Story-Based Effective Length Factors for Unbraced PR Frames

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ABSTRACT

This paper proposes a practical method to evaluate the effective length factor *K* for compressive members in unbraced partially-restrained frames under elastic buckling. In light of story-based buckling and the introduction of the end-fixity factor to characterize the beam-to-column connections of partially-restrained frames, the lateral stiffness of columns with consideration of the effect of axial load is derived. The formulations and procedure of calculating story-based column effective length factors are presented. Numerical examples are then presented to illustrate the effectiveness of the proposed procedure. With the adoption of the end-fixity factor, different member end rotational conditions can be readily modelled by derived formulations. Therefore, the proposed approach is comprehensive and can be applied for both unbraced partially and fully-restrained frames.

INTRODUCTION

The determination of the strength of a compressive member in a frame system or the maximum strength of the frame is of primary importance in structural design. Current design practice recognizes that the maximum strength of an unbraced frame and the maximum strength of a compressive member are interrelated, and the relationship between the two is complicated. The theoretical approach of elastic buckling of unbraced frames under proportional loads that involves solving for the critical load multiplier λ_{cr} from either a highly nonlinear equation or a transcendental equation (Majid, 1972; Livesley, 1975; Bhatt, 1981), which is referred to as the system buckling approach, is generally considered not practical (Galambos, 1988).

In current design practice, the effective length concept for evaluating the strength of a compressive member in a frame is the most widely used method. According to this concept, the strength of a framed compressive member of length *L* is equated to an equivalent pin-ended member of length *KL*, subject to axial load only, by means of *K* factors. The effective length concept is considered as an essential part of many analysis procedures and has been recommended by almost all of the current design specifications (AISC, 1989; AISC, 1999; CSA, 1994).

There are different methods of calculating the *K* factors based on the concept of effective length and different idealizations of the structure. Among them, the most widely adopted procedure for the frame design is the alignment chart method that was originally proposed by Julian and Lawrence in 1959 based on the assumption that all individual columns in a story buckle simultaneously under their individual proportionate share of the total gravity load. This method takes into account the rotational restraints provided by upper and lower beam-column assemblages and provides a direct means to obtain *K* factors. However, since this method involves a number of simplifications and assumptions that are not realistic, the *K* factors evaluated based on this method are inaccurate when these assumptions are not satisfied. Methods based on modification of the *G*-factor to improve the effectiveness of the alignment chart method were proposed (Bridge and Fraser, 1987; Duan and Chen, 1988 and 1989).

Pointing out that sidesway buckling is a total story phenomenon, and a single individual column cannot fail by sidesway without all the columns in the same story also buckling in the same sway mode, the concept of story buckling is introduced (Yura, 1971). A procedure of estimating the frame buckling from the story-buckling manner is illustrated by LeMessurier (1977). This method takes into account the interaction among columns in a story on the lateral stiffness of the frame and the fact that stronger columns brace the weaker columns until story sidesway buckling occurs. In conjunction with the alignment chart, LeMessurier's method resulted in a more accurate estimation of the effective length factors *K*. A simpler method that takes both member stability (p-δ) and frame stability (P-Δ) into account in the calculation of effective length factors *K* was proposed by Lui (1992, 1995). The method needs only first-order frame analysis; and no special charts or iterative procedures are required. Shanmugam and Chen (1995) conducted an assessment of four approaches to determine *K* factors of columns in frames, namely, the alignment chart approach, LeMessurier's approach, Lui's approach, and the system buckling approach. The study concluded that Lui's method is the most appropriate for general use in design

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practice. It was confirmed that the approaches of evaluating effective length factors based on the story-based buckling yield more accurate results, in a parametric study based on variations of bay-width, moment of inertia of columns, loading, and column height for a two-bay three-story frame (Roddis, Hamid, and Guo, 1998). Therefore, the story-based methods are recommended for general use. The AISC LRFD Specification (AISC, 1999) addressed the concept of story-based buckling because the alignment chart method did not consider destabilizing effects due to lean-on columns in a frame. Two methods of determining the story-based effective length factor were presented in the Commentary of the LRFD Specification.

To incorporate connection flexibility in the determination of the effective-length factor K for columns in partially-restrained frames, methods based on modification of the values of the moment of inertia of the restraining beams while using the alignment chart method were proposed (Bjorhovde, 1984; Chen and Lui, 1991; Chen, Goto, and Liew, 1995; Christopher and Bjorhovde, 1999). However, different expressions, which depend on the rotational conditions of the ends of the beam, such as a pinned connection for the "far end" and partially-restrained (PR) connection for the "near end," were used for each of the modification factors. Upon adopting the concept of the end-fixity factor, Xu (1994) derived a comprehensive expression of the modification factor regardless of the rotational conditions at each end of the beam for braced and unbraced frames, respectively. Kishi, Chen, and Goto (1997) proposed to evaluate the effective length factors for columns in unbraced PR frames based on a subassemblage model with two columns, and conducted a comparison with those of the alignment chart method. The effects of the nonlinear Equation of PR connections on the effective length factors were investigated. Story-based buckling and associated K factor for columns in PR frames were investigated based on the stability equation approach (Aristizabal-Ochoa, 1997).

Following the concept of story buckling and the foregoing studies on column and frame stability, this paper proposes a practical method to evaluate the effective length factors K for compressive members in unbraced PR frames. This paper is organized as follows: based upon the discussion of column interactions in story-based buckling and the introduction of the end-fixity factor to characterize the beam-to-column connection of PR frames, the lateral stiffness of PR columns with consideration of the effect of axial load is derived and simplified by using the Taylor series expansions. The formulations and procedure of calculating story-based effective length factors for columns in unbraced PR frames are presented. With the adoption of the end-fixity factor, different member end rotational conditions can be readily modelled by derived formulations. Therefore, the proposed approach is comprehensive and can be applied to both unbraced PR and FR (fully-restrained) frames. Numerical examples are then presented to demonstrate the effectiveness and comprehensiveness of the proposed procedure.

SENSITIVITY OF STORY-BASED EFFECTIVE LENGTH FACTOR

By examining an individual column instead of the framed structure as a whole, the effective length approach is an approximate method for evaluating column buckling. In reality, a column in a frame interacts with other members in the same story and other stories. To adequately assess the interactions, a system stability analysis of the frame is required. Based on the assumption of proportional loading, the axial force in column j can be expressed as

$$P_j = \lambda P_{sj} \tag{1}$$

where λ is the proportional load multiplier, and P_{sj} is the axial force in column j due to the specified loads. Let λ_{cr} be the critical load factor which is associated with the buckling of an unbraced frame system, then the story-based effective length factor for column j is defined as (Lui, 1992)

$$K_j = \pi \sqrt{\frac{EI_j}{\lambda_{cr} P_{sj} L_j^2}} \tag{2}$$

where L_j and I_j are the length and moment of inertia of column j, respectively. E is Young's Modulus of the column material. However, the procedure for obtaining the critical load factor λ_{cr} using frame stability analysis is often complicated because it involves solving either a nonlinear or a transcendental equation.

To investigate the sensitivity of effective length factors evaluated upon story-based buckling to the variations of parameters such as loading, column stiffness, and height, parametrical studies are conducted for the frame shown in Figure 1a (Lui, 1992). Provided that α is a parameter for the flexural stiffness of column BD, η is a parameter for the applied load on node C, and γ is a parameter for the height of column AC, the interaction between columns AC and BD story buckling can be analyzed as follows.

The lateral stiffness S for an individual column in the frame shown in Figure 1b can be obtained as (Livesley, 1975),

$$S = \frac{EI}{L^3} \frac{\phi^3}{\tan\phi - \phi} \tag{3}$$

in which,

$$\phi = \sqrt{\frac{PL^2}{EI}} = \pi\sqrt{P / P_e} = \pi\sqrt{\rho} \tag{4}$$

where $P_e = \pi^2 EI/L^2$ is the Euler buckling load of column AC, and $\rho = P/P_e$ is the load ratio. Let Δ be the lateral sway of the frame. Then, the equilibrium equation corresponding to the sway mode buckling is

$$(S_{AC} + S_{BD})\Delta = 0$$

To have a non-trivial solution, the stability condition is stated as $S_{AC} + S_{BD} = 0$, where, S_{AC} and S_{BD} are the lateral stiffness for columns AC and BD, respectively. From Equation 3, the lateral stiffnesses of columns AC and BD are derived

$$S_{AC} = \frac{EI\phi^3}{L^3}\frac{\sqrt{\eta^3}}{\tan(\gamma\sqrt{\eta}\phi) - \gamma\sqrt{\eta}\phi}$$

$$S_{BD} = \frac{EI\phi^3}{L^3}\frac{1/\sqrt{\alpha}}{\tan\phi/\sqrt{\alpha} - \phi/\sqrt{\alpha}}$$

then, the stability equation of the frame is

$$\frac{1/\sqrt{\alpha}}{\tan\phi/\sqrt{\alpha} - \phi/\sqrt{\alpha}} + \frac{\sqrt{\eta^3}}{\tan(\gamma\sqrt{\eta}\phi) - \gamma\sqrt{\eta}\phi} = 0 \qquad (5)$$

For any given values of parameters α, η, and γ, the value of ϕ can be found by numerically solving Equation 5, hence the corresponding effective length factors K for the columns are obtained as

$$K_{AC} = \frac{\pi}{\gamma\sqrt{\eta\phi}} \qquad (6a)$$

$$K_{BD} = \frac{\pi\sqrt{\alpha}}{\phi} \qquad (6b)$$

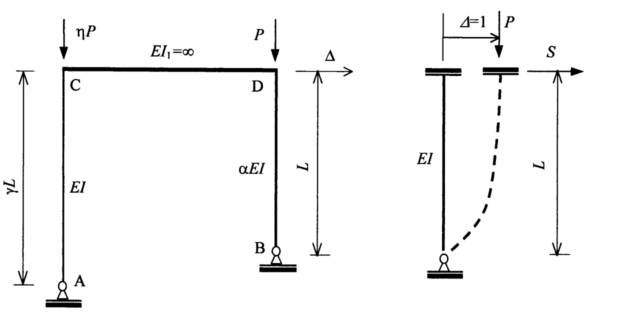
Parametrical studies are carried out by setting two out of the three parameters of α, η and γ as unity and letting one parameter vary between 1 to 7. The corresponding K values for the columns are listed in Table 1 while the plotted K-α, K-γ and K-η curves are shown in Figure 2.

It can be observed from Table 1 that when all parameters are equal to unity, both columns have the same K value. However, if any one of the parameters varies, column AC always has smaller K values than that of column BD. This is because column AC carries a heavier load ($\eta > 1$), and thus, has a smaller value of K according to the definition in Equation 2. The effect of the length variation of column AC on the K factors is more obvious than that of the other two parameters.

The ratio of total critical load P_{tcr} of the frame shown in Figure 3 to column Euler buckling load $P_e = \pi^2 EI/L^2$ can be expressed as

$$\frac{P_{tcr}}{P_e} = (1+\eta)\frac{\phi^2}{\pi^2} \qquad (7)$$

Figure 3 illustrates the ratio of P_{tcr}/P_e in response to the variation of either one of three parameters, α, η or γ. It can be seen from Figure 3 that the total critical load of the frame varies linearly as the increase of the stiffness parameter α while the variation of load parameter η on column AC has almost no effect on the total critical load of the frame. This is because column BD is a stronger column that contributes more to the lateral stability of the frame. The increased value of α will make column BD even stronger, which will subsequently lead to an increase of the total critical load of



(a) Structural Model (b) Lateral Deflection Stiffness

Fig. 1. Simple frame and the associated lateral buckling model.

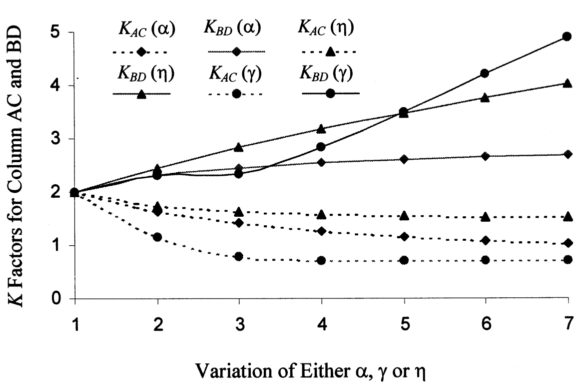


Fig. 2. K *factors due to variation of either* α, γ *or* η

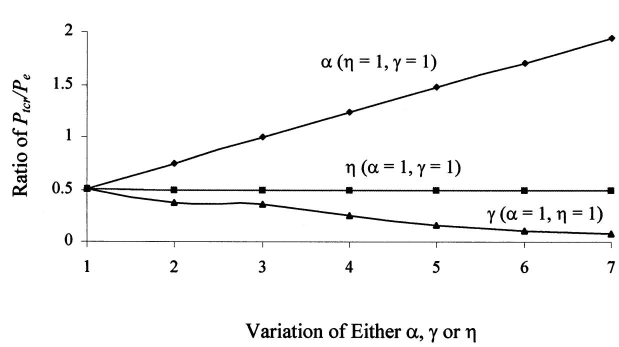


Fig. 3. Critical load ratio due to variation of either α, γ *or* η

Table 1. Values of *K* Factor Associated with Variation of One Parameter

α or η or γ	φ			η = γ = 1		γ = α = 1		η = α = 1	
	η=γ = 1	γ=α = 1	η=α= 1	K_{AC}	K_{BD}	K_{AC}	K_{BD}	K_{AC}	K_{BD}
1	1.5708	1.5708	1.5708	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
2	1.9218	1.2814	1.3533	1.6347	2.3118	1.7336	2.4517	1.1607	2.3215
3	2.2149	1.1085	1.3354	1.4184	2.4567	1.6363	2.8342	0.7842	2.3526
4	2.4705	0.9906	1.1092	1.2716	2.5433	1.5858	3.1715	0.7081	2.8323
5	2.6988	0.9036	0.8949	1.1641	2.6030	1.5548	3.4766	0.7021	3.5106
6	2.9056	0.8365	0.7474	1.0812	2.6485	1.5332	3.7556	0.7006	4.2036
7	3.0943	0.7818	0.6412	1.0153	2.6862	1.5188	4.0182	0.7000	4.8999

the frame. On the other hand, increasing the value of η will only increase the load on column AC, and will not directly affect column BD. As long as column BD can provide the required lateral stiffness to assist column AC in carrying additional load, the total critical load of the frame remains almost constant.

The parametrical studies have again demonstrated that the effective length factor for a column in an unbraced frame is not a constant and it varies depending on many factors such as structural shape, relative dimensions, framing members, stiffness of other columns, and load distribution. For an unbraced frame under sway mode buckling, all columns in a story interact with each other and buckle simultaneously. A strong column or a column with low axial force will brace a weak column or column carrying heavy axial load. The maximum strength of a column is achievable with adequate bracing (Winter, 1960) while the strength of the frame largely relies on the strength and lateral stiffness of columns. Therefore, the assessment of the interaction between the frame and an individual column is the essence of the story-based effective length concept.

MEMBER END-FIXITY FACTOR OF PR FRAMES

The *K* factors of columns of a rigid frame are assumed to be irrelevant to the applied loads when using the alignment chart method. In the story-based effective length approach, the *K* factors of columns are affected by the applied column axial loads as demonstrated in the previous section. In addition, the beam-to-column connections in a PR frame are interrelated to the applied loads. As the frame is loaded, the connection stiffness decreases, and the rotational restraint provided to the columns gradually decreases, causing the column effective-length factor to increase. Consequently, the effective-length factor of the column in the PR frame has to be evaluated by an iterative procedure (Kishi et al., 1997).

Shown in Figure 4 is a partially-restrained member comprised of a finite-length beam member with a rotational spring attached at each end. The effects of connection flexibility are modelled through rotational springs at the ends of the beam. To reflect the relative stiffness of the beam and the rotational end-spring connections, the following "end-fixity factor" is adopted and shown in Figure 5 (Monforton and Wu, 1963),

$$r_j = \frac{\varphi_j}{\theta_j} = \frac{1}{1 + 3EI / R_j L} \qquad (j = 1,\ 2) \tag{8}$$

where R_j is the end-connection spring stiffness, and EI/L is the flexural stiffness of the attached member. The end-fixity factor r_j in Equation 8 defines the stiffness of each end-connection relative to the attached member and can be interpreted as the ratio of the rotation φ_j of the end of the member to the combined rotation θ_j of the member and the connection due to a unit end-moment, as shown in Figure 5 (Cunningham, 1990). For flexible or so-called pinned con-

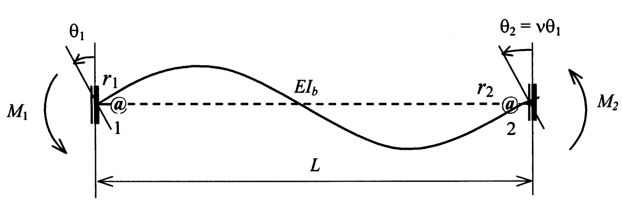


Fig. 4. End rotation of partially restrained beam.

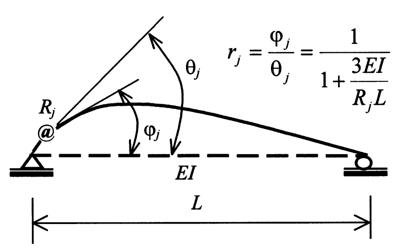


Fig. 5. End fixity-factor.

nections, the rotational stiffness of the connection is zero; thus, the value of the end-fixity factor is zero ($r_j = 0$). For fully-restrained (FR) or so-called rigid connections, the end-fixity factor is unity ($r_j = 1$), since the connection rotational stiffness is taken to be infinite. A PR connection has an end-fixity factor which lies between zero and unity ($0 < r_j < 1$).

According to Equation 8, the relationship between the end-fixity factor and the connection stiffness is nonlinear, as shown in Figure 6. It is also clear that the relationship between the connection stiffness and the end-fixity factor is almost linear when the connection is relatively flexible with a value of the end-fixity factor between 0.0 and 0.5. However, as the end-fixity factor approaches unity, the required increase in connection stiffness becomes substantial. Therefore, designers should keep in mind that with the same percentage increase in the end-fixity factor, the corresponding increment in connection stiffness may be quite different depending on whether the connection is considered to be flexible or rigid.

Experimental tests (Goverdhan, 1983) have demonstrated that most beam-to-column connections exhibit a nonlinear moment-rotational relationship, which is mainly because a connection is an assemblage of several components that interact differently at different levels of applied loads. It is difficult to analyze this nonlinear behavior by rigorous and exact mathematical procedures; hence, the analysis of connection behavior in a practical design is usually approximate in nature with considerable simplifications. Tests of prototype connections are commonly carried out to obtain actual moment-rotational behavior that is then modelled approximately by mathematical expressions. In this study, without loss of generality, no specific nonlinear connection model or connection type is adopted in the buckling analysis of unbraced frames. The procedure developed in this study, therefore, is applicable for any specific connection model through the correlation between the

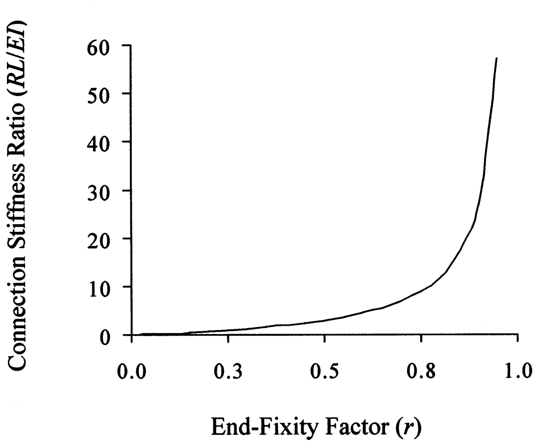


Fig. 6. Relationship between connection stiffness and end-fixity factor.

connection stiffness and the end-fixity factor. When the specified nonlinear connection model is adopted to simulate the nonlinear behavior of the connection, an iterative process has to be developed to compute the connection stiffness since the connection behavior is related to the applied loads. However, this is beyond the scope of this paper.

Upon the introduction of the end-fixity factor, different member-end restraint conditions are readily modelled, such as rigid-pinned, rigid-PR, pinned-PR, simply by setting the end-fixity factors at the two ends of the member to appropriate values. Therefore, the proposed analysis method is comprehensive regardless of member end-rotational conditions and can be applied to the analysis of unbraced frames with any combination of pinned, rigid, and PR connections.

Consider the PR member shown in Figure 4 with various values of connection stiffnesses R_1 and R_2 at ends 1 and 2, respectively. Ends 1 and 2 refer to the so called "near end" and "far end". The slope-deflection equation for such a member can be expressed in terms of the end-fixity factors r_1 and r_2 as (Xu, 1994)

$$M_1 = \frac{3r_1}{4 - r_1 r_2} \frac{EI_b}{L_b} \left(4\theta_1 + 2r_2\theta_2\right) \tag{9}$$

where M_1 is the restraint moment at end 1, EI_b/L_b is the flexural stiffness of the beam, r_1 and r_2 are the end-fixity factors defined in Equation 8, and θ_1 and θ_2 are the joint rotations associated with the column to which the beam ends 1 and 2 are connected, respectively. Let ν be the ratio of θ_2 and θ_1, then Equation 9 becomes

$$M_1 = \frac{6r_1}{4 - r_1 r_2} \frac{EI_b}{L_b} (2 + \nu r_2)\theta_1 \tag{10}$$

in which the counter-clockwise rotation is positive. Therefore, the rotational restraint stiffness of a PR member can be obtained as

$$R_s = \frac{6r_1}{4 - r_1 r_2} \frac{EI_b}{L_b} (2 + \nu r_2) \tag{11}$$

The ratio of θ_2 to θ_1 is associated with the buckling mode of the frame and is unknown until the buckling occurs. The buckling mode of the frame is influenced by many factors such as the configuration of the frame, stiffnesses of members and connections, load distribution, bracing and support conditions, etc. Therefore, an assumption has to be made on the buckling mode of the frames. The current practice of evaluating the effective length factor for rigid frames is based on the alignment chart method, in which a symmetric buckling mode is applied to braced frames with $\nu = -1$, while an asymmetric buckling mode is adopted for unbraced frames with $\nu = 1$. However, in general, the buckling mode of the frame may be neither symmetric nor asymmetric. Therefore, it is understood that such assumptions may result in inaccuracy in some cases. For PR frames, these assumptions have also been adopted to calculate effective length factors using the modified alignment chart method (Bjorhovde, 1984; Chen and Lui, 1991; Xu, 1994; Kishi et al., 1997; Christopher and Bjorhovde, 1999). Figure 7 illustrates the variation of R_s as ν changes from –1 to 1 for the case when the near end is rigid with $r_1 = 1$ and r_2 varies from zero to one for the far end. It can be seen from Figure 7 that the maximum difference of R_s between the symmetric buckling mode ($\nu = 1$) and the asymmetric buckling mode ($\nu = -1$) occurs at $r_1 = r_2 = 1$ for rigid frames. The difference of R_s between $\nu = 1$ and $\nu = -1$ for PR frames diminishes as the connection stiffness decreases. Therefore, in the cases where frames are not subjected to either a symmetric or an asymmetric buckling mode, PR frames may have better accuracy than that of rigid frames when the assumptions of symmetric and asymmetric modes are adopted. A recent study (Xu and Liu, 2001) concludes that the inaccuracy associated with the assumption of $\nu = 1$ is insignificant, which is also demonstrated in Example 2. Therefore, in this study, $\nu = 1$ is adopted for unbraced frames.

Expressed in terms of the end-fixity factors, Equation 11 allows the beam to have different end connection stiffnesses. Moreover, it is applicable for a beam with any combination of pinned, PR, and rigid connections. For instance, Equation 11 will yield a value of $3EI_b/L_b$ for a beam with a rigid connection at the "near end" ($r_1 = 1$) and pinned connection at the "far end" ($r_2 = 0$). When both ends of the beam are rigid, the restrained stiffness is $6EI_b/L_b$. If the "near end" connection is a pinned connection ($r_1 = 0$), the corresponding value of R_s becomes zero, which indicates that the beam will not be able to provide any rotational restraint to the connected column.

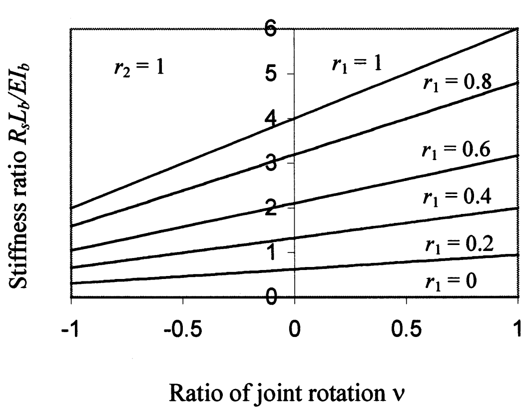


Fig. 7. Relationship between restraining stiffness and ratio of joint rotation.

In the stability analysis of frameworks, the assumption of proportional loading is always adopted in the estimation of the effective length factor. The alignment chart (AISC, 1986 and 1999) is based on a buckling model of the subassemblage of the column under investigation that involves only the immediately adjacent members framing directly to the column. The assumptions that are adopted by the alignment chart method for unbraced frames can be summarized as follows:

1. All of the members have constant cross-section, and the member behavior is purely elastic;
2. All joints are rigid;
3. For unbraced frames, rotations at the far ends of the restraining members are equal in magnitude and opposite in sign, producing reverse-curvature bending;
4. All columns buckle simultaneously, and the column stiffness parameter ϕ defined in Equation 4 must be identical for all columns; and
5. No significant axial compression force exists in the girders.

In this study, the adopted story-based buckling model considers the interactions among the columns in a story. In addition, the assumption of rigid joints is not necessary due to the consideration of partially-restrained construction (AISC, 1999). With regard to PR connections, the following assumptions are adopted in this research:

1. Only the moment-rotation behavior of connections is considered, whereas axial and shear deformations in the connections are ignored.
2. Connection dimensions are assumed to be negligible compared to the lengths of the beams and columns. Thus, the rotational deformation of a connection is considered to be concentrated at a point, which is at the end of a semi-rigid member.
3. The effects of eccentricity at joints are neglected.

LATERAL STIFFNESS OF AN AXIALLY LOADED PR COLUMN

The derivation of the lateral stiffness of an axially loaded column shown in Figure 8 is presented in Appendix A, in which coefficient β is a modification factor that accounts for the effect of the axial load and PR connections. When the axial force $P = 0$, which leads to $\phi = 0$, the modification factor of the lateral stiffness becomes

$$\beta_0 = \lim_{\phi \to 0} \beta = \frac{(r_l + r_u + r_l r_u)}{4 - r_l r_u} \tag{12}$$

where r_l and r_u are end-fixity factors for the upper and lower ends of the column and are defined in Equations A10a, b in Appendix A. In the case that both upper and lower ends are rigidly connected, that is $r_l = r_u = 1$, then Equation A13 becomes

$$\beta = \frac{\phi^3 \sin\phi}{12\left[2(1-\cos\phi) - \phi\sin\phi\right]} \tag{13}$$

For the case that the upper end of the column is rigidly connected and the lower end is a pinned connection, i.e. $r_l = 0$ and $r_u = 1$, Equation A13 becomes,

$$\beta = \frac{\phi^3}{12\left(\tan\phi - \phi\right)} \tag{14}$$

From Equation A13, it is clear that the lateral stiffness of a column in an unbraced frame is affected by both the axial load and the end-fixity factors of the column. The following case is adopted to demonstrate the relationships among the modification factor of the lateral stiffness, the column axial load, and the end-fixity factor.

Consider the case that the upper end of the column is rigidly connected ($r_l = 1$) and let the end-fixity factor at the lower end vary from zero to unity. Then, the relationship between the modification factor of lateral stiffness β and the axial load ratio $\rho = P/P_e$ can be plotted as shown in Figure 9. With the increase of axial load, the magnitude of β will decrease, which indicates the decrease of column lateral stiffness. For a given axial load, the increased value of r_l will lead to the increase of the modification factor of lateral stiffness β. The lower and upper β values that are associated with the variation of r_l for a given axial load ratio ρ are Equations 13 and 14, respectively. Based on the definition given in Equation A12, the positive and negative domains of column lateral stiffness associated with the variation of the axial load shown in Figure 9 clearly indicate whether a column can provide lateral stiffness to maintain the stability of the story. As the axial load increases, the ability of the column to contribute lateral stiffness to support other weaker columns in the same story is reduced and eventually diminishes to zero when the axial load reaches the critical buckling load of the column. With further increase of the axial load, β becomes negative, which signifies that the column relies upon other columns in the same story in order to sustain the axial load. If the required additional lateral stiffness is not available, then all of the columns in the story buckle simultaneously. Therefore, the modification factor of lateral stiffness β provides a quantitative measurement for the interactions among the columns in a story. Consequently, a lean-on column that depends on other columns to maintain its lateral stability can be classified as a column with a negative lateral stiffness as defined in Equation A12. Specifically, for a pinned-end column, the corresponding modification factor can be obtained by setting $r_u = r_l = 0$, and Equation A13 becomes

$$\beta = -\frac{\phi^2}{12} = -\frac{PL^2}{12EI} \tag{15}$$

The negative sign of β indicates that the pinned-end column always relies upon lateral support from other columns, and it is never able to contribute to the lateral stability of the frame regardless of the magnitude of the axial load.

It is noted that when $r_l = r_u = r$ and the lateral stiffness of a column reduces to zero due to an increase in the axial force, this results in $\beta = 0$ in Equation A13. Thus, the relationship between the axial force and column end-fixity factor r becomes

$$r_0 = \frac{\phi\tan(\phi/2)}{3 + \phi\tan(\phi/2)} \tag{16}$$

where r_0 is the critical column end-fixity factor that is associated with zero lateral stiffness of the column. As shown in

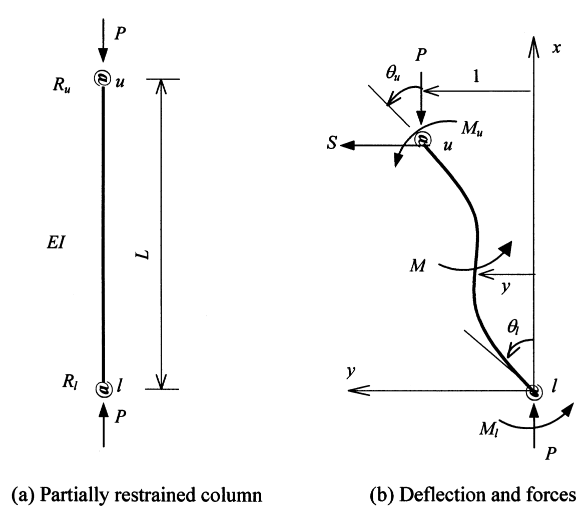


Fig. 8. Analytical model of partially restrained column.

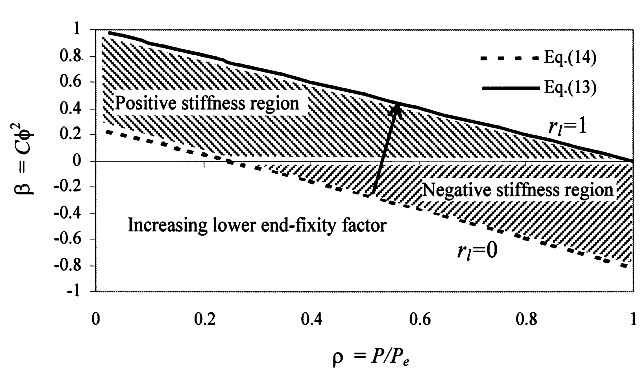


Fig. 9. Relationship between modification factor β and load ratio ρ.

Figure 10, the r_0-P/P_e curve defined by Equation 16 provides a clear boundary of positive and negative domain for lateral stiffness of the column. When the column end-fixity factor is greater than r_0 for a given value of load ratio ρ, then, $\beta > 0$, which signifies that the column can provide lateral bracing to other columns in the same story. If $r < r_{0,}$ which leads to $\beta < 0$, then the column needs to be restrained by other columns.

The transcendental relationship between β and ϕ expressed in Equation A13 is complicated and inconvenient for solving the critical buckling load of the column, especially in a multicolumn case. Equation A13 can be simplified and approximated by means of a Taylor series expansion as

$$\beta = \beta_0 - \beta_1\phi^2 - \beta_2\phi^4 \tag{17}$$

where β_0 is given in Equation 12 and β_1 and β_2 are

$$\beta_1 = \frac{8(5+r_u^2) - (34 - r_u)r_u r_l + (8 + r_u + 3r_u^2)r_l^2}{30(4 - r_l r_u)^2} \tag{18}$$

$$\beta_2 = \frac{f_0 + f_1 r_l + f_2 r_l^2 + f_3 r_l^3}{25200(4 - r_l r_u)^3} \tag{19}$$

where

$$f_0 = 2560 r_u^2 - 1792 r_u^3 \tag{19a}$$

$$f_1 = -4960 r_u + 1844 r_u^2 + 704 r_u^3 \tag{19b}$$

$$f_2 = 2560 + 1844 r_u - 1492 r_u^2 - 41 r_u^3 \tag{19c}$$

$$f_3 = -1792 + 704 r_u - 41 r_u^2 - 17 r_u^3 \tag{19d}$$

Note that β_0, β_1 and β_2 are functions of the end-fixity factors only, and their values are given in Appendix B. As demonstrated in the examples later, Equation 17 provides an adequate approximation for Equation A13.

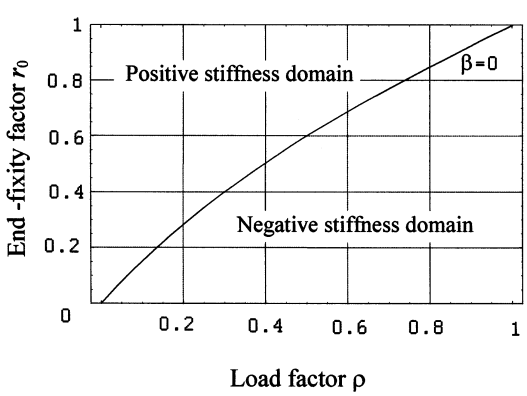


Fig. 10. Relationship between r_0 and ρ.

EVALUATION OF STORY-BASED EFFECTIVE LENGTH FACTORS

Upon the derivation of lateral stiffness of a PR column with consideration of the effect of column compressive load, the story-based effective length factors of columns in an unbraced frame can be readily evaluated. The condition for the multicolumn story-based buckling in a lateral sway mode is that the total lateral stiffness of the story vanishes, thus the stability equation becomes

$$\sum_j S_j = 0 \tag{20}$$

where the summation corresponds to all columns in a story. For column j, substitute Equation 1 into Equation 4, which results in

$$\phi_j = \sqrt{\frac{\lambda P_{sj} L_j^2}{EI_j}} \tag{21}$$

Substituting Equation 21 into Equation 17, the lateral stiffness of column j in Equation A12 can be obtained as

$$S_j = 12\left(\frac{EI_j}{L_j^3}\beta_{0j} - \frac{P_{sj}}{L_j}\beta_{1j}\lambda - \frac{P_{sj}^2 L_j}{EI_j}\beta_{2j}\lambda^2\right) \tag{22}$$

in which L_j and P_{sj} are the length and the axial force due to the specified load of column j, respectively. λ is the proportional load factor. Let

$$a_j = \left(\frac{P_{sj}^2 L_j}{EI_j}\right)\beta_{2j} \tag{23a}$$

$$b_j = \frac{P_{sj}}{L_j}\beta_{1j} \tag{23b}$$

$$c_j = \frac{EI_j}{L_j^3}\beta_{0j} \tag{23c}$$

Substituting Equations 22 and 23 into Equation 20, results in

$$a\lambda^2 + b\lambda - c = 0 \tag{24}$$

where

$$a = \sum_j a_j \tag{25a}$$

$$b = \sum_j b_j \tag{25b}$$

$$c = \sum_j c_j \tag{25c}$$

Therefore, the critical load parameter that corresponds to the story-based lateral sway buckling of the unbraced frame is obtained from the smaller positive root of Equation 24 as

$$\lambda_{cr} = \frac{\sqrt{1+4(a/b)(c/b)}-1}{2(a/b)} \tag{26}$$

Finally, the effective length factor for each column of the frame can be evaluated by Equation 2.

The proposed procedure for evaluation of the column effective length factors for PR-unbraced frames may be summarized as follows:

1. Compute the rotational stiffness of each beam according to Equation 11 and evaluate the rotational stiffnesses R_u and R_l at the upper and lower ends for each column, respectively;
2. Compute the end-fixity factors r_u and r_l from Equation A10 for all of the columns;
3. Obtain coefficients β_{0j}, β_{1j} and β_{2j} from Appendix B for each column or compute them according to Equations 12, 18 and 19;
4. Evaluate coefficients a_j, b_j and c_j according to Equations 23 and corresponding summations a, b and c from Equations 25 for each story; and
5. Solve the critical loading multiplier λ_{cr} from Equation 26 and compute corresponding effective length factors K_j from Equation 2 for columns.

It is noted that if only the first two terms of the Taylor series expansion in Equation 17 are adopted, then, the corresponding stability equation Equation 24 is reduced to

$$b\lambda - c = 0 \tag{27}$$

and the solution of the critical load factor becomes

$$\lambda_{cr} = \frac{c}{b} \tag{28}$$

NUMERICAL EXAMPLES

Example 1

A one-story and two-bay frame with lean-on columns was investigated by other researchers (Cheong-Siat-Moy, 1986; Aristizabal-Ochoa, 1997) as shown in Figure 11. Although the frame is not a PR frame, the proposed approach still applies because both rigid and pinned connections can be treated as special cases of a partially-restrained connection upon the adoption of the end-fixity factor to characterize rotational behavior of connections. The column height L_c is 5.4864 m and the beam span length L_b is 18.288 m. Youngs Modulus is taken to be E = 200 000 MPa, the moment of inertia of columns and beams are $I_1 = I_2 = I_3 = 1.781\times10^{-4}$ m^4 and $I_4 = I_5 = 24.558\times10^{-4}$ m^4, respectively. Following the procedure proposed in the previous section, the story-based effective length factors for columns of the frame are determined as follows:

1. Compute the rotational stiffness of each beam according to Equation 11 and evaluate rotational stiffness R_u and R_l at the upper and lower end for each column, respectively.

   Due to the pinned-end condition, there are no end rotational restraints to columns 1 and 3; therefore, $R_{l1} = R_{l3} = R_{u1} = R_{u3} = 0$. Note that the end-fixity factors for beams 4 and 5 are $r_1 = 1$ and $r_2 = 0$, thus the restraint stiffness provided by beams 4 and 5 to column 2 at the upper end can be evaluated based on Equation 11 as

$$R_{u2} = \frac{3EI_2}{L_b} + \frac{3EI_4}{L_b} = \frac{6EI_4}{L_b}$$

$$= \frac{6E\times24.558\times10^{-4}}{18.288} = 8.0571E\times10^{-4}$$

   and $R_{l2} = 0$.

2. Compute the parameters r_u and r_l from Equation A10 for all of the columns. From Equation A10, $r_{l1} = r_{l2} = r_{l3} = r_{u1} = r_{u3} = 0$, and

$$r_{u2} = \frac{1}{1+3EI_2/(L_cR_{u2})}$$

$$= \frac{1}{1+3\times0.3246/8.0571} = 0.8922$$

3. Obtain values of coefficients β_{0j}, β_{1j} and β_{2j} from Appendix B by linear interpolation for each column:
   • For columns 1 and 3, $\beta_{01} = \beta_{03} = 0$, $\beta_{11} = \beta_{13} = 1/12$, and $\beta_{21} = \beta_{23} = 0$.
   • For column 2, $\beta_{02} = 2.2304\times10^{-1}$, $\beta_{12} = 9.660\times10^{-2}$, and $\beta_{22} = 4.740\times10^{-4}$
4. Evaluate coefficients a_j, b_j and c_j according to Equations 23 with $P_{sj} = 1$ (j = 1, 2, 3) and the corresponding summations a, b and c from Equations 25 for each story:

$$a_2 = \frac{P_{s2}^2 L_c}{EI_2}\beta_{22} = \frac{1^2\times5.4864}{1.781\times10^{-4}E}4.740\times10^{-4} = \frac{14.602}{E}$$

$$a_1 = a_3 = 0$$

$$b_1 = b_3 = \frac{P_{s1}}{L_c}\beta_{11} = \frac{1}{5.4864}\times\frac{1}{12} = 0.0152$$

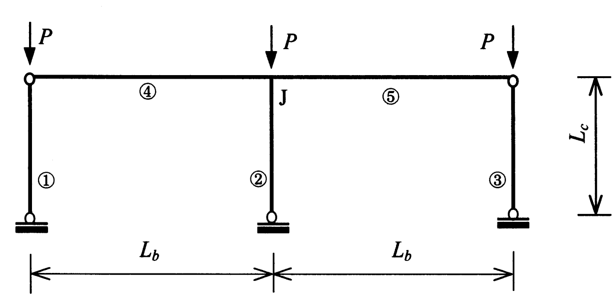


Fig. 11. Example 1: Lean-on column frame.

$$b_2 = \frac{1}{5.4864} \times 0.0966 = 0.0176$$

$$c_2 = \frac{EI_1}{L_c^3}\beta_{02}$$

$$= \frac{1.781 \times 10^{-4} E}{5.4864^3} \times 0.22304 = 2.4054 \times 10^{-7} E$$

$$c_1 = c_3 = 0$$

Thus, the parameters a, b and c are

$$a = \sum a_j = \frac{14.602}{E}$$

$$b = \sum b_j = 0.0480$$

$$c = \sum c_j = 2.4054 \times 10^{-7} E$$

5. Solve λ_{cr} from Equation 26 and compute the corresponding effective length factors K_j from Equation 2 for columns:
   For columns 1 and 3, the effective length factors are unity due to the pinned ends condition, and for column 2,

$$\lambda_{cr} = \frac{\sqrt{1+4(a/b)(c/b)} - 1}{2(a/b)}$$

$$= \frac{E(\sqrt{1+4(14.602/0.048) \times 5.0113 \times 10^{-6}} - 1)}{2 \times 14.602/0.048}$$

$$= 5.0037 \times 10^{-6} E$$

Therefore, from Equation 2 the effective length factor K_2 can be obtained as

$$K_2 = \pi\sqrt{\frac{EI_2/L_c^2}{\lambda_{cr}P_{s2}}} = \pi\sqrt{\frac{1.781E \times 10^{-4}/5.4864^2}{5.0037 \times 10^{-6} E \times 1}} = 3.4162$$

In the case that Equation 28, instead of Equation 26, was used for the evaluation of λ_{cr}, the corresponding effective length factor K_2 is

$$\lambda_{cr} = \frac{c}{b} = \frac{2.4054 \times 10^{-7} E}{0.0480} = 5.0129 \times 10^{-6} E$$

$$K_2 = \pi\sqrt{\frac{EI_2/L_c^2}{\lambda_{cr}P_{s2}}}$$

$$= \pi\sqrt{\frac{1.781E \times 10^{-4}/5.4864^2}{5.0129 \times 10^{-6} E \times 1}} = 3.4131$$

Comparing the results obtained from Equation 28 and Equation 26, which are associated with the first- and second-order approximations of λ in the Taylor series expansion of Equation 22, respectively, the difference of the K factor is only 0.09 percent, and that of λ_{cr} is 0.18 percent. This may suggest that Equation 28 will yield sufficient accuracy of the K factor for engineering practice. The result obtained from this study is very close to the value of 3.40 reported by Cheong-Siat-Moy (1986) and the value of 3.416 evaluated by Aristizabal-Ochoa (1997).

Example 2

The steel frame shown in Figure 12, which was investigated by LeMessurier (1977), Lui (1992), and Shanmugam and Chen (1995), is adopted in this study for comparison of the results for the case of a rigid frame and is further extended to the investigation of the case of a PR frame. Young's modulus of steel is taken to be E = 200 000 MPa, the moments of inertia are $I_1 = 2.8886 \times 10^{-5}$ m^4, $I_2 = 9.4485 \times 10^{-5}$ m^4, $I_3 = 7.6586 \times 10^{-5}$ m^4, and $I_4 = I_5 = 136.12 \times 10^{-5}$ m^4, respectively.

Considering the case of rigid frames, the end-fixity factors are taken to be unity, i.e. $r_1 = r_2 = r_3 = r_4 = 1$, for the beams in Figure 12. Based on Equation 11, the restraint stiffness provided by beams 4 and 5 to their connected columns are $6EI_4/L_4$ and $6EI_5/L_5$, respectively. The other coefficients that are associated with the columns are evaluated and listed in Table 2.

Thus, the summation of the coefficients a_j, b_j and c_j are

$$a = \sum a_j = 0.0423, \quad b = \sum b_j = 25.1007, \quad c = \sum c_j = 192.2950$$

From Equations 28 and 26, the evaluated critical load multiplier λ_{cr} is 7.6608 and 7.5643, respectively. The effective length factors obtained from this study and from other researchers are listed in Table 3, in which values of the K-factor associated with the alignment chart and LeMessurier's methods are directly computed based on the transcendental equations in order to obtain accurate results. It is shown that except for those obtained with the alignment chart method, all of the results are close to the theoretical solutions obtained from frame stability analysis. The differences between the results of Equations 26 and 28 are insignificant.

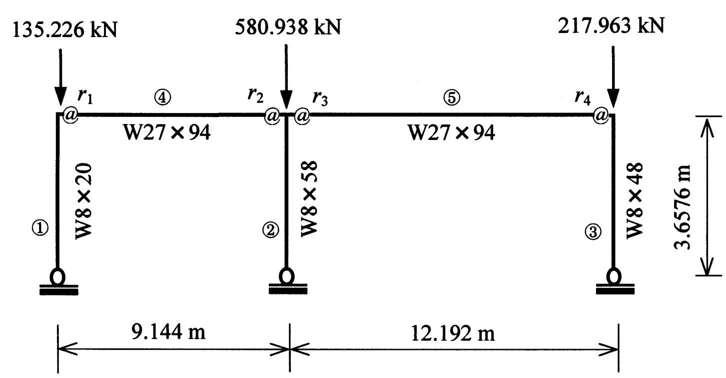


Fig. 12. Example 2: Two bays and one story frame.

Table 2. Example 2: Values of Coefficients

Column	r_l	r_u	$\beta_0(10^{-1})$	$\beta_1(10^{-2})$	$\beta_2(10^{-4})$	$a_j(10^{-3})$	b_j	c_j
1	0	0.9742	2.435	9.9150	4.7915	5.5471	3.6657	28.7540
2	0	0.9528	2.382	9.8463	4.7991	3.1249	15.639	91.9874
3	0	0.9143	2.286	9.7767	4.3633	5.4188	5.7962	70.5490

Table 3. Example 2: Comparison of *K* Factor for the FR Frame

Column	Theoretical	Alignment chart	LeMessurier (1977)	Lui (1992)	1st-order approximation	2nd-order approximation
1	2.05	2.02	2.04	2.04	2.03	2.04
2	1.79	2.03	1.78	1.78	1.77	1.78
3	2.63	2.06	2.62	2.62	2.60	2.62

For the case of PR frames, when $r_1 = r_2 = r_3 = r_4 = r$, a similar calculation computation was carried out for the given identical value of the end-fixity factors for both beams. The critical load multipliers obtained are shown in Figure 13; the variation of the value of the end-fixity factor is between zero and unity. It was found that there is little difference (less than 1.3 percent) in the values of critical load multipliers evaluated by Equations 26 and 28. The maximum difference of the multipliers between the theoretical and the first-order approximation is 1.9 percent. For the second-order approximation, the maximum difference is 0.63 percent. In the case where $r = 0$ for all of the beams, the effective length factor of each column becomes infinity, the frame becomes geometrically unstable and a discussion of elastic stability is meaningless. Therefore, a small value, $r = 0.01$, was used to simulate the purely pinned connection in the evaluation of the values of λ_{cr} and K to avoid excessive numerical difficulty. The values of column effective length factors associated with variations of beam end-fixity factors are shown in Figure 14. Both Figures 13 and 14 reveal that the effect of PR connection on either the critical load multiplier or the effective length factor is insignificant when the value of r approaches unity. However, the effect of the PR connection is considerable when the value of r is very small. Listed in Table 4 is a comparison of the column effective length factors associated with various values of the end fixity factors for beams. In Table 4, the theoretical results were computed based on the condition when the determinant of the stiffness matrix vanishes, in which the effect of PR connections was taken into account. For the modified alignment chart method, stiffness modification factors were applied for beams with PR connections (Bjorhovde, 1984; Chen and Lui, 1991; Xu, 1994; Kishi et al., 1997; Christopher and Bjorhovde, 1999). While applying LeMessurier's method, the values of K factors obtained from the modified alignment chart were used; therefore, the

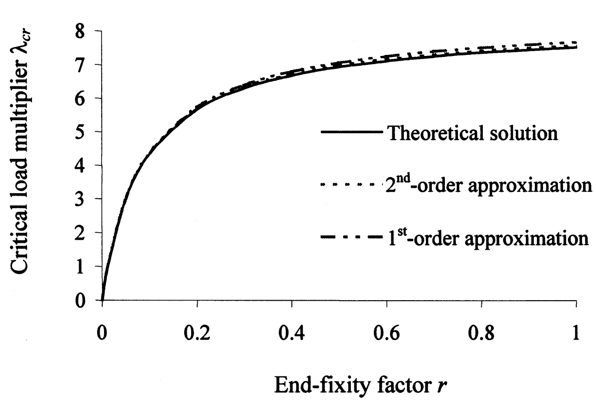


Fig. 13. Example 2: λ_{cr} *of frame with equal value of* r *for beams.*

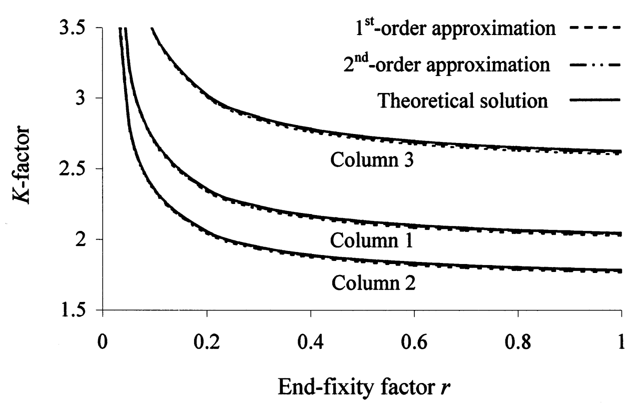


Fig. 14. Example 2: K *factors associated with equal value of* r *for beams.*

Table 4. Example 2: Comparison of *K* Factor for PR Frames

r_j	Theoretical			Alignment Chart			LeMessurier (1977)			1^{st}-order approximation			2^{nd}-order approximation		
	K_1	K_2	K_3	K_1	K_2	K_3	K_1	K_2	K_3	K_1	K_2	K_3	K_1	K_2	K_3
0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0.01	6.19	5.40	7.93	4.56	5.98	8.04	6.19	5.40	7.93	6.19	5.40	7.93	6.19	5.40	7.93
0.05	3.27	2.86	4.20	2.65	3.15	3.93	3.27	2.86	4.20	3.27	2.85	4.19	3.27	2.86	4.20
0.1	2.70	2.36	3.46	2.33	2.60	3.07	2.70	2.35	3.46	2.69	2.35	3.45	2.70	2.35	3.46
0.2	2.36	2.06	3.03	2.16	2.29	2.54	2.36	2.06	3.02	2.35	2.05	3.01	2.36	2.06	3.02
0.3	2.24	1.95	2.87	2.10	2.19	2.35	2.23	1.95	2.86	2.22	1.94	2.85	2.23	1.95	2.86
0.7	2.08	1.82	2.67	2.03	2.06	2.12	2.08	1.81	2.67	2.07	1.80	2.65	2.08	1.81	2.66
0.9	2.06	1.80	2.64	2.02	2.04	2.08	2.05	1.79	2.63	2.04	1.79	2.61	2.05	1.79	2.63

effect of PR connections is considered. Using the first-order Taylor series approximation proposed, the maximum difference of *K* factors between the theoretical results and that of Equation 26 is −1.14 percent. For the second-order Taylor series approximation, the maximum difference is reduced to –0.56 percent. The effective length factors obtained from both the methods of LeMessurier and the proposed second-order Taylor series approximation have an adequate agreement with theoretical results. Obviously the errors associated with the modified alignment chart methods are much greater than those of other methods. For the first-order approximation, it yields satisfactory results; therefore, it is recommended for use in practice due to the simplicity of the method.

It can be seen from both Figures 13 and 14 that both the critical load multiplier and the *K* factor are more sensitive to changes in r when the value of r is small. A small increase in the connection stiffness will dramatically increase the critical load multiplier λ_{cr}. For example, for connections with $r = 0.1$, the associated critical load multiplier λ_{cr} increases more than seven times as that of when $r = 0.01$. However, such benefit of the increase of frame capacity could not be achieved without acknowledging PR construction since the connections with $r = 0.1$ would be normally treated as purely pinned connections ($r = 0$) in current design practice. If $r \geq 0.7$, further increase in connection stiffness will not result in a considerable increase of the load carrying capacity of the frame since the r values have a trivial effect on λ_{cr}.

Consider the case that $r_1 = r_3$, $r_2 = r_4$ and $r_1 \neq r_2$ for the frame shown in Figure 12, in which end-fixity factors are not identical for beam-to-column connections due to connection load reversal (Christopher and Bjorhovde, 1999). The corresponding critical loading multipliers of the frame, which are illustrated in Figure 15, have shown good accordance between the theoretical solution and the solutions obtained by the first- and second-order approximations. Table 5 lists the errors in λ_{cr} due to the adoption of the Taylor series approximations. The maximum error of λ_{cr} associated with the first-order approximation is 4.211 percent when $r_1 = 1.0$ and $r_2 = 0.093$. The error associated with the second-order approximation reaches its maximum of 3.245 percent when $r_1 = 1.0$ and $r_2 = 0.086$. Take into account that the approximations are based on the assumption of $\theta_1 = \theta_2$. Therefore, the assumption of $\theta_1 = \theta_2$ is acceptable for design practice.

CONCLUSION

This paper discusses the elastic stability for unbraced PR frames based on the concept of story-based buckling. To illustrate the characteristics of the interaction between columns in a story, the theoretical stability analyses for two frames were conducted. The fact that the columns in a story will interact with one another in resisting increasing gravity loads has demonstrated that the stability of unbraced frames should be investigated based on system buckling. It is inappropriate to have the stability of frames evaluated simply based on the buckling capacity of the isolated column

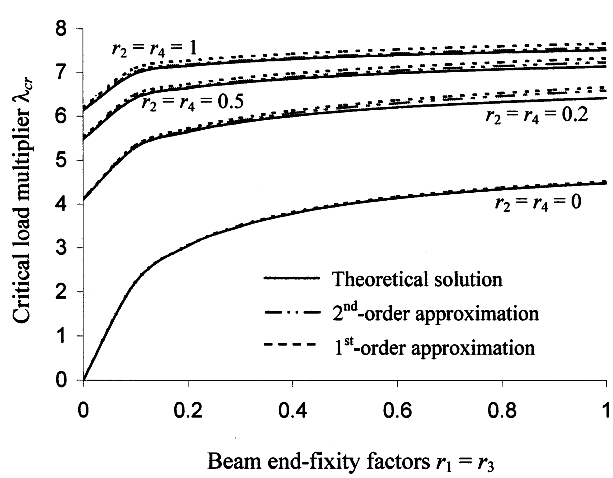


Fig. 15. Example 2: λ_{cr} *of frame with different values of* r *for beams.*

Table 5. Example 2: Errors of λ_{cr} Associated with 1st and 2nd-order Approximations

$r_1 = r_3$	$r_2 = r_4$ =0		$r_2 = r_4$ =0.2		$r_2 = r_4$ =0.5		$r_2 = r_4$ =0.8		$r_2 = r_4$ =1	
	1st (%)	2nd (%)	1st (%)	2nd (%)	1st (%)	2nd (%)	1st (%)	2nd (%)	1st (%)	2nd (%)
0.0	0.00	0.00	0.45	0.05	0.86	0.10	0.99	0.11	1.04	0.12
0.1	0.13	0.01	0.86	0.15	1.29	0.25	1.57	0.43	1.73	0.54
0.2	0.30	0.02	1.23	0.39	1.35	0.25	1.45	0.26	1.51	0.29
0.3	0.43	0.04	1.57	0.66	1.51	0.37	1.50	0.28	1.51	0.27
0.4	0.52	0.05	1.90	0.93	1.67	0.50	1.58	0.35	1.56	0.31
0.5	0.59	0.05	2.21	1.20	1.82	0.64	1.67	0.43	1.62	0.37
0.6	0.64	0.06	2.52	1.48	1.97	0.77	1.76	0.51	1.69	0.42
0.7	0.67	0.06	2.82	1.75	2.12	0.90	1.85	0.59	1.75	0.48
0.8	0.70	0.06	3.11	2.02	2.26	1.03	1.93	0.66	1.81	0.54
0.9	0.72	0.06	3.40	2.29	2.39	1.16	2.00	0.73	1.86	0.58
1.0	0.73	0.07	3.68	2.56	2.52	1.27	2.07	0.80	1.92	0.63

1st: 1st-order approximation; 2nd: 2nd-order approximation.

assemblage such as that in the alignment chart without considering column interaction.

To extend the concept of story-based buckling to unbraced PR frames, a so-called end-fixity factor was applied to incorporate the rotational behavior of beam-to-column connections into the frame stability analysis. With the beam-to-column rotational stiffness and the column lateral stiffness expressed by the associated end-fixity factors, a story-based stability equation is obtained from the summation of lateral stiffnesses for all columns in a story. A Taylor series expansion is employed to simplify the stability equation as a quadratic equation for the load multiplier, λ. After the critical load multiplier, λ_{cr}, is found from the solution of the equation, the effective length factors of the columns are then obtained from Equation 1. The quadratic equation may be further reduced to a linear equation. The results obtained from the examples have demonstrated that the critical load multiplier obtained from the linear equation provides an accurate estimation of column effective length factors; therefore, it is recommended for use in design practice.

It is worthwhile to point out that upon the introduction of the end-fixity factor, various member-end restraint conditions may then be readily modelled, such as rigid-pinned, rigid-PR, pinned-PR, simply by setting the end-fixity factors at the two ends of the member to the appropriate values. Therefore, the proposed approach is comprehensive regardless of member end-rotational conditions and can be applied to the stability analysis of unbraced frames with any combination of pinned, PR and rigid connections. The simplicity and the efficiency of this approach have been demonstrated by means of several examples.

When the connection stiffness is large, very significant changes in stiffness produce only very small changes in the end-fixity factor, as shown in Figure 6. Consequently, such changes have a trivial influence on both the critical load multiplier and the effective length factors of columns (Table 4). Conversely, with a lower value of connection stiffness, even small increases in the stiffness would result in appreciable increases in the end-fixity factor. Therefore, it will have significant effects on the λ_{cr} and K factors based on Table 4, and Figures 13 and 14. From the design point of view, a pinned connection in practice always has some rotational stiffness that may lead to considerably enhanced load capacity and stability of the frame that could be of great benefit to a structure. Cost efficiency may also be achieved with large reductions in connection stiffness from full fixity ($r \approx 1.0$) since it has little effect on the λ_{cr} (Figure 13) in that there is only a small change in the end-fixity factor (Figure 5). The reduction in connection stiffness from FR connections may result in potential savings in connection fabrication costs.

ACKNOWLEDGMENT

This work was funded by a grant from the Natural Science and Engineering Research Council of Canada.

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APPENDIX A. LATERAL STIFFNESS OF AXIALLY LOADED COLUMN

To evaluate the lateral stiffness S of a column in an unbraced frame, the deformation model as shown in Figure 8a with the consideration of the effect of compression force P is analysed. Let R_l and R_u be the total rotation restraint stiffness provided by its immediately adjacent PR beams at upper and lower joints, respectively. With a unit lateral deflection at the upper end B, the corresponding deformation and the forces of the column are shown in Figure 8b. Then, the internal moment at location x along the column can be expressed as

$$M = -M_u - P(1 - y) - S(L - x) \qquad \text{(A1)}$$

where M is the moment in the column, M_u is the end moment at the upper end, P is axial load, L is the height of column, and y is the lateral deflection of the column. Then, the equilibrium equation condition yields

$$EI_y'' = -M_u - P(1 - y) - S(L - x) \qquad \text{(A2)}$$

The general solution of Equation A2 is expressed as

$$y = c_1\cos(\phi x/L) + c_2\sin(\phi x/L) + 1 + M_u/P + S(L - x)/P \qquad \text{(A3)}$$

where ϕ is the stiffness parameter defined in Equation 4, and c_1 and c_2 are coefficients to be determined by the boundary conditions of the column. The end moment at the lower end can be obtained from Equation A1 as

$$M_l = -M_u - P - SL \qquad \text{(A4)}$$

Let θ_l and θ_u be the end rotations of the column associated with the lower and upper ends, respectively. Thus, the boundary conditions of the column are:

$$M_u = R_u\theta_u;\; M_l = R_l\theta_l \qquad \text{(A5a, b)}$$

$$y\big|_{x=0} = 0;\; y\big|_{x=L} = 1; \qquad \text{(A6a, b)}$$

$$y'\big|_{x=0} = \theta_l;\; y'\big|_{x=L} = \theta_u \qquad \text{(A6c, d)}$$

Substituting Equation A4 and the boundary conditions from Equations A6 into Equation A3, the coefficients c_1 and c_2 can be determined as

$$c_1 = \frac{\phi(1+C) - C\sin\phi}{\phi(\cos\phi - 1) - C_l\sin\phi}$$
$$c_2 = \frac{C(\cos\phi - 1) - C_l(1+C)}{\phi(\cos\phi - 1) - C_l\sin\phi} \qquad \text{(A7)}$$

and c_1 and c_2 satisfy the following relationship

$$-(\phi\sin\phi + C_u)c_1 + \phi\cos\phi c_2 = (1 + C_u)C + C_u \qquad \text{(A8)}$$

in which

$$C = \frac{SH}{P} = \frac{SL^3}{12EI\phi^2} \qquad \text{(A9a)}$$

$$C_l = \frac{1 - r_l}{3r_l}\phi^2 \qquad C_u = \frac{1 - r_u}{3r_u}\phi^2 \qquad \text{(A9b,c)}$$

where r_l and r_u are the end-fixity factors for the upper and lower ends of the column, respectively, which are defined as

$$r_l = \frac{1}{1 + 3EI/R_lL}; \qquad r_u = \frac{1}{1 + 3EI/R_uL} \qquad \text{(A10a,b)}$$

The coefficient C can then be determined from Equations A5 - A10 as follows

$$C = \frac{(C_l + C_u)\phi\cos\phi + (\phi^2 - C_lC_u)\sin\phi}{2\phi - \phi(2 + C_l + C_u)\cos\phi + (C_l + C_u - \phi^2 + C_lC_u)\sin\phi} \qquad \text{(A11)}$$

Thus, from Equation A9a, the lateral stiffness of the column can be expressed as

$$S = 12\frac{\phi^2 CEI}{L^3} = \beta\frac{12EI}{L^3} \qquad \text{(A12)}$$

where $\beta = C\phi^2$ is the modification factor of the lateral stiffness that takes the effects of axial force and column end rotational restraints into account. The expression of the modification factor β in terms of the end-fixity factors can be derived from Equations A9 to A12 as

$$\beta = \frac{\phi^3}{12}\left[\frac{a_1\phi\cos\phi + a_2\sin\phi}{18r_lr_u - a_3\cos\phi + a_4\phi\sin\phi}\right] \qquad \text{(A13)}$$

where

$$a_1 = 3(r_l + r_u - 2r_ur_l) \qquad \text{(A14a)}$$
$$a_2 = 9r_lr_u - (1 - r_l)(1 - r_u)\phi^2 \qquad \text{(A14b)}$$
$$a_3 = 18r_lr_u + a_1\phi^2 \qquad \text{(A14c)}$$
$$a_4 = a_1 - a_2 \qquad \text{(A14d)}$$

Equation (A13) is the lateral stiffness offered by the column.

APPENDIX B. VALUES OF COEFFICIENTS β_0, β_1, AND β_2

For the convenience of design practice, the coefficients of column lateral stiffness (β_0, β_1 and β_2) are computed based on the variation of column end-fixity factors and are presented in the following tables. The values of β_0, β_1 and β_2 are evaluated from Equations 12, 18 and 19, respectively. The end-fixity factors r_i and r_j in Tables B1 to B3 can be either r_l or r_u of the column for the reason of symmetry.

Table B1. β_0 ($\times 10^{-1}$) Values Associated with Column End-fixity Factors

r_j \ r_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0.000	0.250	0.500	0.750	1.000	1.250	1.500	1.750	2.000	2.250	2.500
0.05	0.125	0.388	0.652	0.916	1.181	1.447	1.713	1.980	2.247	2.516	2.785
0.10	0.250	0.526	0.804	1.083	1.364	1.646	1.929	2.214	2.500	2.788	3.077
0.15	0.375	0.665	0.957	1.252	1.548	1.847	2.148	2.452	2.758	3.066	3.377
0.20	0.500	0.804	1.111	1.421	1.735	2.051	2.371	2.694	3.021	3.351	3.684
0.25	0.625	0.943	1.266	1.592	1.923	2.258	2.597	2.941	3.289	3.642	4.000
0.30	0.750	1.083	1.421	1.765	2.113	2.468	2.827	3.193	3.564	3.941	4.324
0.35	0.875	1.223	1.578	1.938	2.306	2.680	3.061	3.449	3.844	4.247	4.658
0.40	1.000	1.364	1.735	2.113	2.500	2.895	3.298	3.710	4.130	4.560	5.000
0.45	1.125	1.504	1.893	2.290	2.696	3.113	3.539	3.976	4.423	4.882	5.352
0.50	1.250	1.646	2.051	2.468	2.895	3.333	3.784	4.247	4.722	5.211	5.714
0.55	1.375	1.787	2.211	2.647	3.095	3.557	4.033	4.523	5.028	5.549	6.087
0.60	1.500	1.929	2.371	2.827	3.298	3.784	4.286	4.804	5.341	5.896	6.471
0.65	1.625	2.071	2.532	3.009	3.503	4.014	4.543	5.092	5.661	6.252	6.866
0.70	1.750	2.214	2.694	3.193	3.710	4.247	4.804	5.385	5.988	6.617	7.273
0.75	1.875	2.357	2.857	3.377	3.919	4.483	5.070	5.683	6.324	6.992	7.692
0.80	2.000	2.500	3.021	3.564	4.130	4.722	5.341	5.988	6.667	7.378	8.125
0.85	2.125	2.644	3.185	3.752	4.344	4.965	5.616	6.300	7.018	7.774	8.571
0.90	2.250	2.788	3.351	3.941	4.560	5.211	5.896	6.617	7.378	8.182	9.032
0.95	2.375	2.932	3.517	4.132	4.779	5.461	6.181	6.942	7.747	8.601	9.508
1.00	2.500	3.077	3.684	4.324	5.000	5.714	6.471	7.273	8.125	9.032	10.00

Table B2. β_1 ($\times 10^{-2}$) Values Associated with Column End-fixity Factors

r_j \ r_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	8.333	8.350	8.400	8.483	8.600	8.750	8.933	9.150	9.400	9.683	10.00
0.05	8.338	8.340	8.376	8.446	8.550	8.689	8.862	9.071	9.314	9.594	9.909
0.10	8.350	8.338	8.360	8.416	8.508	8.635	8.798	8.998	9.235	9.509	9.822
0.15	8.371	8.345	8.353	8.395	8.474	8.589	8.741	8.932	9.162	9.431	9.742
0.20	8.400	8.360	8.354	8.383	8.449	8.551	8.693	8.874	9.096	9.360	9.668
0.25	8.438	8.384	8.364	8.380	8.432	8.522	8.652	8.824	9.037	9.296	9.600
0.30	8.483	8.416	8.383	8.385	8.424	8.502	8.621	8.782	8.987	9.239	9.540
0.35	8.538	8.458	8.411	8.400	8.426	8.491	8.598	8.749	8.945	9.191	9.488
0.40	8.600	8.508	8.449	8.424	8.438	8.490	8.585	8.725	8.913	9.152	9.444
0.45	8.671	8.567	8.495	8.458	8.459	8.499	8.582	8.712	8.891	9.122	9.411
0.50	8.750	8.635	8.551	8.502	8.490	8.519	8.590	8.709	8.879	9.103	9.388
0.55	8.838	8.712	8.617	8.556	8.532	8.549	8.609	8.717	8.878	9.096	9.376
0.60	8.933	8.798	8.693	8.621	8.585	8.590	8.639	8.738	8.889	9.101	9.377
0.65	9.038	8.893	8.778	8.696	8.650	8.643	8.682	8.770	8.914	9.119	9.392
0.70	9.150	8.998	8.874	8.782	8.725	8.709	8.738	8.817	8.952	9.151	9.421
0.75	9.271	9.112	8.980	8.879	8.813	8.787	8.806	8.877	9.005	9.199	9.467
0.80	9.400	9.235	9.096	8.987	8.913	8.879	8.889	8.952	9.074	9.264	9.531
0.85	9.538	9.367	9.223	9.107	9.026	8.984	8.987	9.043	9.160	9.347	9.615
0.90	9.683	9.509	9.360	9.239	9.152	9.103	9.101	9.151	9.264	9.449	9.719
0.95	9.838	9.661	9.508	9.383	9.291	9.238	9.230	9.277	9.387	9.573	9.847
1.00	10.00	9.822	9.668	9.540	9.444	9.388	9.377	9.421	9.531	9.719	10.00

Table B2. β_2 ($\times 10^{-4}$) Values Associated with Column End-fixity Factors

r_j \ r_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0.000	0.148	0.546	1.129	1.829	2.579	3.314	3.967	4.470	4.757	4.762
0.05	0.038	0.041	0.309	0.778	1.386	2.066	2.753	3.380	3.879	4.180	4.214
0.10	0.148	0.011	0.150	0.507	1.020	1.626	2.260	2.856	3.343	3.652	3.709
0.15	0.320	0.049	0.064	0.309	0.726	1.255	1.832	2.391	2.863	3.173	3.247
0.20	0.546	0.150	0.045	0.181	0.502	0.952	1.469	1.987	2.437	2.744	2.830
0.25	0.818	0.305	0.086	0.116	0.343	0.713	1.167	1.641	2.065	2.364	2.457
0.30	1.129	0.507	0.181	0.108	0.243	0.534	0.924	1.351	1.746	2.033	2.130
0.35	1.468	0.748	0.322	0.153	0.199	0.412	0.738	1.116	1.479	1.751	1.847
0.40	1.829	1.020	0.502	0.243	0.205	0.342	0.604	0.933	1.262	1.516	1.609
0.45	2.202	1.315	0.715	0.373	0.254	0.319	0.519	0.799	1.094	1.328	1.415
0.50	2.579	1.626	0.952	0.534	0.342	0.338	0.479	0.710	0.970	1.184	1.263
0.55	2.953	1.943	1.206	0.721	0.461	0.393	0.477	0.663	0.889	1.081	1.152
0.60	3.314	2.260	1.469	0.924	0.604	0.479	0.510	0.652	0.845	1.017	1.079
0.65	3.655	2.567	1.732	1.137	0.764	0.587	0.571	0.672	0.834	0.986	1.039
0.70	3.967	2.856	1.987	1.351	0.933	0.710	0.652	0.717	0.851	0.985	1.030
0.75	4.241	3.117	2.225	1.557	1.102	0.841	0.746	0.779	0.889	1.008	1.046
0.80	4.470	3.343	2.437	1.746	1.262	0.970	0.845	0.851	0.941	1.046	1.079
0.85	4.645	3.525	2.613	1.908	1.404	1.088	0.938	0.923	0.996	1.092	1.121
0.90	4.757	3.652	2.744	2.033	1.516	1.184	1.017	0.985	1.046	1.135	1.163
0.95	4.799	3.716	2.819	2.111	1.589	1.246	1.068	1.025	1.078	1.163	1.191
1.00	4.762	3.709	2.830	2.130	1.609	1.263	1.079	1.030	1.079	1.163	1.190