

Hyperboloid Space Frames for Tower Structures

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IN RECENT YEARS the quest for esthetics in structural design has resulted in a number of new forms, utilizing shells, folded plates and other sophisticated shapes in reinforced concrete and metals. Rather pleasing effects have been achieved by exposing the structural framing on the exterior of building walls in checkered and diamond patterns. In the design of towers, graceful shapes and patterns of framing have been obtained by following the geometry of hyperboloids and other curved surfaces. While glamour, display and advertisement may be the factors inspiring some lavish architecture, there is still a need for esthetic treatment of more prosaic structures serving industrial and utilitarian purposes. One problem recognized by many architects and engineers is the water tower or elevated tank of conventional utilitarian design, an uninspiring sight at its best, and more often a blot on the landscape and a blight to the neighborhood.

More attention is being given to improvement in appearance of water towers. Generally, better architectural treatment involves greater costs. More pleasing shapes of the tank proper, such as spheres and spheroids are not costly in themselves. The supporting tower structure, however, is more of a problem. Among recent designs, esthetic solutions are provided by rigid frames of steel or concrete, by metal shells as main supporting structure or as cover for interior framing, and by massive architectural masonry.

Similar considerations apply to other structures, such as radar towers and airport control towers, which are becoming frequent and prominent landmarks.

It is believed that the hyperboloid configurations, with rectilinear elements, offer interesting solutions to the problem of tower design, and realize considerable esthetic values, combined with simplicity of construction and moderate cost. Perhaps the obstacle to wider use of the hyperboloid is the lack of familiarity with its geometry and the impression of complexity and indeterminacy of stress analysis.

In this paper, the geometric properties of the hyperboloid are derived and a simple method of stress analysis is developed. A whole group of space frames, based on the hyperboloid, is described; these are self braced and capable of supporting vertical, horizontal and torsional loads. For the critical cases of loading commonly used in design, it is shown that the frames are statically determinate.

DEFINITIONS AND GEOMETRY

A hyperbola $x^2/r^2 - z^2/c^2 = 1$ revolving about the Z -axis generates a surface $x^2/r^2 + y^2/r^2 - z^2/c^2 = 1$, which is a hyperboloid of one sheet (see Fig. 1). All sections of the hyperboloid parallel to the XY plane are circles of varying radii. The circle of minimum radius has its center at the origin and is in the XY plane. This is the waist line of the hyperboloid. If bounded by two circles equidistant from the XY plane, the hyperboloid is symmetrical about this plane. If not equidistant, the circles are not equal and the hyperboloid is not symmetrical.

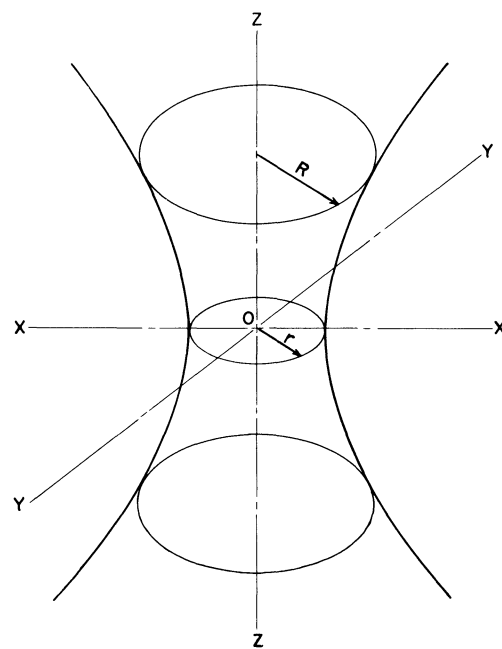


Fig. 1. The hyperboloid of revolution

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The hyperboloid is also defined as a ruled surface, containing two systems of straight lines. The surface may be regarded as generated by a straight line inclined to the XY plane and revolving about the Z -axis, while maintaining a constant inclination to the XY plane.

In Fig. 2, line **A** of length L revolves clockwise about a vertical axis, so that the inclination of the line to the horizontal (angle β) remains constant. The lower end of the line describes a circle of radius R_1 ; the upper end describes a circle of radius R_2 . Angle ϕ is the constant phase angle between the two ends of the line. Similarly

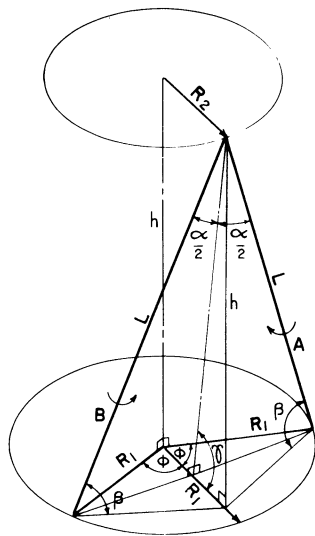


Fig. 2. Straight line generators of the hyperboloid

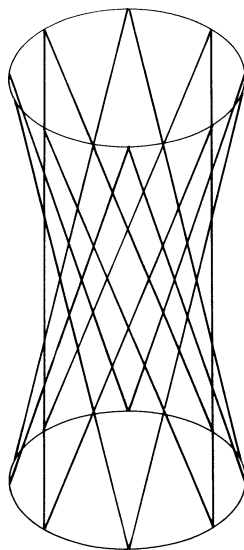


Fig. 3. The hyperboloid as a ruled surface

line **B**, of the same length and inclination, revolves counterclockwise. The pair of lines **A** and **B** represent the two systems of lines of the generated surface of the hyperboloid (Fig. 3). It may be observed that the slimness of the waist line depends on the radii R_1 , R_2 and angle ϕ . When $\phi = 0$ the generated surface becomes a frustum of a cone. When $\phi = 0$ and $R_1 = R_2$ it becomes a cylinder. When either R_1 or R_2 is zero the surface is a cone. When $\phi = 180^\circ$ the surface is a quadric cone.

The angle formed by lines **A** and **B** is α and the plane defined by the two lines has an inclination γ to the horizontal.

The following relationships are obtained from Fig. 2:

$$L = \frac{R_1 \sin \phi}{\sin \alpha/2} \quad (1)$$

$$\tan \frac{\alpha}{2} = \frac{R_1 \sin \phi}{\sqrt{(R_2 - R_1 \cos \phi)^2 + h^2}} \quad (2)$$

$$\sin \beta = \frac{h}{L} \quad (3)$$

$$\tan \gamma = \frac{h}{R_2 - R_1 \cos \phi} \quad (4)$$

HYPERBOLOID SPACE FRAMES

Based on the concept of the hyperboloid and its generators, a space frame may be constructed as shown in Fig. 4. Two horizontal circles of radii R_1 and R_2 , spaced a distance h apart, are divided into n equal parts. In Fig. 4, twelve parts are shown, marked U_1, U_2 , etc., on the upper circle and correspondingly L_1, L_2 , etc., on the lower circle. From U_{12} a pair of inclined lines are drawn to points on the lower circle, such as L_3 and L_9 . Thus a phase angle ϕ (90°) is established. Similarly, pairs of lines are drawn from all upper points to all lower points, with the same phase angles. The points on each circle are joined by chords, forming regular polygons. This completes the space frame having polygonal top and bottom and straight line legs.

In Figs. 5, 6, 7, 8 and 9, other frames are shown with base polygons having 3, 4, 5, 6 and 8 sides, respectively. The triangular frame has the least number of legs (six in all). The hexagonal frame has twelve legs, the octagonal frame has sixteen. Since ϕ must be in multiples of the chords of the chosen polygon, and must be less than 180° , the choice of ϕ is limited. For the triangular frame only one value is possible, $\phi = 120^\circ$ (Fig. 5). For the square, $\phi = 90^\circ$ (Fig. 6). For the pentagon, $\phi = 72^\circ$ or 144° (Fig. 7). For the hexagon, $\phi = 60^\circ$ or 120° (Fig. 8). For the octagon, $\phi = 45^\circ, 90^\circ$, or 135° (Fig. 9). Except for the triangle, which is stable in itself, all other poly-

gons must be stabilized either by making their joints rigid or by subdivision into triangles, with additional members. In lieu of the polygons, the top and bottom of the frames may be flat plates. The connections of the inclined legs to the upper and lower polygons (or plates as the case might be) need not be rigid, since the pairs of legs form A-frames, which are stable even with hinged connections.

The inclined legs form intersections at several intermediate levels between the upper and lower bases. The number of intersections depends on the number of legs and the phase angle. At the intersections, the two systems of members will touch, and may be bolted, riveted or welded to provide mutual bracing and reduce the unsupported lengths of members in compression. At each level the points of intersection are in one horizontal plane (i.e., parallel to the two base polygons) and by introducing intermediate polygons, more substantial lateral bracing may be obtained.

Figure 10 shows a water tower having an octagonal frame with three intermediate horizontal octagons for lateral bracing. Note that concrete foundations take the place of the lower structural polygon. Figure 11 illustrates a tower concordant with a pyramidal base of steel or masonry.

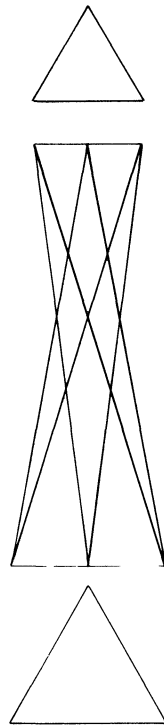


Fig. 5. Triangular frame ($\phi = 120^\circ$)

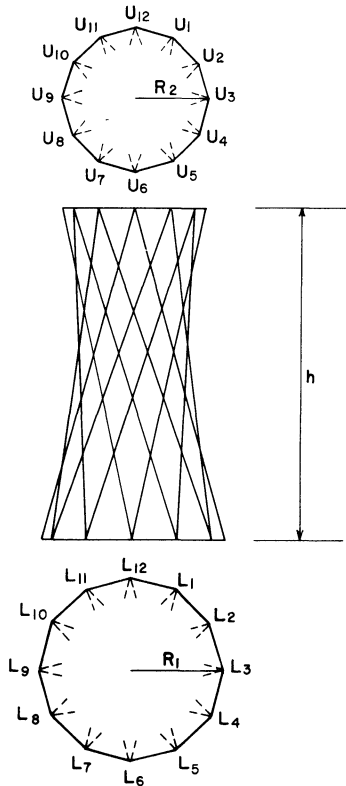


Fig. 4. Construction of a hyperboloid space frame

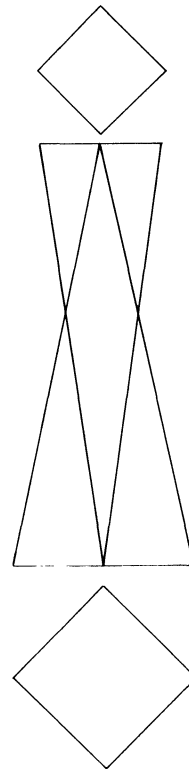


Fig. 6. Square frame ($\phi = 90^\circ$)

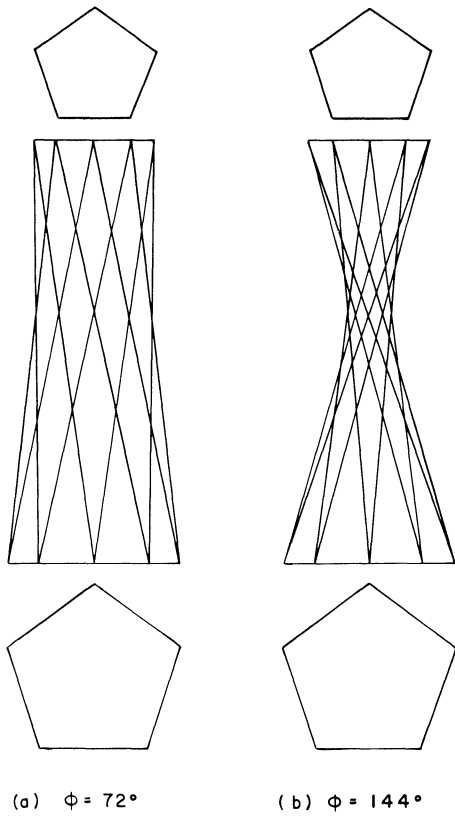


Fig. 7. Pentagonal frames

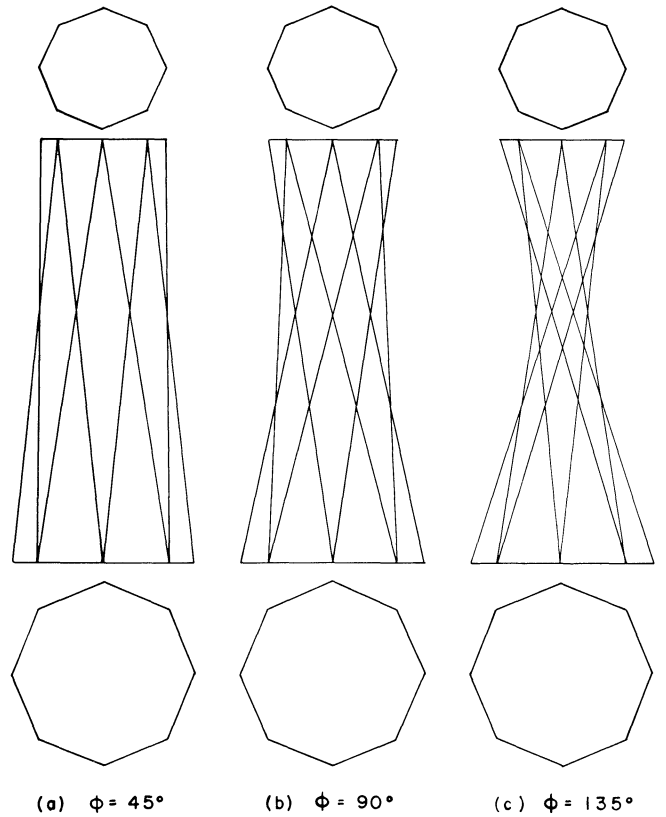


Fig. 9. Octagonal frames

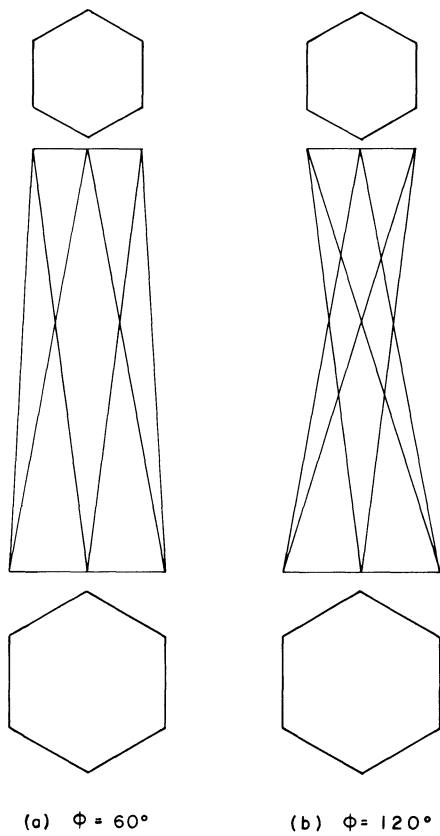


Fig. 8. Hexagonal frames

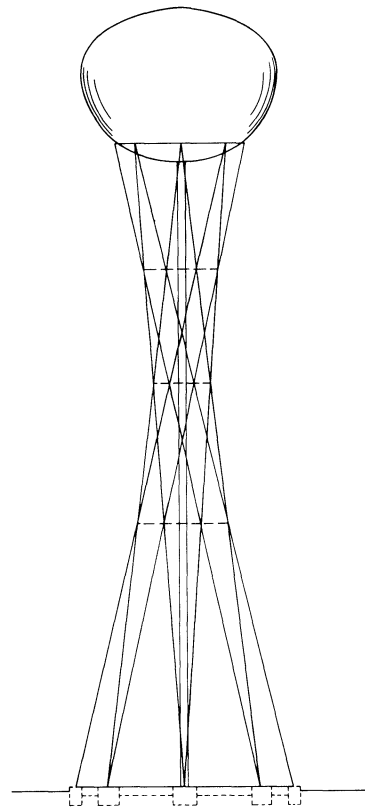


Fig. 10. Octagonal hyperboloid frame for a water tower

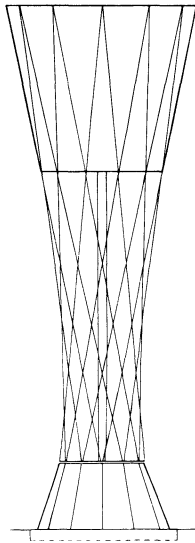


Fig. 11. Hyperboloid tower with pyramidal base

STRESS ANALYSIS

It may appear that with multiple points of support and multiple intersections, this type of frame would be highly indeterminate for stress analysis. However, due to the symmetry and regularity of the frame, the problem is greatly simplified. For most loading conditions encountered in practice and used in design, the primary stresses can be obtained by methods of statics. The analysis of the tower frame of Fig. 4 is given here in general terms for several cases of loading.

Case 1—Torsion about vertical axis. Moment M applied at the upper polygon of the frame (Fig. 12).

Since all A-frames comprising the space frame (12 A-frames in Fig. 12) are identical and equally oriented

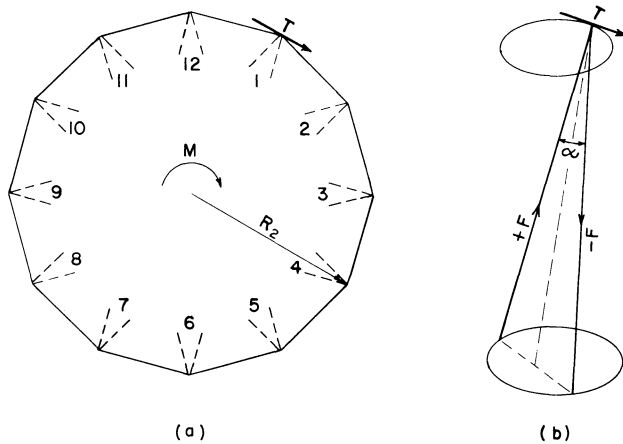


Fig. 12. Analysis for torsion

with respect to the vertical axis of rotation, they all equally resist the torsion. The moment M may be represented by tangential forces T acting at each vertex of the polygon and delivering stresses to each A-frame. Resolving T in the plane of each A-frame, one leg receives tension F and the other equal compression.

$$T = \frac{M}{R_2 n} \quad (5)$$

where n is the number of A-frames. In this case $n = 12$. From Fig. 12b:

$$F = \pm \frac{T}{2 \sin \alpha/2} \quad (6)$$

Case 2—Eccentric horizontal load H , applied in the plane of the upper polygon (Fig. 13).

The eccentric load is resolved into a concentric load H and a moment He , where e is the eccentricity. The effect of torsion is analyzed as in Case 1.

Due to the concentric load H the structure sways and, since the polygon is rigid, all vertices are displaced an equal amount in the direction of the applied load. The individual A-frames take up the load in different proportions, depending on their orientation with respect to direction of H , shown by different angles θ in Fig. 13. The maximum stresses are in the frame with minimum θ . In Fig. 13, the maximum stresses are in frames 12 and 6 where $\theta = 0$, while frames 3 and 9 do not receive any stress from the concentric load H . This is obvious, since the swaying of the structure is accompanied by rotation of the plane of the upper polygon,

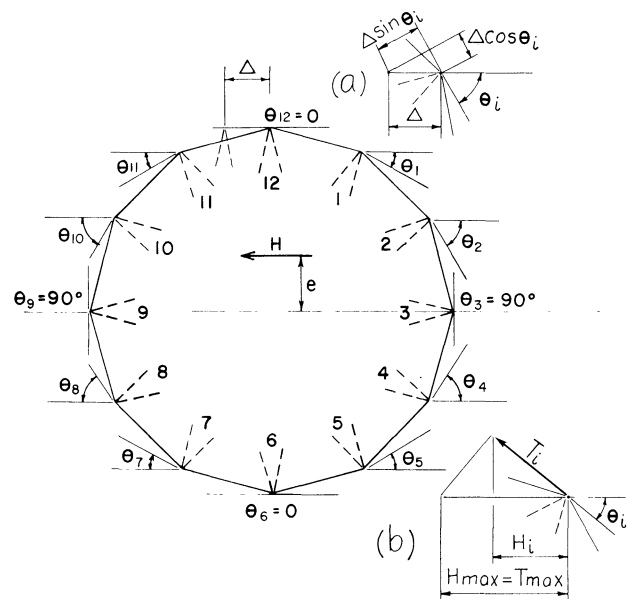


Fig. 13. Analysis for horizontal load

so that apex 3 is raised and apex 9 lowered, which relieves these frames from taking up a load. The tangential force T_i at the vertex of an individual frame, as derived in the Appendix, is:

$$T_i = \frac{H \cos \theta_i}{\Sigma \cos^2 \theta} \quad (7)$$

and this force is resolved into the legs of the A-frame by Equation (6).

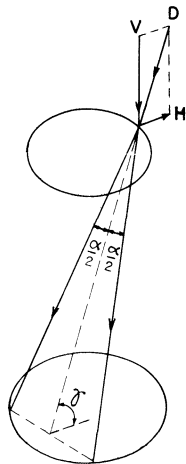


Fig. 14. Analysis for vertical load

Case 3—A downward vertical load V applied at a vertex of an A-frame (Fig. 14).

The vertical load V is resolved into a load D in the plane of the A-frame and a horizontal load H .

$$D = \frac{V}{\sin \gamma} \quad (8)$$

$$H = \frac{V}{\tan \gamma} \quad (9)$$

Load D produces equal compression in the two legs of the A-frame:

$$C = \frac{D}{2 \cos \alpha/2} \quad (10)$$

H is treated as in Case 2.

Case 4—Uniform vertical loading on all vertices of the upper polygon. The vertical loads are resolved as in Case 3, producing equal stresses in all legs, by Equation (10). The radial horizontal loads have a zero resultant; there is no sway and no stresses in the legs due to the uniform radial loading. However, there is a uniform tension in the members of the upper polygon, which is obtainable by methods of statics.

In any of these cases of loading, the legs carry their stresses to the points of the lower polygon, which is the foundation of the structure. In Fig. 10 the foundation is represented by concrete footings at each point, connected by concrete ties. From the loads delivered by the leg members, the loads on each footing and the tensile stresses in the ties are obtainable from statics. For a final check, $\Sigma V = \Sigma H = \Sigma M = 0$ is applied by taking moments about the plane of the foundation.

In the cases discussed, the loads were acting at the level of the top polygon. If there are considerable loads acting below that level, such as wind or earthquake loads acting on the legs of the tower, it is necessary to provide intermediate polygons previously mentioned. Two or three such polygons will usually be sufficient (Fig. 10). The horizontal loads may then be assumed as concentrated at the levels of the intermediate polygons. The portion of the tower below an intermediate polygon may be regarded as a free body, as a frame with its component A-frames, and analysed by the methods described.

SECONDARY STRESSES

In the previous analysis the assumption has been made that there is free action of the A-frames, not affected by the connections at the intersections.

For the case of uniform dead and live load (Case 4) all leg members are equally stressed, there is no tendency of relative motion between members at any of the intersections and there is no shear on the bolts or rivets. Therefore, there are no secondary stresses induced by the connections.

For the case of torsion (Case 1) deformations are such that all intersection points describe arcs of circles and all frame legs remain straight lines. Therefore there are no secondary bending stresses induced.

For non-uniform vertical loading, as well as for horizontal wind and earthquake loading, there is a drift and a tendency to slide at the intersection points, producing shear in the connections and hence secondary bending stresses in the leg members. However, these secondary stresses are not significant and may be disregarded in practical designs, where the leg members are long and slender. For very short and stiff frames it will be desirable to check secondary stresses by means of model tests.

ECONOMY AND ESTHETICS

From the illustrations it is apparent that the selection of the polygonal pattern and inclination of the legs (phase angle ϕ) will greatly influence the appearance of the structure. It will also influence the cost. Fewer (and larger size) legs will have a lower handling and construction cost than a larger number of lighter members.

On the other hand, a sparse frame may not look well. An over-dense frame may not be agreeable either, and may be costly.

Where vertical dead and live loads govern the design, a large ϕ would not be economical. For dominant horizontal loads, large ϕ is desirable. Again, the selection of ϕ , and hence the "waist line", should be moderated by architectural considerations.

Within the range of polygons from pentagon to decagon, there should be a range of moderate costs, combined with desirable appearance of the structure. In this range, the total number of members of a tower frame will be comparable to the number of main and bracing members in a conventional multi-column X-braced tower. The construction costs and the maintenance painting costs should be also comparable.

CONCLUSIONS

A method of framing for tower structures has been described, which utilizes straight members arranged in a pattern on a hyperboloid surface. A group of such frames has been developed and their geometric properties derived.

It has been demonstrated that these frames are statically determinate and a stress analysis presented for several common cases of loading conditions. The frames consist of main members which are mutually braced and do not require any secondary bracing. They may be designed in steel or reinforced concrete.

Economy and esthetics have been discussed.

Use of these frames for water towers, radar and airport towers has been suggested. Other possible uses may be for architectural elements in building design.

APPENDIX

Distribution of Horizontal Load Between A-Frames—

The assumption involved in this analysis is the rigidity of the upper polygon joining the vertices of the A-frames, so that all are displaced an equal distance Δ in the direction of the applied concentric load H (Fig. 13). Frames in planes parallel to H receive the greatest share of the load and the tangential force at their vertices is denoted as T_{\max} ($= H_{\max}$). Any other frame turned at an angle θ to H has two components of Δ , one in its own plane

($\Delta \cos \theta_i$) and another at right angles to it ($\Delta \sin \theta_i$). See Fig. 13a. The former component produces stress in the frame. The latter does not; it is a free rotation on the hinges at the bottom of the legs.

The tangential force acting at the vertex of the frame is therefore, (Fig. 13b):

$$T_i = T_{\max} \cos \theta_i$$

and its component parallel to H is:

$$H_i = T_i \cos \theta_i = T_{\max} \cos^2 \theta_i$$

The summation of all H 's is equal to the applied load H

$$T_{\max} \Sigma \cos^2 \theta = H$$

or

$$T_{\max} = \frac{H}{\Sigma \cos^2 \theta}$$

hence

$$T_i = \frac{H \cos \theta_i}{\Sigma \cos^2 \theta} \quad (7)$$

Numerical Example—

Given: In the polygon of Fig. 13, $R = 10$ ft, $e = 3$ ft, and $H = 1$ kip.

Solution:

$M = 3$ kip-ft

Due to moment:

$$T = \frac{3}{10 \times 12} = 0.025 \text{ kips}$$

Due to concentric H :

$$\Sigma \cos^2 \theta = (2 \times 0) + (2 \times 1) + (4 \times 0.25) + (4 \times 0.75) = 6.000$$

$T_3 = T_9 = 0$

$T_6 = T_{12} = \frac{1}{6} = 0.166$ kips

$T_1 = T_{11} = T_5 = T_7 = 0.866 \times 0.167 = 0.144$ kips

$T_2 = T_{10} \Rightarrow T_4 = T_8 = 0.500 \times 0.167 = 0.083$ kips