

Generalized Superposition Method in Plastic Analysis

GEORGE PINCUS

A RECENT DESCRIPTION OF the method of combining mechanisms in plastic analysis has been presented.¹ Further refinements in the use of the method have been developed and are presented herein. These include the use of a sign convention in determining plastic rotation angles and a tabular arrangement for the ultimate load of independent and combined mechanisms arising out of two independent mechanisms. Constant comparison of the magnitude of the ultimate loads is then possible while the use of the sign convention in calculating the ultimate loads of composite mechanisms immediately eliminates some of the possible combinations from consideration.

INTRODUCTION

Several textbooks on plastic analysis describe the superposition of independent mechanisms method in calculating the ultimate load of combined mechanisms.^{2, 3, 4, 5} The method consists of adding appropriate independent mechanisms, noting cancelled hinges, and subtracting the corresponding internal work from the right side of the work equations. It may be shown that whenever a hinge cancellation occurs, the ultimate load of the composite mechanism is smaller than the ultimate load for any or all of the component independent mechanisms. This means that if a hinge cancellation occurs when combining any number of independent mechanisms, the particular combination must be investigated. If no hinges are cancelled, then the resulting ultimate load is no smaller than the individual ultimate loads and no analysis is necessary. Deciding if a cancellation can take place is greatly simplified using a convenient sign convention for the plastic hinge rotations.

The sign convention consists of placing a dashed line on one side of all the members in the structure sketch. If the plastic rotation is such that the original member-to-member or member angle decreases on the side of the dashed line, the rotation is negative. If the original angle increases, the rotation is positive. Since the internal

plastic moment always acts in the direction of the corresponding plastic rotation, the internal work at each plastic hinge (plastic strength times plastic hinge rotation) is always positive.

It should be noted that generally the independent mechanism geometric configurations and work equations must both be multiplied by appropriate factors to obtain the desired cancellations.¹

EXAMPLE

The superposition method is used to find the failure load for the gable frame shown in Fig. 1. The dashed line is arbitrarily drawn for use as a sign convention for the rotations.

The failure load is first calculated for the four independent mechanisms shown in Fig. 2. The plastic hinge rotations, with appropriate signs, are also shown in the figure. For example, in Mechanism 4, Fig. 2(d), the total rotation at the corner 6 is the algebraic sum of the rotation of the column 6-7, $-\theta$ because the angle on the side of the dashed line decreases when the column rotates, plus the rotation of member 4-5-6 about its instantaneous center of rotation, $+\theta$ because it also decreases the angle. The total rotation at the corner 6 is then $-\theta$. At the gable, the total hinge rotation will be $+2\theta$ ($+\theta$ due to the rotation of member 4-5-6 plus $+\theta$ due to the rotation of member 2-3-4, both increasing the original inside gable angle).

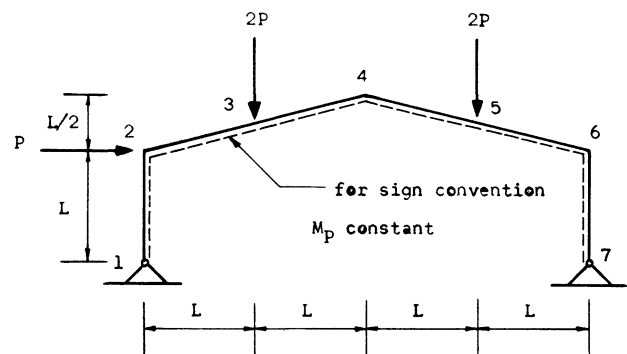


Fig. 1. Gable frame

George Pincus is Associate Professor, Department of Civil Engineering, University of Kentucky, on leave. He is presently Chief-of-Party, USAID/University of Houston Graduate Engineering Program, Rio de Janeiro, Brazil.

The work equations for the independent mechanisms are:

$$\text{Mechanism 1: } 2P\theta L = 4M_p\theta \quad (1)$$

$$\text{Mechanism 2: } 2P\theta L = 4M_p\theta \quad (2)$$

$$\text{Mechanism 3: } P\theta L = 2M_p\theta \quad (3)$$

$$\text{Mechanism 4: } 4P\theta L = 5M_p\theta \quad (4)$$

The ultimate loads for these four independent mechanisms are listed in Table I at the intersection of 1 and 1; 2 and 2; 3 and 3; 4 and 4.

The combination 1-2 is seen to produce no cancellation, since only the top gable hinge is common to both, and the individual rotations are both negative at that point (see Figs. 2(a) and 2(b)). A symbol **NC** is then entered in line 1, column 2, meaning no cancellation occurs when superposing mechanisms 1 and 2 and no further calculation is necessary.

The combined mechanism 1-3 must be considered, since a left top column cancellation results ($-\theta + \theta = 0$). Using the superposition method:

$$\text{Mechanism 1: } 2P\theta L = 4M_p\theta - M_p\theta \quad (5)$$

$$\text{Mechanism 3: } P\theta L = 2M_p\theta - M_p\theta \quad (6)$$

$$\hline 3P\theta L = 4M_p\theta \quad (7)$$

and this ultimate load is also entered in Table I.

Table I. Ultimate Loads of Independent and Complex Mechanisms Derived from Any 2 Independent Mechanisms

Mechanism	1	2	3	4
1	$2M_p/L$	NC	$4M_p/3L$	$9M_p/8L$
2		$2M_p/L$	NC	$9M_p/8L$
3			$2M_p/L$	M_p/L
4				$5M_p/4L$

The combined mechanism 1-4 must be considered since a top gable cancellation is possible. However, either mechanism 1 must be multiplied by a factor of 2 or mechanism 4 must be multiplied by a factor of $1/2$:

$$2 \times \text{Mechanism 1: } 4P\theta L = 8M_p\theta - 2M_p\theta \quad (8)$$

$$\text{Mechanism 4: } 4P\theta L = 5M_p\theta - 2M_p\theta \quad (9)$$

$$\hline 8P\theta L = 9M_p\theta \quad (10)$$

The corresponding ultimate load is entered in Table I.

By the same procedure, the combinations 2-3 and 2-4 are analyzed and the ultimate loads entered in the table. Since the reverse combinations yield the same ultimate loads, the spaces below the diagonal in the table will have the same values as the symmetrical ones above the diagonal and may be left blank.

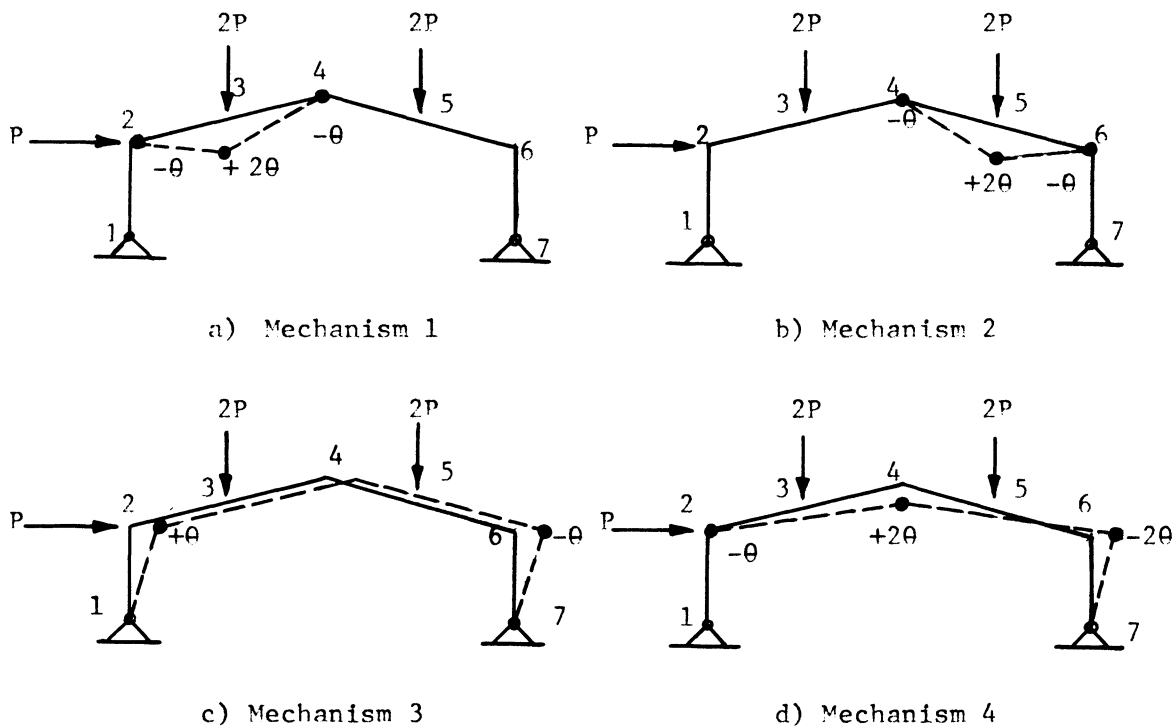


Fig. 2. Independent mechanisms

The combinations of any three independent mechanisms must also be considered. The calculation is simplified by eliminating those combinations that do not produce additional cancellations beyond those in a two-mechanism combination. For example, **1**, **2** and **3** need not be investigated because **1** and **2** produce no cancellation while **1-3** and **2-3** have been investigated. However, others must be investigated. For example, the combination of **1**, **3** and **4** will produce hinge cancellations at the top left column and at the gable. The factors that must multiply the geometry and the corresponding work equations are found by noting the canceling conditions. (Mechanism **1** is multiplied by a factor M while Mechanism **3** is multiplied by a factor N .) In order to obtain a cancellation at the top left column and at the gable, the following conditions must be true:

$$\text{Top left column: } M(-\theta) + N(+\theta) + (-\theta) = 0 \quad (11)$$

$$\text{Gable: } M(-\theta) + N(0) + (2\theta) = 0 \quad (12)$$

from which $M = 2$ and $N = 3$. The first term in each equation is the rotation in mechanism **1**, the second term is the rotation in mechanism **3**, and the third term is the rotation in mechanism **4**. The modified equations are:

$$2 \times \text{Mech. 1: } 4P\theta L = 8M_p\theta - 2M_p\theta - 2M_p\theta \quad (13)$$

$$3 \times \text{Mech. 3: } 3P\theta L = 6M_p\theta - 3M_p\theta \quad (14)$$

$$\text{Mech. 4: } 4P\theta L = 5M_p\theta - M_p\theta - 2M_p\theta \quad (15)$$

$$11P\theta L = 9M_p\theta \quad (16)$$

All other combined mechanisms may be analyzed in an analogous manner. The combination **1-3-4** happens to yield the smallest ultimate load and is therefore the true failure mechanism.

CONCLUSIONS

A sign convention for plastic hinge rotations may be used in calculating the ultimate loads of combined mechanisms when using the superposition method. One major advantage of the sign convention is clear identification of those combined mechanisms where canceled plastic hinges do not occur and can therefore be immediately eliminated. A tabular arrangement of the magnitude of ultimate load for independent and two-independent mechanisms combinations is also useful. A gable frame solution has been presented to illustrate the use of the sign convention and as a review of the general superposition method. With the additional features, the method becomes an extremely powerful tool in calculating the ultimate loads of complex mechanisms.

REFERENCES

1. Pincus, G. Method of Combining Mechanisms in Plastic Design, *AISC Engineering Journal*, Vol. 2, No. 3, July 1965, p. 85.
2. Neal, B. G. The Plastic Method of Structural Analysis, *Chapman and Hall, London, 1963*.
3. Lothers, J. E. Advanced Design in Structural Steel, *Prentice-Hall, Englewood Cliffs, New Jersey, 1960*.
4. Hodge, P. G. Plastic Analysis of Structures, *McGraw-Hill Book Co., New York, 1959*.
5. Beedle, L. S. Plastic Design of Steel Frames, *John Wiley and Son, New York, 1958*.