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A Practical Look at Frame Analysis, Stability and Leaning Columns

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INTRODUCTION

Analysis and design of unbraced moment frames is a fairly regular activity in structural engineering practice yet it can be a complex structural engineering problem. Numerous analysis methodologies are available and the many commercial software packages used in practice provide a variety of approaches to the problem. Some of the questions that arise during frame design include:

a) Is a first-order or second-order analysis more appropriate for a particular design?
b) Should an elastic or inelastic analysis be carried out?
c) What moment magnifiers should be used when axial load and moment act together?
d) Should effective length factors or some other approach be used to evaluate column capacity?

Frame analysis may be approached by a variety of methods. Linear elastic analysis is perhaps the most common, although the least complete. A second-order inelastic analysis, while perhaps the most comprehensive, is also the most complex. And there are many approaches between these. Whichever analysis method is chosen, the design approach must be compatible.

Stability of a column, although often expressed as a function of the individual column, is actually a function of all of the members in the story. Thus, column design is a system problem, not an individual member problem. When unbraced moment frames support pin-ended columns, additional problems arise. These pin-ended columns do not participate in the lateral resistance of the structure, but instead, rely on the unbraced frame for their lateral stability. Thus, the frame must be designed to accommodate the loads that are applied as well as the influence of these leaning columns.

Numerous approaches have been presented in the literature to address the design of frames both with and without leaning columns. Although a direct buckling analysis may be performed, the most common approaches still appear to be those that utilize some form of simplification.

This paper will briefly review a wide range of analytical approaches including elastic buckling analysis, as well as first- and second-order elastic and inelastic analytical methods. Once these analytical approaches have been presented, the design process will be addressed, including the use of effective length factors. Effective length calculations will be reviewed with particular attention to the approaches presented by Yura, Lim, and McNamara, LeMessurier, and the equations found in the AISC LRFD Commentary. The results from these approaches will be compared to those of an elastic stability analysis for simple frames that have been found in the literature.

This paper is an expansion of an earlier paper by Geschwindner (1994). It is hoped that it will help the engineer develop an understanding of these aspects of structural behavior in order to better understand new approaches that are currently being investigated and will likely impact future specifications.

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ANALYSIS

The state of the art of structural analysis encompasses a wide range of possible approaches for the determination of system response to structural loading. Each new approach adds or subtracts some aspect of frame or member behavior in an attempt to properly model the true behavior of the structure. It will be helpful to categorize these analysis approaches and discuss their characteristics. Figure 1 shows a comparison between the load-displacement curves of several analysis approaches. These approaches are well documented by McGuire, Gallagher, and Ziemian (2000) as well as in the individual references cited.

First-Order Elastic Analysis (West, 1989)

The first and most common approach to structural analysis is the first-order elastic analysis, which is also called simply elastic analysis. In this case, deformations are assumed to be small so that the equations of equilibrium may be written with reference to the undeformed configuration of the structure. Additionally, superposition is valid and any inelastic behavior of the material is ignored. Thus, the resulting load-displacement curve shown in Figure 1 is linear. This is the approach used in the development of the common analysis tools of the profession, such as slope-deflection, moment distribution and the stiffness method that is found in most commercial computer software.

Elastic Buckling Analysis (Galambos, 1968)

An elastic buckling analysis will result in the determination of a single critical buckling load for a system. The critical buckling load may be determined through an eigenvalue solution or through a number of iterative schemes based on equilibrium equations written with reference to the deformed configuration. This analysis can provide the critical buckling load of a single column and is the basis for the effective length factor. It can be seen in Figure 1 that the results of this analysis do not provide a load-displacement curve but rather the single value of load at which the structure buckles.

Second-Order Elastic Analysis (Galambos, 1968)

When the equations of equilibrium are written with reference to the deformed configuration of the structure and the deflections corresponding to a given set of loads are determined, the resulting analysis is a second-order elastic analysis. This is the analysis generally referred to as a P-delta analysis. Two components of these second-order effects should be included in the analysis. When the influence of member curvature is included, it is said that the P-δ effects or *member effects* are included and when the sidesway effects are included it is said that the P-Δ effects, also referred to as the *story sway* or *frame effects* are included. The load-displacement history obtained through this analysis may approach the critical buckling load obtained from the eigenvalue solution as shown in Figure 1. This analysis usually requires an iterative solution so it is a bit more complex than the first-order elastic analysis. Because of the problems inherent with iterative solutions, many researchers have proposed one-step approximations to the second-order elastic analysis. It should also be noted that not all commercial computer analysis software includes both the member effects and the frame effects.

First-Order Plastic-Mechanism Analysis (Disque, 1971)

As the load is increased on a structure, it is assumed that defined locations within the structure will reach their plastic capacity. When that happens, the particular location continues to resist that plastic moment but undergoes unrestrained deformation. These sections are called plastic hinges. Once a sufficient number of plastic hinges have formed so that the structure will collapse, it is said that a mechanism has formed and no additional load can be placed on the structure. Thus, a plastic-mechanism analysis can predict the collapse load of the structure. This limit can be seen in Figure 1.

First-Order Elastic-Plastic Analysis (Chen, Goto, and Liew, 1996)

If the determination of the collapse mechanism tracks the development of individual hinges, more information, such

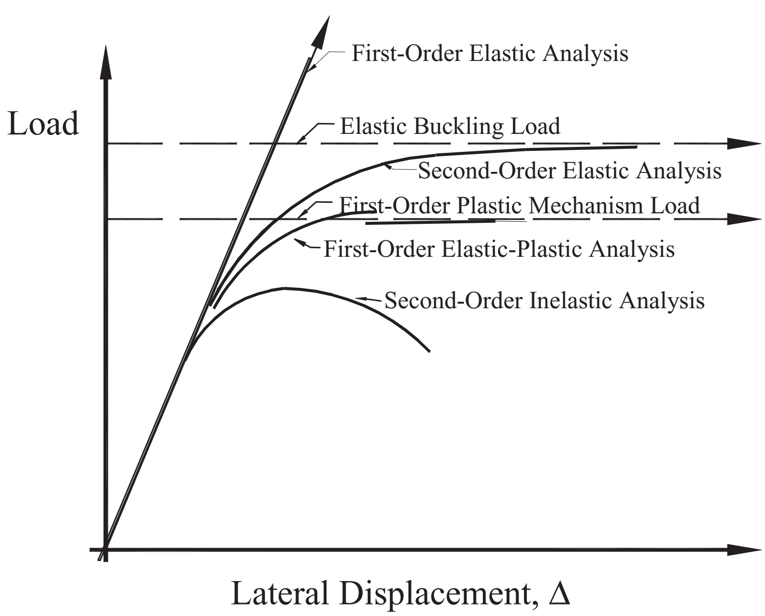


Fig. 1. Load-displacement history.

as deflections and member forces, is obtained from this analysis than from the mechanism analysis. It is clear that if zero length hinges are assumed and the geometry is maintained, the limit of the elastic-plastic analysis will be the mechanism analysis as seen in Figure 1.

Second-Order Inelastic Analysis (Chen and Toma, 1994)

This analytical approach combines the same principles of second-order analysis discussed previously with the plastic hinge analysis. This category of analysis is more complex than any of the other methods of analysis discussed thus far. It does, however, yield a more complete and accurate picture of the behavior of the structure, depending on the completeness of the model that is used. This type of analysis is often referred to as "advanced analysis." The load-displacement curve for a second-order inelastic analysis is shown in Figure 1.

In summary, it can be seen that as more realistic and hence more complex behavior is taken into account in the analysis, the predicted critical load level is reduced or the calculated lateral displacement for a given load is increased. Thus, designers need to be aware of the assumptions utilized in any analytical approach that they employ. This is particularly important when using commercially available software.

DESIGN

The approach taken for member design must be consistent with the approach chosen for analysis. Currently, three design approaches are acceptable for steel structures under US building codes as they incorporate AISC specifications (AISC, 1999; AISC, 1989). The most up-to-date tool for steel design is the load and resistance factor design specification (LRFD). However, the plastic design (PD) approach is also permitted and the allowable stress design specification (ASD) is still used.

The LRFD *Specification* stipulates, in Section C1, that "Second order effects shall be considered in the design of frames." The comparable statement in the ASD *Specification* states, in Section A5.3, "Selection of the method of analysis is the prerogative of the responsible engineer," and in Section C1 that "frames…shall be designed to provide the needed deformation capacity and to assure overall frame stability." Since the typical analysis method is first-order, satisfying deformation capacity requirements and assuring stability are left to the engineer.

Thus, regardless of the specification used, the engineer is required to address stability and second-order effects. When using the AISC *Specifications*, stability is usually addressed through an estimate of column buckling capacity while second-order effects may be addressed through a first-order analysis coupled with a code-provided correction for second-order effects or through direct use of a second-order analysis.

Difference between Second-Order Elastic Analysis and Elastic Buckling Analysis

The frame shown in Figure 2 will be used to demonstrate the difference between the results of a first-order elastic analysis, a second-order elastic analysis, and an elastic buckling analysis. All three of these analyses were carried out using GTSTRUDL (1999) including axial, flexural, and shearing deformations. The equivalent shear area used in these calculations is the area of the web as defined by AISC. The frame is composed of three W8×24 members with gravity load, P, applied as shown and a single lateral load of $0.01P$. For this simplified problem, the column supports are treated as pins.

The results of the three analyses are shown in Figure 3. The first-order elastic analysis yields a straight-line load-

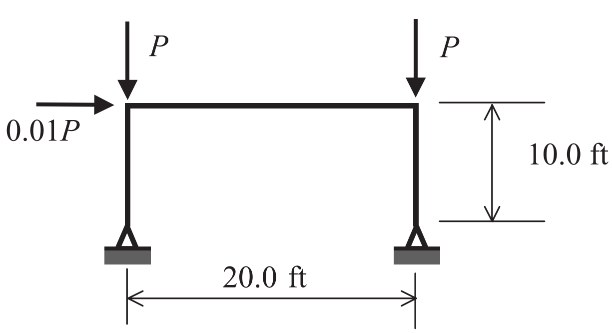


Fig. 2. Frame for comparison of analysis results.

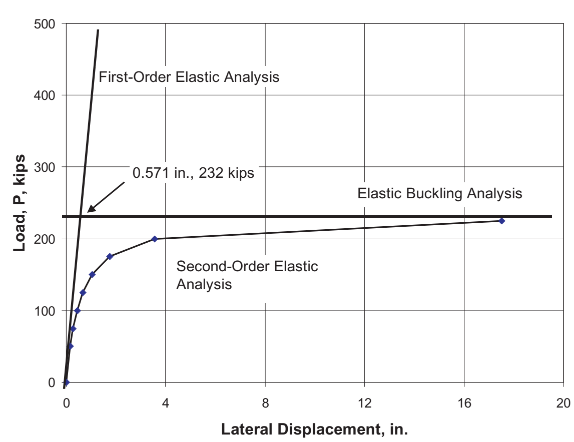


Fig. 3. Comparison of load/lateral displacement results for frame of Fig. 2.

displacement relationship as shown. An elastic buckling analysis yields a critical load of P_{cr} = 232 kips with the frame buckling in a sidesway mode. The intersection of the first-order analysis with P_{cr} = 232 kips is a displacement of 0.571 in.

The results of the second-order elastic analysis are also shown in Figure 3. This analysis was carried out at eight different load levels. It can be seen that as the magnitude of the load P is increased, the lateral displacement increases at a progressively greater rate. This reflects the influence of the additional moments induced as the structure deflects. As the load approaches 232 kips, the slope of the load-displacement curve approaches zero and the displacement tends toward infinity, confirming that a second-order elastic analysis can be used to approximate the results of an elastic buckling analysis.

IMPACT OF SECOND-ORDER EFFECTS ON A SINGLE COLUMN

Two different second-order effects will impact the design of a single column. The first, illustrated in Figure 4a for a column in which the ends are prevented from displacing laterally with respect to each other, is the result of the bending deflection along the length of the column. If the moment equation is written with reference to the displaced configuration, it can be seen that the moments along the column will be increased by an amount $P\delta$. As already discussed, this increase in moment due to chord deflection is referred to as the $P\delta$ or member effect.

The column in Figure 4b is part of a structure that is permitted to sway laterally an amount Δ under the action of the lateral load, H. As a result, the moment required on the end of the column to maintain equilibrium in the displaced configuration is given as $HL + P\Delta$. This additional moment, $P\Delta$, is referred to as the frame effect, since the lateral displacement of the column ends is a function of the properties of all of the members of the frame participating in the sway resistance.

The deflections, δ and Δ, shown in Figure 4 are second-order deflections, resulting from the applied loads plus the deflections resulting from the additional second-order moments. These displacements are not the displacements resulting from a first-order elastic analysis but from a second-order elastic analysis. Although second-order deflections are more complicated to determine than first-order deflections, they appear to be straightforward for the individual column of Figure 4. However, when columns are combined to form frames, the interaction of all of the members of the frame significantly increases the complexity of the problem. The addition of gravity only columns that do not participate in the lateral frame resistance brings further complexity to the problem.

For engineers using commercial software packages to carry out a second-order elastic analysis, it is important to fully understand the assumptions made in the development of that software. For instance, most commercial applications include only the $P\Delta$ or frame effects and do not include the $P\delta$ or member effects. In addition, as with the results presented in this paper, GTSTRUDL includes axial,

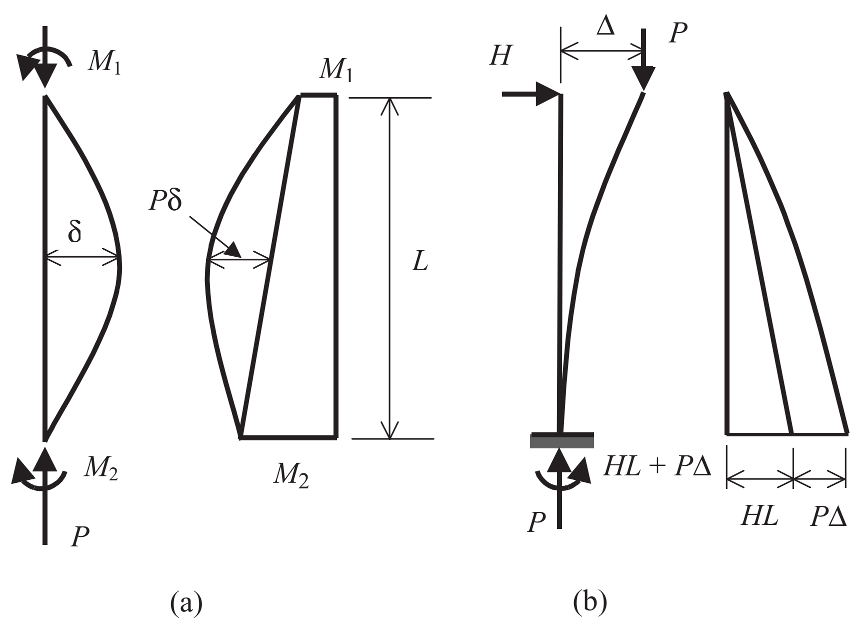


Fig. 4. Influence of second-order effects.

flexural, and shearing deformations in the analysis when member properties are selected from the property table and material is specified as steel. This may or may not be important depending on the particular situation.

PREDICTING THE CRITICAL ELASTIC BUCKLING LOAD

When an analysis tool is available to determine the critical elastic buckling load of a frame, there is no need to predict that load through some other means. Thus, it might be said that if all structural analysis were carried out using an elastic buckling analysis, there would be no need to spend time discussing the correct approach for determining an elastic *K*-factor to use in design. It seems that ever since the *K*-factor was introduced into the 1961 AISC *Specification*, it has generated extensive discussion and misunderstanding (Higgins, 1964). To understand the debate over the *K*-factor, one must understand what the *K*-factor is intended to accomplish. The critical buckling load of a column, determined by one of the elastic buckling analysis programs is taken as P_{cr}. It will be helpful to remember that the critical buckling load of the perfect column, as derived by Euler, is given as

$$P_e = \frac{\pi^2 EI}{L^2} \tag{1}$$

Since the column in a steel frame is not likely to have perfectly pinned ends, but rather some end restraint and the possibility of sidesway, its critical buckling capacity can be said to be somewhat different than the Euler column, thus

$$P_{cr} = P_e \times (\text{modification factor}) \tag{2}$$

If that modification factor is defined as $1/K_{exact}^2$, it is seen that

$$P_{cr} = \frac{\pi^2 EI}{(K_{exact} L)^2} \tag{3}$$

Thus, the *K*-factor is simply a mathematical adjustment to the perfect column equation to try to predict the capacity of an actual column. Every method or equation that is proposed for the determination of the *K*-factor or effective length factor is simply trying to accurately predict the actual column capacity as a function of the perfect column.

Perhaps the most commonly used approach for the determination of *K*-factors is the nomograph found in the commentary to both the LRFD and ASD *Specifications* (AISC, 1999; AISC 1989). The equation upon which the sidesway permitted nomograph is based is given in Equation 4 (Galambos, 1968).

$$\frac{G_A G_B (\pi/K)^2 - 36}{6(G_A + G_B)} = \frac{\pi/K}{\tan(\pi/K)} \tag{4}$$

and

$$G = \frac{\Sigma I_c / L_c}{\Sigma I_b / L_b}$$

The *A* and *B* subscripts refer to the ends of the column under consideration.

The many assumptions used in the development of the nomograph are detailed in the Commentary to the Specification (AISC, 1999). One of these important assumptions is that "all columns in a story buckle simultaneously." Although this assumption was essential in the derivation of this useful equation, it is also one that is regularly violated in practical structures. This assumption is critical since it eliminates the possibility that any column in an unbraced frame might contribute to the lateral sway resistance of any other column. A reasoned analysis of the behavior of columns in actual structures would indicate that columns loaded below their capacity should be able to help restrain weaker columns. Thus, other approaches to determining the *K*-factor should be considered.

BUCKLING ANALYSIS VS. NOMOGRAPH

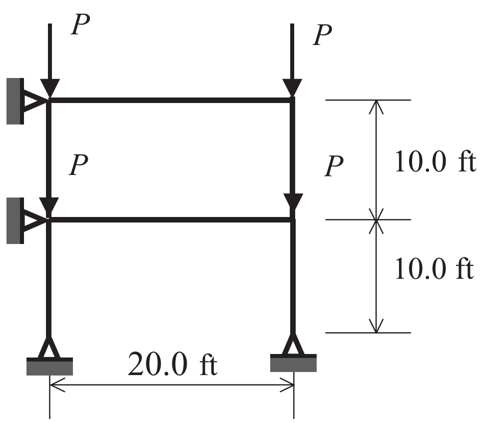
First, a comparison of results from a first-order elastic buckling analysis and the nomograph equation, Equation 4, will be presented. To make this comparison through the use of effective length factors, Equation 3 can be rearranged as follows:

$$K_{exact} = \sqrt{\frac{\pi^2 EI}{P_{cr} L^2}} \tag{5}$$

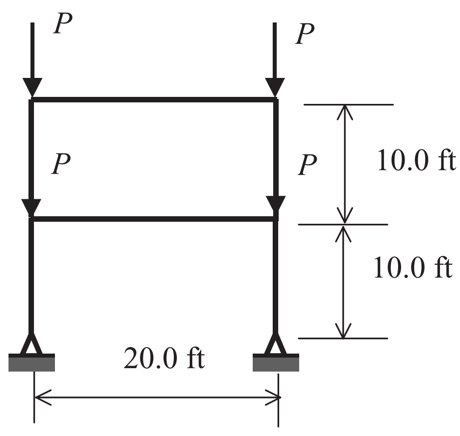
The frame from Figure 2 will again be considered, this time without the lateral load. An elastic buckling analysis using GTSTRUDL yields a critical buckling load, P_{cr} = 232 kips. For this critical load, Equation 5 yields K_{exact} = 2.66. Since the GTSTRUDL analysis includes flexural, axial, and shearing deformations while the nomograph solution includes only flexural deformations, a more accurate comparison would be expected if axial and shearing deformations were excluded from the elastic buckling analysis. In this case, P_{cr} = 237 kips and K_{exact} = 2.63. The nomograph equation also gives K = 2.63. Since the structure of Figure 2 and the elastic buckling analysis without axial and shearing deformations satisfy the assumptions of Equation 4, it is not surprising to find that the effective length factors are the same. The total buckling load for this frame is 474 kips, the sum of the two column buckling loads.

Table 1. Results of Elastic Buckling Analysis for Frame of Figure 5

Case	Loads	Lateral Restraint	Flexural Deformations			Axial and Flexural Deformations			Axial, Flexural and Shearing Deformations		
			P_{cr} (kips)	K_{upper}	K_{lower}	P_{cr} (kips)	K_{upper}	K_{lower}	P_{cr} (kips)	K_{upper}	K_{lower}
1	Top & Bottom	Yes	1145	1.20	0.85	1145	1.20	0.85	1038	1.26	0.89
2	Top & Bottom	No	111	3.85	2.72	125	3.63	2.57	122	3.67	2.60
3	Top	No	209	2.81	2.81	237	2.64	2.64	232	2.66	2.66
4	Bottom	No	233	-	2.66	260	-	2.52	255	-	2.54



(a) Sidesway Prevented



(b) Sidesway Permitted

Fig. 5. Buckling of a two-story frame.

Quite a different situation results, however, if the load is removed from one of the columns. The frame buckling load considering only flexural deformations is found to be P_{cr} = 472 kips, which, using Equation 5, yields K_{exact} = 1.87. The nomograph effective length factor is unchanged from the previous case since it is unable to account for load patterns. Comparing K_{exact} with that predicted by the nomograph shows that the frame could actually carry a much higher individual column load at buckling when only one column is loaded than would be predicted by use of the nomograph. Since the unloaded column becomes a restraining member rather than a buckling member, the loaded column capacity is increased. The total frame buckling load is 472 kips, approximately the same as when both columns were loaded.

Another interesting example is the two-story frame shown in Figure 5. The frame is modeled in GTSTRUDL with nodes at member intersections and at the mid-height of each column. Again, all members are W8×24. The elastic buckling analysis results for three different analyses are presented in Table 1. First are the results when only flexural deformations are considered. Second are the results when axial and flexural deformations are included and third are the results when axial, flexure and shearing deformations are included.

Case 1 is the sidesway-prevented frame with load *P* at each beam column intersection as shown in Figure 5a. Again, since the nomograph equation is based on flexural deformations only, the results from Table 1 for this analysis will be discussed. With P_{cr} = 1145 kips, Equation 5 yields an effective length factor for the upper story columns K_{upper} = 1.20. Recognizing that the lower story columns carry $2P_{cr}$, Equation 5 yields K_{lower} = 0.85. From the nomograph, K_{upper} = 0.85 and K_{lower} = 0.95. The nomograph *K*-factors are lower than the worst case of K=1.0 for a braced frame as expected. However, the elastic buckling analysis yields a K-factor for the upper column greater than 1.0. This indicates that the upper column actually needs less restraint than would be provided by pinned ends. Thus, this column is not a buckling column but rather, like the unloaded column in the previous example, it is helping to restrain the lower columns. Since buckling is a system phenomenon and the relative load magnitudes and columns sizes cannot be changed, design using the nomograph would show that the system capacity is controlled by the lower story and the upper column would be the same size.

Case 2 is a sidesway-permitted frame, as shown in Figure 5b. For this case, the elastic buckling capacity reduces to P_{cr} = 111 kips or approximately 10 percent of that which could be carried by the sidesway-prevented frame. The effective length factors based on this critical buckling load are K_{upper} = 3.85 and K_{lower} = 2.72. From the nomograph equation, the columns would have K_{upper} = 1.79 and K_{lower} = 3.18. As for the sidesway-prevented case, the nomograph and the elastic buckling analysis give quite different results.

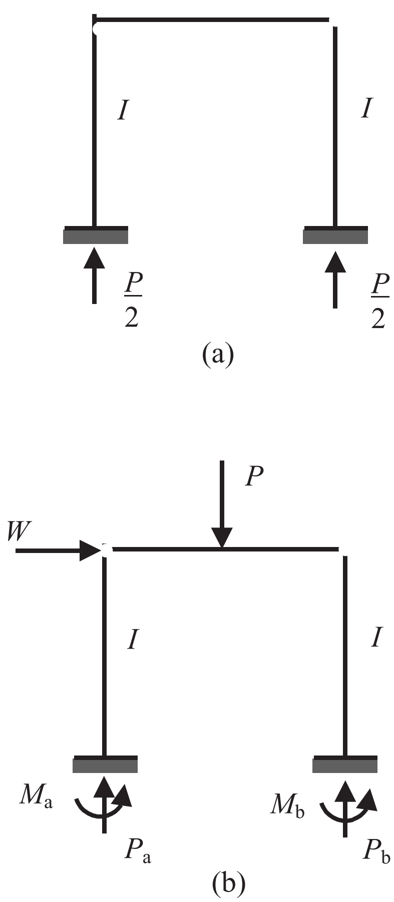


Figure 6. Symmetric portal frame.

If the loading arrangement is changed to include only loads on the top story in the sidesway-permitted frame, Case 3, the elastic buckling load is P_{cr} = 209 kips and K_{upper} = K_{lower} = 2.81 since both columns carry the same load. As discussed earlier, the nomograph cannot account for load placement, so those K-factors remain unchanged. If the load is applied only to the lower story columns, Case 4, the elastic buckling load is P_{cr} = 233 kips and K_{lower} = 2.66. Since the upper columns carry no load, they actually provide restraint to the lower columns and there is no need for a K-factor. Again, the nomograph K-factors remain unchanged.

Although the results from the other two analyses are close to those obtained when only flexure is included, they are clearly different. It is interesting to note the extent to which the inclusion of these deformations influence the results. It should also be clear from these examples that care must be exercised, as with any analysis approach, when using an elastic buckling analysis for the determination of K-factors.

From these four cases it can be seen that the determination of the system buckling capacity is not only a function of geometry and member properties but also a function of load arrangement, something the nomograph cannot accommodate. It should also be noted that any member that is loaded to less than its buckling capacity provides restraint to the other members framing into the same joint.

For any reasonable steel material strength, the columns in this frame will behave inelastically due to the existence of residual stresses. An inelastic buckling analysis should be carried out to account for this behavior. When using the nomograph approach, the AISC Specification provides a stiffness reduction factor to account for inelastic behavior. To obtain the correct design values for this frame, an inelastic buckling analysis should be performed.

INFLUENCE OF LEANING COLUMNS

Returning to the second-order analysis, the impact of leaning columns can be evaluated through the use of two simple sidesway-permitted frames.

Without leaning columns: The symmetric frame shown in Figure 6a is subjected to a symmetrically placed gravity load. A first-order analysis yields the forces shown. Note that there are no column moments or lateral deflections. Thus, there will be no axial force and moment interaction of the type shown in Figure 4 and a second-order analysis will yield the same results as the first-order analysis. When a lateral load is added as shown in Figure 6b, forces and moments as shown result from a first-order analysis and the structure sways laterally. In this case, all moment is due to the lateral load. These moments would be increased if a second-order analysis were performed.

With Leaning Columns: When a leaning column is added to the frame of Figure 6a, the resulting structure is as shown in Figure 7a. A first-order analysis will yield the same member forces for the unbraced frame as had been determined for that portion of the structure in Figure 6a. Thus, it appears that the leaning column has no impact on the original structure. If the structure is subjected to a second-order analysis, there will be no change in the results.

If a leaning column is added to the frame of Figure 6b, as shown in Figure 7b, and both the gravity and lateral loads shown are applied, a first-order analysis will again repeat the results from the frame of Figure 6b. If a second-order analysis is performed, however, the results will be different from those previously determined, since there are bending deflections and there will be load-displacement interaction. These new results will account for the amplification of moment due to sidesway of the structure and both loads P and Q.

When there are no first-order deflections, as for the frames in Figures 6a and 7a, a second-order analysis will produce the same results as the first-order analysis. For frames that do exhibit first-order deflections, a second-order analysis performed at a given load level will yield corresponding second-order forces and moments. Thus, for systems with no first-order deflections, some fictitious lateral force or displacement must be introduced to permit the determination of second-order effects. It can also be seen that a second-order analysis performed for the applied loads will not provide the information needed to determine the critical buckling load of the system.

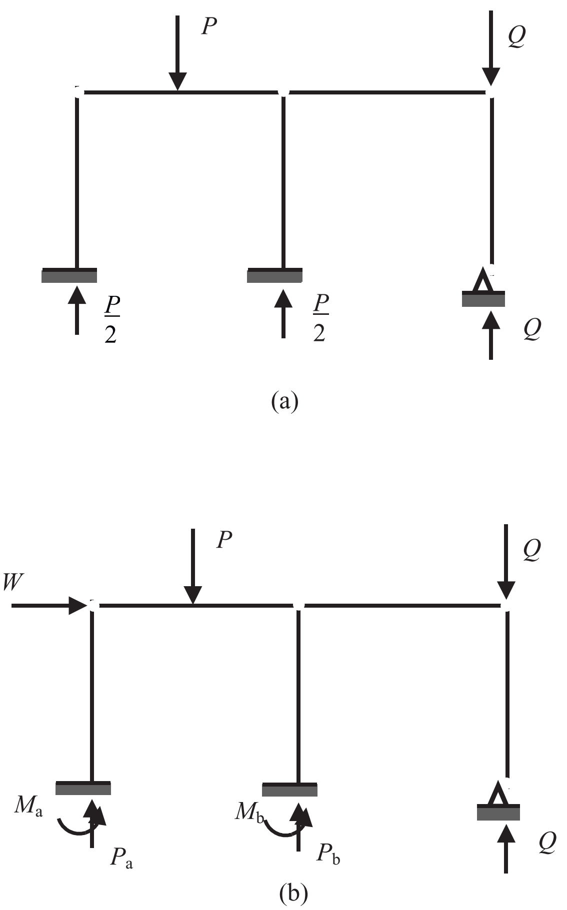


Fig. 7. Symmetric frame with leaning column.

OTHER APPROACHES FOR DETERMINING EFFECTIVE LENGTH

If the buckling load for a frame member is to be determined through an approach other than a complete elastic buckling analysis, a model that will reasonably predict the capacity of the frame, including leaning columns and the variety of possible loading arrangements, is needed. Numerous approaches intended to account for the effect of leaning columns and the sharing of lateral resistance have been presented in the literature and were reviewed by Geschwindner (1994). These approaches offer a wide range of mathematical complexity and practical usefulness. Four approaches that have been presented in the literature for including the leaning column in the determination of column capacity will be discussed along with some simplified equations that are included in the Commentary of the LRFD *Specification* (AISC, 1999). As always, the designer is called upon to decide on the appropriate approach to use in a particular design situation.

Modified Nomograph Equation (Geschwindner, 1994): The derivation of Equation 4 is available in numerous references, including (Galambos, 1968). Following the same procedures and assumptions, with the addition of the leaning column, as shown in Figure 8, a new equation may be developed.

Viewing the structure in its displaced equilibrium configuration, the restraining column and the leaning column are separated as shown in Figure 8b and 8c respectively. The load Q on the leaning column CD must be balanced by the horizontal force, $Q\Delta/L$, at D, for equilibrium of the leaning

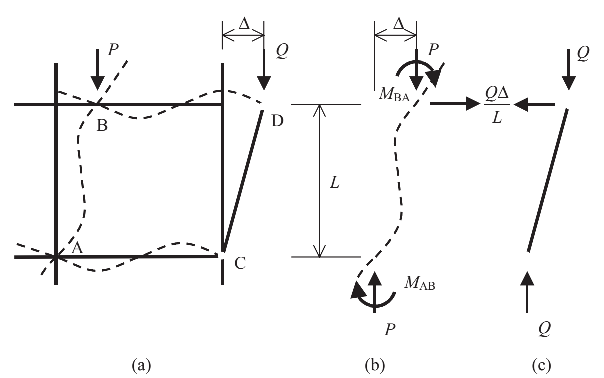


Fig. 8. Restraining and leaning columns.

column. This force must then be applied as a load at B on the restraining column AB.

Equations of equilibrium at the joints of column AB and the sway equilibrium equation can be written for the structure in the displaced configuration. Member end moment equations are then written using the slope deflection method, incorporating the stability functions (Chen and Lui, 1991) necessary to account for the influence of axial load on column AB. Combining these equations and setting the determinate of the coefficients equal to zero will yield the following buckling condition equation.

$$\begin{aligned} &\frac{G_A G_B (\pi / K)^2 - 36}{6(G_A + G_B)}\left(1+\frac{Q}{P}\right) \\ &-\frac{\pi / K}{\tan(\pi / K)}\left(1+\frac{Q}{P}\right) \\ &+\frac{6\tan(\pi / 2K)}{(G_A + G_B)(\pi / 2K)}\left(\frac{Q}{P}\right) \\ &+\frac{Q}{P} = 0 \end{aligned} \tag{6}$$

If the leaning column load is zero, $Q = 0$, Equation 6 reduces to Equation 4. Since neither of these equations can be solved explicitly, an iterative approach may be used or, in the case of the frame without leaning columns, the nomograph already discussed may be used.

The Yura Approach (Yura, 1971): This is perhaps the easiest approach to develop since it relies on a straightforward interpretation of the physical problem. For the unbraced frame shown in Figure 9, equilibrium will be established for the structure in the undeflected configuration and again in the deflected configuration. The first-order, undeflected equilibrium configuration forces are shown in Figure 9a. If the frame is permitted to displace an amount Δ through bending, equilibrium in this displaced configuration will be as shown in Figure 9b. In order for column CD to be in

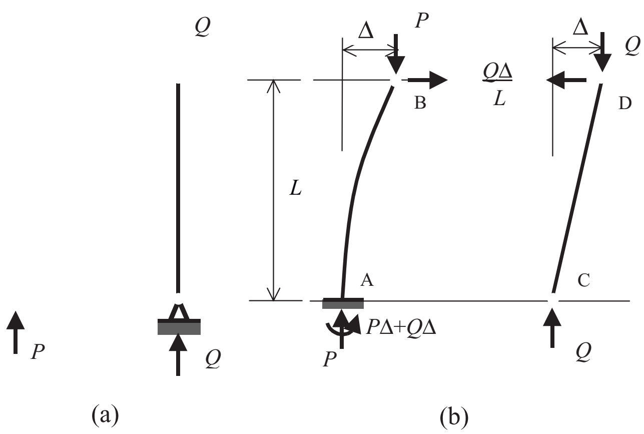


Fig. 9. Equilibrium forces for Yura derivation.

equilibrium, a lateral force, $Q\Delta/L$ as shown at D is required. This force must be equilibrated by an equal and opposite force shown at B. Thus, when column AB sways, it requires a moment of ($P\Delta + Q\Delta$) at its base for equilibrium. It is observed that this is the same moment that would result if the individual column AB were supporting an axial load of ($P + Q$) without the leaning column. The assumption that the buckling load is ($P + Q$) is only slightly conservative for the individual column AB, since the buckled shape due to an axial load and the deflected shape due to a lateral load differ only slightly. In order to ensure sufficient lateral restraint for column CD, column AB must be designed to carry a fictitious load ($P + Q$) in the plane of the frame. Out of the plane of the frame, the column would be designed to carry the load P unless the frame is also unbraced in that direction.

In order to compare this approach to others presented in the literature, it is helpful to convert it to an effective length approach. If column AB is to be designed to carry the load P but have the capacity ($P + Q$), a modified effective length factor will be required. K_o is defined as the effective length factor that would be determined from the nomograph or Equation 4, which does not account for the leaning column. In this case $K_o = 2$. K_n is defined as the effective length factor that will account for the leaning column. Thus, based on the sway buckling load being ($P + Q$)

$$(P+Q) = \frac{\pi^2 EI}{K_o^2 L^2} \tag{7}$$

If the column is to be designed to carry the actual applied load, P, with the leaning column accounted for through K_n, then

$$P = \frac{\pi^2 EI}{K_n^2 L^2} \tag{8}$$

Solving Equations 7 and 8 for their corresponding K's and taking the ratio K_n^2/K_o^2 yields

$$\frac{K_n^2}{K_o^2} = \frac{P+Q}{P} \tag{9}$$

which may be solved for K_n as

$$K_n = K_o\sqrt{\frac{P+Q}{P}} \tag{10}$$

If column AB from Figure 9a were designed to carry the load P using the effective length factor K_n, it would provide sufficient lateral restraint to permit column CD to be designed to carry the load Q using $K = 1.0$.

For frames with more than one leaning column and more than one restraining column, ΣP and ΣQ will replace P and Q. It should also be noted that this approach maintains the assumption that all restraining columns in a story buckle in a sidesway mode simultaneously.

Lim & McNamara Approach (Lim and McNamara, 1972): Another approach that will account for the leaning column was proposed by Lim and McNamara for columns of unbraced tube buildings. Their development is also based on the assumption that all columns in the restraining frame buckle in a sidesway mode simultaneously; however, they developed the sway buckling equation through the use of stability functions and an eigenvalue solution.

The resulting effective length factor, accounting for leaning columns is given in their paper as

$$K_n = K_o\sqrt{1+n\frac{F_o}{F_n}} \tag{11}$$

where

K_n and K_o are as defined earlier

n = $\Sigma Q/\Sigma P$

F_o = eigenvalue solution for a frame without leaning columns

F_n = eigenvalue solution for a frame with leaning columns.

The authors suggest that for normal column end conditions, $F_o/F_n = 1.0$ should provide a K-factor that is conservative by at most two percent. Substituting for n and using $F_o/F_n = 1.0$, the Lim and McNamara approach gives the same K-factor as the modified Yura approach where

$$K_n = K_o\sqrt{1+\frac{\Sigma Q}{\Sigma P}} = K_o\sqrt{\frac{\Sigma P+\Sigma Q}{\Sigma P}} \tag{12}$$

Thus, for this story-buckling approach, a single multiplier for each story will be sufficient to modify the individual nomograph K-factors to account for leaning columns.

LeMessurier Approach (LeMessurier, 1977): In his landmark paper, LeMessurier presented a more complex, yet still very practical approach for frames with and without leaning columns. The basic equations were developed for a single cantilever column and then extended to the general frame. Where the previous approach determined a constant value for a story by which the nomograph value of K_o was modified, this approach determines a constant value for a story which then multiplies the individual column moment of inertia divided by the column load, I_i/P_i, for each column, i. Thus, the contribution of each column to the lateral resistance is accounted for individually along with the magnitude of the load on that column. The effective length factor for each column that participates in resisting sidesway buckling, Equation 46c from the original paper, expressed in the notation of this paper, is given by

$$K_i^2 = \frac{I_i}{P_i}\pi^2\frac{\Sigma P+\Sigma Q+\Sigma(C_L P)}{\Sigma(\beta I)} \tag{13}$$

where

$$\beta = \frac{6(G_A+G_B)+36}{2(G_A+G_B)+G_A G_B+3} \tag{14}$$

$$C_L = \frac{\beta K_o^2}{\pi^2}-1 \tag{15}$$

K_i = effective length of column i, accounting for leaning columns

C_L = 0 for leaning columns

P_i = load on restraining column, i

I_i = moment of inertia for column, i

ΣP = total load on the restraining columns in a story

ΣQ = total load on the leaning columns in a story

$\Sigma(C_L P)$ = sum of $(C_L P)$ for each column in the story

$\Sigma(\beta I)$ = sum of (βI) for each column participating in lateral sway resistance

Commentary Equations (AISC, 1999): Although use of Equation 13 is not particularly complex, the third edition of the Commentary to the 1999 LRFD Specification presents two modified LeMessurier equations that may be of value to the practicing engineer. One is based on the story-buckling model while the other is based on a story-stiffness model.

For the story-buckling model, it is assumed that there is no reduction in column stiffness due to the presence of axial load. This is accomplished by taking $C_L = 0$ for all columns, which leads to $\beta = \pi^2/K_o^2$. Substitution of these values into Equation 13 yields:

$$K_i^2 = \frac{I_i}{P_i}\pi^2\frac{\Sigma P+\Sigma Q}{\Sigma(\pi^2 I/K_o^2)} \tag{16}$$

which reduces to

$$K_i = \sqrt{\frac{I_i}{P_i}\frac{\Sigma P+\Sigma Q}{\Sigma(I/K_o^2)}} \tag{17}$$

Equation 17 can be recast into the form of LRFD Commentary Equation C-C2-6 as

$$K_i' = \sqrt{\frac{P_e}{P_{ui}}\left(\frac{\Sigma P_u}{\Sigma P_{e2}}\right)} \tag{18}$$

where

$$\Sigma P_u = \Sigma P + \Sigma Q$$

$$P_e = \frac{\pi^2 E I_i}{L^2}$$

$$P_{ui} = P_i$$

$$\Sigma P_{e2} = \sum \frac{\pi^2 E I_i}{(K_o L)^2}$$

For a structure in which only one column is providing lateral stability, the summations in Equation 16 are unnecessary and the equation reduces to

$$K_i = K_o \sqrt{\frac{P+Q}{P}} \tag{19}$$

which is the same as the equation that resulted from the modified Yura and Lim and McNamara approaches, Equations 10 and 12 respectively.

For the story-stiffness model, the stiffness reduction due to axial load is included as though all columns were cantilevers with a buckled shape in the form of a half sine curve, thus C_L = 0.216. Since the leaning columns have no lateral stiffness of their own, C_L = 0 for all leaning columns. The equation given in this paper as Equation 13 is just one form of the effective length factor equations given by LeMessurier. Another form is also available through the same derivation (LeMessurier, 1977). This equation uses the ratio of lateral displacement to lateral load as a measure of buckling stiffness. Equation 46d from the original paper, in the notation of this paper, is given as

$$K_i^2 = \frac{I_i}{P_i} \frac{\pi^2 E}{L^3} \frac{\Delta_{oh}}{\Sigma H} (\Sigma P_T + \Sigma C_L P_T) \tag{20}$$

where ΣH is the total lateral load supported by the level under consideration, Δ_{oh} is the corresponding lateral displacement of the level and $\Sigma P_T = \Sigma P + \Sigma Q$ is the total load on the given story. In order to account for C_L = 0 on the leaning columns and C_L = 0.216 on all others, the load on the leaning columns must be subtracted from the total load on the story so that $(\Sigma P_T + \Sigma C_L P_T) = (\Sigma P_T + 0.216\,(\Sigma P_T - \Sigma Q))$. Making this substitution and factoring out ΣP_T yields

$$K_i^2 = \frac{I_i}{P_i} \frac{\pi^2 E}{L^3} \Sigma P_T \frac{\Delta_{oh}}{\Sigma H} \left(1.216 - 0.216 \left(\frac{\Sigma Q}{\Sigma P_T}\right)\right) \tag{21}$$

This equation was somewhat simplified in the Commentary to the 1993 LRFD *Specification* (AISC, 1993) as

$$K_i^2 = \frac{I_i}{P_i} \frac{\pi^2 E}{L^3} \Sigma P_T \frac{\Delta_{oh}}{\Sigma H} \left(\frac{1}{0.85 + 0.15 \Sigma Q / \Sigma P_T}\right) \tag{22}$$

Geschwindner presented a comparison between Equations 21 and 22 (Geschwindner, 1994).

If the leaning columns are not excluded and the stiffness reduction due to axial load is applied to all columns, C_L = 0.216 would be applied to the total load on the story and the separation taken to arrive at Equation 21 would not be necessary. Thus, Equation 20 becomes

$$K_i^2 = \frac{I_i}{P_i} \frac{\pi^2 E}{L^3} \Sigma P_T \frac{\Delta_{oh}}{\Sigma H} (1.216) \tag{23}$$

This equation, recast in the form of the 1999 LRFD Commentary Equation C-C2-5 is

$$K_i' = \sqrt{\frac{P_{ei}}{0.822 P_{ui}} \Sigma P_u \left(\frac{\Delta_{oh}}{\Sigma HL}\right)} \tag{24}$$

where

$$P_{ei} = \frac{\pi^2 E I_i}{L_i^2}$$

and

$$P_{ui} = P_i$$

These simplifications may not be necessary since, in the original form, the equations presented by LeMessurier are not much more complex and will yield more accurate results.

EXAMPLES

The following examples will show how these approaches may be used to evaluate columns in unbraced frames.

Example 1

The frame shown in Figure 10, introduced by Geschwindner (1995), will be used to compare the simplified methods for determining effective length factors with an elastic buckling analysis. The frame is supported in such a way that in-plane behavior will be critical. The columns AB and CD as well as the beam BC are W12×136. The other members are of such a size that their individual characteristics will not control. The results of a GTSTRUDL buckling analysis and three simplified equations are presented in Table 2. Comparisons will be made for the analysis including flexural deformations only. For equal loads on columns AB and CD and no loads on the other columns, P_{cr} = 1302 kips. Using Equation 5 this is equivalent to K = 2.18.

When equal loads are also applied to columns EF, GH, and JK, GTSTRUDL yields P_{cr} = 568 kips or K = 3.29. The

Table 2. Elastic Buckling Analysis Results for Frame of Example 1

Case	Leaning Columns	Flexure		Flexure, Axial		Flexure, Axial, Shear		Eq. 6		Eq. 12		Eq.13	
		P_{cr} (kips)	K	P_{cr} (kips)	K	P_{cr} (kips)	K	P_{cr} (kips)	K	P_{cr} (kips)	K	P_{cr} (kips)	K
1	no	1302	2.18	1299	2.18	1237	2.23						
2	yes	568	3.29	559	3.32	529	3.41	569	3.29	524	3.43	566	3.30

loading on the structure shows two equal loads on the restraining columns and three equal loads on the leaning columns. This gives P = 2 and Q = 3. From the modified nomograph equation, Equation 6, K = 3.29 and P_{cr} = 569 kips. Since the assumptions made in the derivation of Equation 6 are satisfied with this model, it is expected that Equation 6 and the buckling analysis will yield the same results. The modified Yura equation, Equation 12, yields K = 3.43 and P_{cr} = 524 and the LeMessurier equation, Equation 13, using G = 100,000 to represent the pin end, yields K = 3.30 and P_{cr} = 566 kips.

No matter what approach is taken to account for the leaning columns, it is clear that leaning columns have a significant impact on the stability of the structure. It is also evident, from earlier discussion, that a second-order elastic analysis for this frame will yield the same forces for members AB, CD, and BC, whether there are loads on the leaning columns or not. This is true since, through a first order analysis; there will be no lateral displacement of the frame. Thus, more than a second-order elastic analysis for the given loads is needed to complete design of the structure.

Example 2

Factored loads, including a lateral load, are now applied to the frame of Example 1, as shown in Figure 11. First- and second-order elastic analyses are performed and, along with the results from Example 1, a check on column CD, with F_y = 50 ksi, is performed.

Using the results from the first-order elastic analysis, for column CD, P_u = 240 kips and M_u = 200 kip-ft. From Example 1, the LeMessurier analysis including the leaning columns yields, K = 3.30. Using the LRFD Specification (AISC, 1999), the column behaves elastically and its capacity is ϕP_n = 421 kips. Thus, $P_u/\phi P_n$ = 240/421 = 0.57 > 0.2 so LRFD Equation H1-1a must be satisfied.

Since the column moment is from a first order analysis, it must be amplified. This will be accomplished using the first suggested equation for B_2, Equation C1-4.

$$B_2 = \frac{1}{1 - \Sigma P_u \left(\dfrac{\Delta_{oh}}{\Sigma HL} \right)}$$

The non-sway analysis generates no moments so the second-order moment becomes, $M_u = B_2$ (200). The results of the first-order analysis give a lateral deflection due to the 20 kip load of 1.74 in. Using these values, B_2 = 1.57 so that M_u = 1.57(200) = 314 kip-ft. The interaction equation becomes

$$\frac{240}{421} + \frac{8}{9}\left[\frac{314}{803}\right] = 0.57 + 0.35 = 0.92$$

Since this is less than 1.0, the column will be adequate.

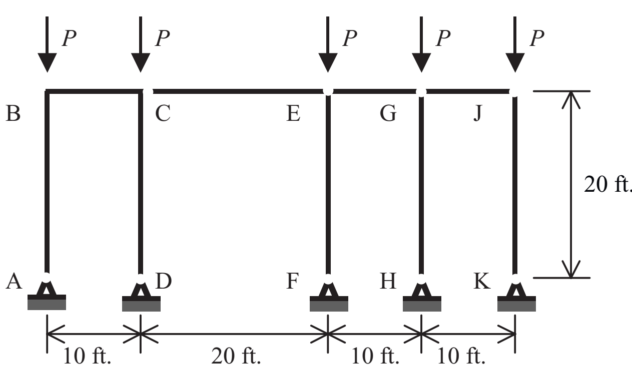


Fig. 10. Frame for Example 1 with leaning columns.

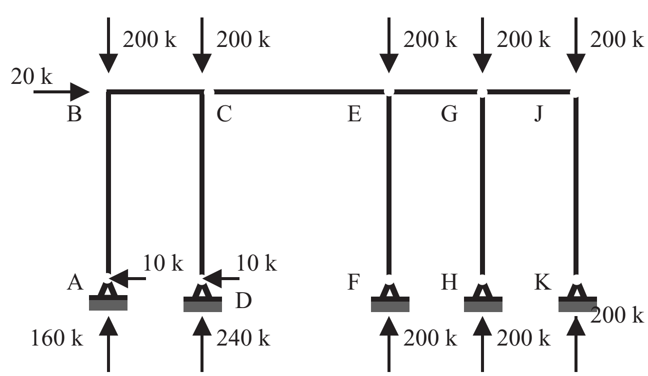


Fig. 11. Example 2 frame with gravity and lateral load and reactions from a first order analysis.

Table 3. Summary of Effective Length Calculations for Example 3

Column Mark	I_i	P_{ui}	Eq. 4	Eq. 6	Eq. 13	Eq. 20	Eq. 17	Eq. 21	Eq. 22	Eq. 23	Elastic Buckling
i	in^4	kips	K_o	Effective length, K_i							
1	425	150	1.81	3.49	3.21	3.29	3.35	3.33	3.30	3.57	3.30
2	350	50	1.73	3.30	5.04	5.17	5.27	5.23	5.19	5.61	5.19
3	475	275	1.74	3.33	2.50	2.57	2.62	2.60	2.58	2.79	2.58
4	350	25	1.72	3.27	7.13	7.31	7.45	7.39	7.34	7.94	7.33
5	350	125	1.78	3.43	3.19	3.27	3.33	3.30	3.28	3.55	3.28
K_i^2 = story constant $\times \dfrac{I_i}{P_i}$					$3.63\dfrac{I_i}{P_i}$	$3.81\dfrac{I_i}{P_i}$	$3.96\dfrac{I_i}{P_i}$	$3.90\dfrac{I_i}{P_i}$	$3.85\dfrac{I_i}{P_i}$	$4.50\dfrac{I_i}{P_i}$	$3.85\dfrac{I_i}{P_i}$

If the results of a second-order analysis are used, P_u = 261.9 kips and M_u = 308.2 kip-ft. Again, the effect of the leaning columns will be included from the LeMessurier analysis so that, $P_u/\phi P_n = 261.9/421 = 0.62 > 0.2$ and LRFD Equation H1-1a is used again. Since the column moment results from a second-order analysis, there is no need to amplify it prior to using the interaction equation, thus

$$\frac{261.9}{421}+\frac{8}{9}\left[\frac{308.2}{803}\right]=0.62+0.34=0.96$$

Again, the column is seen to be adequate. It is interesting to note that there is an increase in the column axial load due to the second order effects that is not included in the simplified code approach to second-order analysis. This increase is the direct result of the leaning columns gaining stability from the frame and, in effect, adding a lateral load to the system. In addition, the second-order moments obtained from the two approaches are quite similar. It also must be recognized that even though a second-order analysis was used, the influence of the leaning columns on the axial capacity of the restraining column must still be accounted for. This was accomplished using the LeMessurier equation, Equation 13.

Example 3

An interesting structure was presented by Baker (1997) to demonstrate the problems associated with determining effective length when the assumptions of the nomograph are violated. The frame shown in Figure 12 represents one of two frames that provide lateral resistance for a fairly large footprint building. This frame carries a gravity load of ΣP = 625 kips and provides lateral stability to columns carrying an additional gravity load of ΣQ = 1875 kips. This is one-half of the total load for the building. Column bases are modeled as rotational springs with stiffnesses of $\dfrac{6EI_i}{G_B L}$.

The AISC Commentary recommended value of G_B = 10 for the pinned bases was used throughout the analysis. Lateral deflection of the frame due to ΣH =12 kips, is Δ_{oh} = 0.362 in. taken as the average deflection at the top of the 5 columns. This analysis includes axial, flexural and shearing deformations. The total gravity load is $\Sigma P_T = (\Sigma P + \Sigma Q)$ = 2500 kips and F_y = 50 ksi.

The results presented by Baker were in the form of column capacities, P_e. However, a review of his equations shows that his solution actually uses Equation 22. Table 3 shows the K-factors for each of the five columns in this frame as determined through the nomograph equation, the modified nomograph equation, six simplified equations, and an elastic buckling analysis. It can be seen from Table 3 that the use of the nomograph, Equation 4, does not predict effective length values that would subsequently produce accurate elastic buckling values for the columns of this frame. This is due to the fact that the structure significantly violates the assumptions used to develop the nomograph. If this structure had been designed with those values, the columns would have had an expected capacity significantly larger than their true capacity.

The modified nomograph equation, Equation 6, also provides K-factors that would be poor predictors of column buckling capacity for this frame. Although this equation is able to account for the leaning columns, it is not able to account for the distribution of load or individual column contribution to lateral resistance.

Since the lateral displacement used in Equations 17, 20, 22, and 23 was calculated as a function of axial, flexural and shearing deformations, it is appropriate to determine the elastic buckling capacity using these same displacements. The results from GTSTRUDL converted to effective length, are given in Table 3. The LeMessurier equations, Equations 13 and 20, provide results that are close to this elastic buckling analysis. The values determined from

Equations 17 and 21 are fairly consistent although somewhat above the elastic buckling values while the results from Equation 23 are significantly higher than the elastic buckling values and will yield a more conservative solution. For this specific example, Equation 22 yields the same results as the elastic buckling analysis.

CONCLUSIONS

This paper presented a brief discussion of the full range of approaches that might be used to carry out a structural analysis. The elastic buckling load of a frame can be determined through an eigenvalue analysis that predicts system buckling behavior. This buckling load is the load that the K-factor is attempting to predict for individual columns. Although the K-factor has been a controversial topic from its initial introduction into the AISC *Specification*, it remains a useful tool to predict column capacity. Perhaps the most troubling aspect associated with the use of the K-factor has been overlooking the assumptions included in the development of the most common predictor equations. Unbraced moment frames that do not meet the restrictive assumptions, which permit use of the nomograph pose interesting problems for the structural engineer. Such factors as leaning columns, columns supporting less than their full capacity, and inelastic behavior all must be considered in any design.

Four approaches from the literature for determining K-factors were presented, along with several simplified equations derived from those procedures. It was shown that an iterative solution of Equation 6 produced a more accurate value of K_n than that from the nomograph, Equation 4, when leaning columns were present. Using Equation 6, the leaning column loads are accounted for; however, the other limitations of the nomograph solution are still present and, as shown in Example 3, the modified nomograph equation will not be a good predictor of K-factors for these systems.

The equations proposed by LeMessurier, Equations 13 and 20, are generally recognized as the most accurate of those presented, even though in Example 3, Equation 23 yields the same results as the elastic buckling analysis. There are two approaches to the use of the LeMessurier equations. One requires the determination of K_o, which may be accomplished through the nomograph, as is normally done, or by an iterative solution of Equation 4. The other approach uses the lateral stiffness of the frame, as measured by its lateral deflection due to a lateral load. Either of these approaches will provide a practical method of determining column capacity. Based on Example 3, Equations 13 and 20 provide K-factors that closely approximate the elastic buckling analysis results. It is not unrealistic to use the LeMessurier equations for effective length factors in normal engineering practice.

The commentary to the LRFD *Specification* provides simplified equations, based on the LeMessurier equations. The examples presented here allow for a comparison of results between several of these equations. It appears that there is some wide variation in results, depending on the choice of the simplified equation. The assumptions used to develop these simplified equations are presented so the engineer will be in a better position to decide which expressions should be used in a particular situation.

Although the LeMessurier approach is not overly complicated to use, designers wishing to use a simpler approach may find that the Lim and McNamara equation for K_n pro-

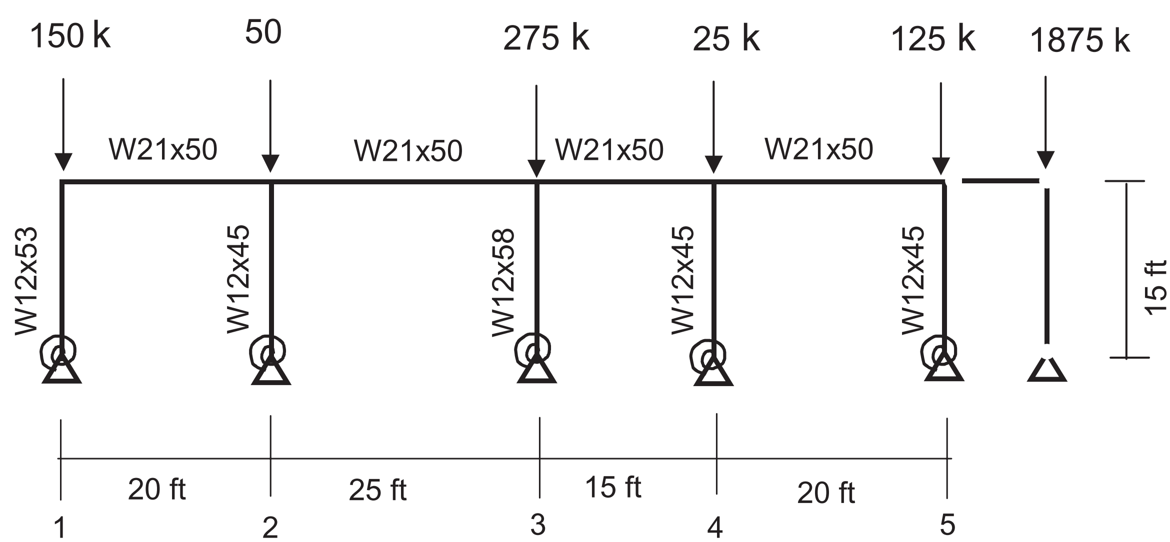


Fig. 12. Example 3 Frame from Baker (1997).

vides a sufficiently accurate way to account for leaning columns, particularly in preliminary stages of design. However, the LeMessurier equation based on the lateral deflection of the frame provides a straightforward and accurate approach for use in design as suggested by Baker. Once the elastic-buckling load of the frame has been determined and the appropriate amount attributed to the individual columns, design by any approved method may proceed.

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