42	$12 \times 6\frac{1}{2}$ WF 31	132	132	Reduced 5 plf
4	114	38	10×8 WF39	141	125	No change
5	241	80	12×10 WF58	260	260	No change

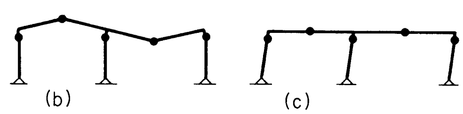


Figure 7

Were it possible to choose columns with precisely the required resisting moments, the frame would develop hinges at the points indicated in Fig. 7. Although there are enough hinges to produce a mechanism, the possible displacements, two of which are shown in Figs. 7(b) and 7(c), involve rotations at certain hinges which are opposed in sense to the moments at these points. Therefore, this system of hinges does not produce a collapse mechanism for the given system of loads, and it may be possible to develop a lighter frame.

The collapse configuration in Fig. 7(b) suggests that a hinge is needed in at least one of the beams at the interior column. This is easily accomplished by using the distribution of Fig. 2(b). Thus, adding +34, 0, +34 to the final moments −127, +161, +127 in Fig. 7(a) results in values of −93, +161, +161. However, the off-

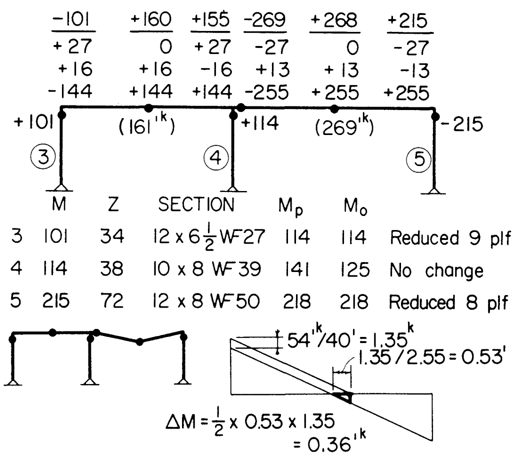


Figure 8

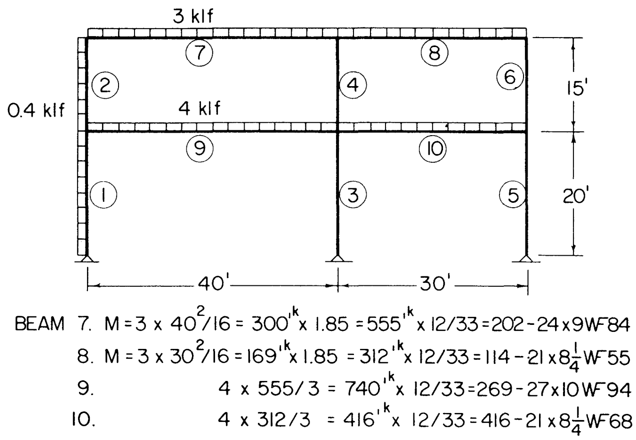


Figure 9

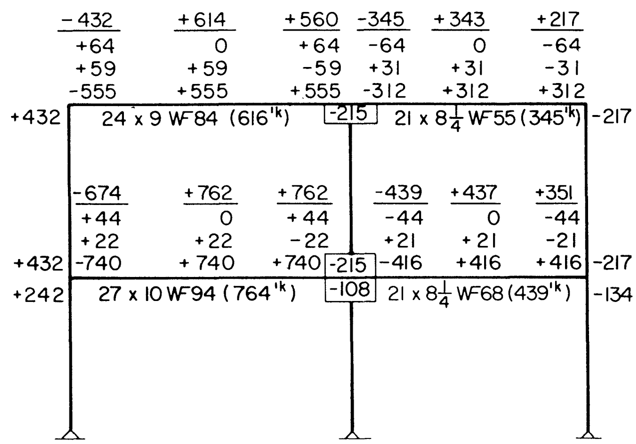


Figure 10

center location of the interior hinge should be anticipated; therefore, adjustments +16, +16, −16 and +13, +13, −13 are made in Fig. 8, which results in values 1 ft-kip short of the hinges at midspan. Furthermore, note that the +, 0, + distribution which is added to one beam must be accompanied by a −, 0, − distribution (with the same absolute values) in the other span in order to maintain equilibrium of sway moments. It turns out that the right span controls. That is, the left span cannot develop an interior hinge and one at the interior column without exceeding the moment capacity of the 18WF45 in the right span. The check in the shear diagram shows that the maximum moment in the right span is only 0.36 ft-kip more than the moment at the center, so that the 1 ft-kip allowed was sufficient.

Assuming that the columns can be proportioned to have precisely the required moment, hinges will develop at the locations shown. A possible mechanism displacement shows local collapse of the right span, with all rotations in the same sense as the moments. The fact that there are also hinges in the left span which do not participate in the mechanism displacement is incidental. Since the hinge pattern gives the frame two degrees of freedom, a beam mechanism displacement could also be drawn in the left bay, independently of the one in the right bay.

It will be noted that the new distribution of moments saves 9 lbs/lin ft in the left column and 8 lbs/lin ft in the right. The web of the 12WF27 is adequate with respect to local buckling.

TWO-STORY TWO-BAY FRAME

The frame in Fig. 9 is designed in A7 steel for a load factor of 1.85 on gravity loads. Figure 10 shows the beam-mechanism moments and, in parentheses, the moment capacities of the lightest beams that can be used. The excess capacity is turned to account by developing near-hinges at mid-span and a hinge at the column in whichever beam it can be developed. The distributions are made as in Fig. 8. In this example, moments at mid-span are held 2 ft-kips below the moment capacity. In balancing joints at the floor, full advantage is taken of the moment capacities of the upper-tier columns, which are dictated by equilibrium of the joints at the roof.

The required column sections are shown in Fig. 11. They are chosen so as to furnish, with minimum weight, the required section modulus and, at the same time, a flange width adequate for groove welding of the beam flange. Although the flange of the 18WF is narrower than that of the 27WF94, it is wide enough to develop the required moment (674 ft-kips). This is easily shown by calculating the moment which is lost by the difference between the 10-in. flange and an 8¾-in. butt weld:

$$33 \times 1.25 \times 0.747 \times 19.44/12 = 50 \text{ ft-kips}$$

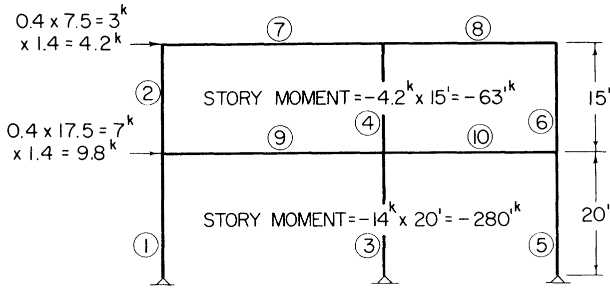
Reduction in M_p because of concurrent axial force was checked according to Formula 22 (Case II column).

COL.	M	P	SECTION	M_p	P/P_y	M_o
1	242	259	18 x $8\frac{3}{4}$ WF64 §	362	0.418	233 *
2	432	111	18 x $8\frac{3}{4}$ WF77	441	0.149	441
3	108	453	14 x 12 WF78	368	0.598	154
4	215	194	14 x 10 WF68	282	0.328	212 *
5	134	194	12 x 10 WF53	215	0.378	147
6	217	83	12 x 10 WF58	238	0.148	238

§ $d/w \leq 70 - 100\ P/P_y = 70 - 42 = 28$,
but need not be less than 43
$d/w = 17.87/0.403 = 44 > 43$
Could change section to 18 x $8\frac{3}{4}$ WF70

* $M_o < M$ required

Figure 11



BEAM 7. $M = 555 \times 1.4/1.85 = 420'^k$
8. $312 \times 1.4/1.85 = 236'^k$
9. $740 \times 1.4/1.85 = 560'^k$
10. $416 \times 1.4/1.85 = 315'^k$

Figure 13

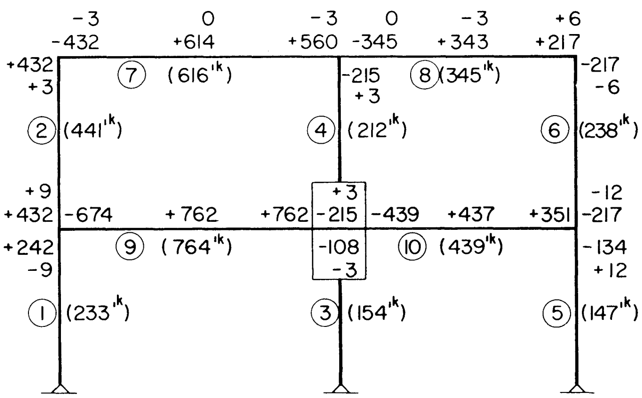


Figure 12

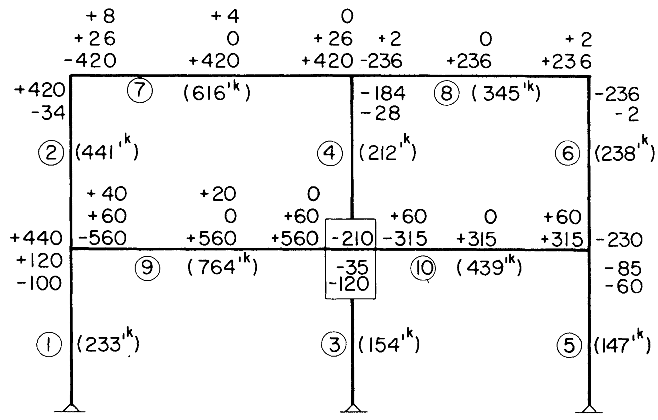


Figure 14

This is conservative for the upper-story columns, which can be classified as Case I. The 18WF64 is slightly deficient in respect to local buckling of the web. Columns **1** and **4** turn out to be somewhat deficient in moment capacity. However, before rejecting them, a check is made to see if moments can be reassigned to overcome the deficiency. This is done in Fig. 12.

The final moments of Fig. 10 are repeated in Fig. 12, together with (in parentheses) the moment capacity of each member. Moments in columns **1** and **4** exceed the capacity of the members chosen. The moment at the top of column **4** can be reduced to -212 ft-kips by adding $+3$. Nothing can be assigned to beam **8** to balance the joint because the beam is already hinged at the joint. Therefore, the joint is balanced through beam **7**, which must be accommodated by a new distribution of moment in the beam. The distribution of Fig. 2(a) would be simplest, since it would not unbalance the sway moment. However, it cannot be used because the moment at midspan would then exceed the capacity of beam **7**. Use the distribution of Fig. 2(b), which adds $+3$ to the top of the left column and unbalances the story moment by $+3 + 3 = 6$ ft-kips. Balance is restored by adding -6 to column **6**, balancing the joint with $+6$ in the beam, and then adjusting the moment in beam **8** with the distribution of Fig. 2(d).

The moment at the top of column **1** is reduced to 233 ft-kips and the joint is balanced with $+9$ at the bottom of column **2**. A similar operation reduces the moment at the bottom of column **4** to the required value. Story-moment balance is restored by adding -12 to column **6** and $+12$ to column **5**. The resulting moment distribution nowhere exceeds the plastic-moment capacity of the members and it is in equilibrium with the loads. Therefore, the frame as sized in Figs. 10 and 11 is adequate.

The fact that two of the lower-tier columns (**1** and **5**) are lighter than the second-tier columns might present erection difficulties, unless the 35-ft length is erected in one piece. The splice between the two must be below the top flange of the beam a distance sufficient to reduce the moment to the capacity of the bottom-tier column.

The frame is checked in Figs. 13 and 14 for the combination of wind and gravity load at the load factor 1.4. The reduced beam-mechanism moments are balanced at

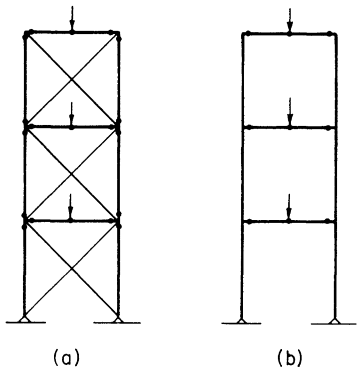


Figure 15

the joints (Fig. 14) before proceeding with the wind moments. At the floor, joints are balanced by assigning most of the moment to the upper-tier columns. This is done to reserve as much moment capacity as possible to take care of the wind moment in the lower tier.

The required upper-story moment (63 ft-kips) is developed at the roof level, beginning with the distribution of Fig. 2(b) in both beams. A hinge is developed in column **6** by using the distribution +2, 0, +2 in beam **8**. Column **4** limits the distribution in beam **7** to +26, 0, +26. These two produce only 56 ft-kips of story moment. An additional 8 ft-kips is obtained by the distribution +8, +4, 0 in beam **7**. Distributions in the same manner enable the lower-tier moment, −280 ft-kips, to be developed by additional moments of −100, −120, and −60 ft-kips in columns **1**, **3**, and **5**, respectively. This results in −155 ft-kips in column **3**, which exceeds its capacity by 1 ft-kip. However, this is easily corrected by the distribution −1, +1, +1 in beam **10** (not shown), which has the effect of transferring 1 ft-kip from column **3** to column **5**, so that the final moments in these columns would be −154 and −146 ft-kips, respectively. The result is a distribution of moments which is in equilibrium with the loads and which does not exceed the capacity of any member of the frame. Therefore, the frame is adequate for gravity and wind loads.

The column moment capacities which were computed in Fig. 11 were used in the preceding analysis for wind and gravity loads. This is on the safe side. The reason is that the reductions in moment capacity in Fig. 11 are based on axial forces resulting from gravity load multiplied by the load factor 1.85, whereas the load factor need be only 1.4 for gravity loads plus wind. Of course, this could have been taken into account by recomputing the reduced column moments in Fig. 11. However, this would entail some allowance for the additional column axial forces due to the wind forces. These would have to be estimated, since they are not known until final distribution of moments has been determined.

HIGH-RISE FRAMES

It is obvious that this procedure can be used to develop moment distributions for frames of any height. Thus, each additional floor involves nothing more than an arbitrary allocation of the beam moments to the columns meeting at the floor. If the frame has a trussed bracing system, or if it is supported by a shear-wall core through diaphragm action of floors or horizontal trusses, it can be considered restrained against sidesway. This means that all columns, except those in the first tier, can be classified as Case I (reversed curvature), which are the simplest to size. Of course, the analysis for lateral forces (Figs. 13 and 14) is eliminated. For the self-braced frame, however, axial forces (due to wind) in the columns which are part of the bracing system must be calculated in checking the adequacy of the frame for wind and gravity loads. In addition, axial forces in the girders may become significant at the lower levels.

The unbraced high-rise frame, i.e., one which is not supported against sidesway in the manner just described, presents certain complications which will be discussed briefly. Such frames develop certain moments which are ignored in the preceding examples. These are the additional moments of the gravity loads caused by the increased moment arms resulting from sidesway. Thus, in Fig. 1(f), with the horizontal displacement of the beam denoted by Δ, there is a moment $P\Delta$ which is not accounted for in the examples. This effect can be approximated by an equivalent lateral force $P\Delta/h$ at each floor. This involves a preliminary estimate of the story deflections Δ, which can be computed subsequently and the analysis revised if necessary.

Gravity load alone will determine the design of only the upper stories of the unbraced high-rise frame, after which the combination of wind and gravity load at the reduced load factor must be considered in establishing a distribution of moments for preliminary purposes. Heyman gives the formula[2]

$$\frac{n}{m} = \frac{1}{2}\frac{WL}{Hh}\left(\frac{F_G}{F_W} - 1\right)$$

where

n	=	Number of upper stories likely to be controlled by gravity load
m	=	Number of bays
F_G, F_W	=	Load factors, gravity load, wind load
W	=	Service load per bay, dead plus live
H	=	Lateral force per story
h	=	Story height

Figure 15(a) shows the system of hinges which could develop in a braced frame. The joints are held in essentially their unloaded positions. With respect to lateral forces, the frame at this stage is equivalent to a pinjointed truss. However, in Fig. 15(b) it is seen that, in the absence of bracing, the columns tend to become vertical cantilevers. The stiffness of the frame deteriorates progressively as the plastic hinges form, and, as a result, the unbraced frame may assume a new equilibrium configuration involving sidesway before the complete mechanism is developed. Of course, if side loads act in conjunction with the gravity loads, lateral displacements begin immediately upon application of load and increase until a peak load is reached, after which equilibrium can be maintained at increased deflections only if the loads are reduced. Procedures for investigating these phenomena have been developed.[3]

CONCLUSION

The procedure illustrated in this paper is called "moment balancing" by some and "moment distribution" by others. It is significant that it bears little or no resemblance to moment distribution based on elastic behavior. In elastic design one must fit the moment diagram to the structure; this is because the distribution of moments depends on relative stiffnesses and cannot be determined until the frame has been proportioned. On the other hand, plastic design allows one to construct a moment diagram and then proportion a frame to fit it; this can be done because the distribution of moments is according to plastic-moment capacities rather than relative stiffnesses. Thus, whether computations are by hand or by digital computer, plastic design offers greater freedom in the design of frameworks.

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