Simplified Procedure to Determine Maximum Beam Deflection

BENJAMIN KOO

THE MAXIMUM DEFLECTION of a beam occupies an important role in discussions concerning structural design. Building codes such as ACI-63 and the AISC Specification limit the deflection caused by a live load to 1/360 of the beam span. Beam design calculations to meet the specifications usually involve tedious and lengthy computations. In this paper, the following simplified procedure is proposed: For any beam of variable moment of inertia, subject to end moments and lateral loads, first use matrix multiplication to determine a segment where maximum deflection will occur. Then, using boundary conditions of the segment, repeat matrix multiplication to obtain the maximum deflection.

THEORETICAL BACKGROUND

To introduce the technique, a brief review of general theoretical background is presented. For a beam subjected to any arbitrary transverse loads, the differential equation of deflection is expressed as:

$$\frac{d^2y}{dx^2} = \frac{M_n}{EI_n} \qquad (1)$$

where

- y = deflection (negative sign points downward)
- M_n = moment at point **n** produced by transverse loads (conventional beam sign, i.e., positive moment tends to bend upward and causes top fiber to be in compression)
- E = modulus of elasticity
- I_n = moment of inertia of cross-section of a beam at point **n.**

By dividing a beam into equal intervals of length h and designating equally spaced points, starting from left support, such as 0, 1, 2, . . . , etc., Equation (1) can be expanded by finite difference expression as:

$$Y_{n-1} - 2Y_n + Y_{n+1} = h^2 \frac{M_n}{EI_n} \qquad (2)$$

Benjamin Koo is Associate Professor in Structural Engineering, University of Toledo, Toledo, Ohio.

For each pivotal point one equation can be written, so that a set of simultaneous linear equations equal to the number of unknown deflection points on the beam is obtained. In most common practice, the beam is divided into four (at most five) equal segments with three equally spaced pivotal points (Y_{n-1}, Y_n and Y_{n+1}) located between the two end support points (Y_{n-2} and Y_{n+2}). Application of Equation (2) results in the following simultaneous equations:

$$Y_{n-2} - 2Y_{n-1} + Y_n = h^2 \frac{M_{n-1}}{EI_{n-1}} \qquad (2a)$$

$$Y_{n-1} - 2Y_n + Y_{n+1} = h^2 \frac{M_n}{EI_n} \qquad (2b)$$

$$Y_n - 2Y_{n+1} + Y_{n+2} = h^2 \frac{M_{n+1}}{EI_{n+1}} \qquad (2c)$$

These equations can be written in compact matrix form as:

$$[A]\{Y\} = \{C\}$$
$$\{Y\} = [A]^{-1}\{C\} \qquad (3)$$

where $[A]$ is a coefficient matrix and $[A]^{-1}$ is its inverse, $\{Y\}$ is the deflection matrix, and $\{C\}$ is the constant matrix. If there is no settlement for each end support, then

$$Y_{n-2} \text{ and } Y_{n+2} = 0$$

Thus for

$$[A] = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} \text{ and } [A]^{-1} = -\frac{1}{4}\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\text{and } C = \begin{vmatrix} C_{n-1} \\ C_n \\ C_{n+1} \end{vmatrix}$$

where C_{n-1}, C_n and C_{n+1} equal, respectively,

$$h^2 \frac{M_{n-1}}{EI_{n-1}}, \quad h^2 \frac{M_n}{EI_n} \quad \text{and} \quad h^2 \frac{M_{n+1}}{EI_{n+1}}$$

then

$$\begin{vmatrix} Y_{n-1} \\ Y_n \\ Y_{n+1} \end{vmatrix} = -\frac{1}{4}\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}\begin{vmatrix} C_{n-1} \\ C_n \\ C_{n+1} \end{vmatrix} \qquad (4)$$

Equation (4) has the merit of simplicity. After matrix multiplication, if it is found that the numerical value of Y_{n-1} is the smallest among the three, then the maximum deflection must occur in the interval Y_n, Y_{n+1}. Then subdivide Y_n, Y_{n+1} again into four equal subintervals of h' and designate pivotal points as Y'_{n-1}, Y'_n and Y'_{n+1}. Use the same $[A]^{-1}$ but assign different values to the elements of $\{C\}$, i.e.,

$$C'_{n-1} = \frac{M'_{n-1}}{EI'_{n-1}} - Y_n$$

$$C'_n = \frac{M'_n}{EI'_n}$$

$$C'_{n+1} = \frac{M'_{n+1}}{EI'_{n+1}} - Y_{n+1}$$

For ordinary design accuracy, the maximum deflection can usually be found among one of the Y' values after matrix multiplication.

In order to explain the procedure more clearly, a numerical example is given.

EXAMPLE

Given: A simply supported beam of variable moment of inertia subject to two end moments, concentrated loads and uniformly distributed load is shown in Fig. 1. ($E = 29 \times 10^6$ psi; $I = 270$ in.^{4}) Assume the end supports are on the same level.

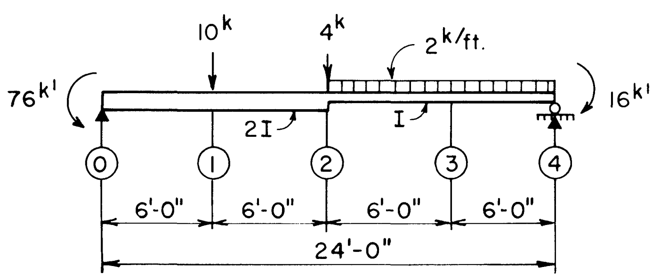

Figure 1

Solution:

Step 1: Divide the beam into four equal segments of $h = 6$ ft-0 in. and designate the interior pivotal points **1, 2,** and **3,** and left and right supports, **0** and **4,** respectively.

Determine the moment at each pivotal point by statics (or by matrix algebra and finite difference methods as explained in the Appendix).

Apply Equation (2) to each interior pivotal point:

$$Y_0 - 2Y_1 + Y_2 = (6)^2\left(\frac{32}{2EI}\right) = 576/EI \qquad (2a)$$

$$Y_1 - 2Y_2 + Y_3 = (6)^2\left(\frac{80}{1.5EI}\right) = 1920/EI \qquad (2b)$$

$$Y_2 - 2Y_3 + Y_4 = (6)^2\left(\frac{68}{EI}\right) = 2448/EI \qquad (2c)$$

As there is no deflection at the end supports, the boundary conditions are $Y_0 = Y_4 = 0$.

From these three simultaneous equations, the matrix $\{C\}$ can be written as:

$$\{C\} = \begin{vmatrix} 5 & 7 & 6 \\ 1{,}9 & 2 & 0 \\ 2{,}4 & 4 & 8 \end{vmatrix}\frac{1}{EI}$$

Substituting into Equation (4), the complete solutions in the following format are:

$$\begin{vmatrix} Y_1 \\ Y_2 \\ Y_3 \end{vmatrix} = -\frac{1}{4EI}\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}\begin{vmatrix} 5 & 7 & 6 \\ 1{,}9 & 2 & 0 \\ 2{,}4 & 4 & 8 \end{vmatrix} = \frac{1}{EI}\begin{vmatrix} -2{,}0 & 0 & 4 \\ -3{,}4 & 3 & 2 \\ -2{,}9 & 4 & 0 \end{vmatrix}$$

It is obvious that a maximum value can occur in a segment between pivotal points **2** and **3**, as shown in Fig. 2.

Step 2: Repeat the previous procedure by dividing the segment **2,3** into four equal subdivisions of $h' = 1.5$ ft and designating pivotal points, Y'_1, Y'_2 and Y'_3. Again the moment at the pivotal points can be obtained by statics (or by matrix algebra — see Appendix).

The three simultaneous equations formed by repeat application of Equation (2) to the pivotal points are:

$$\begin{aligned} Y_2 - 2Y'_1 + Y'_2 &= (1.5)^2\,(83.75)/EI \\ &= 188.4375/EI \end{aligned} \qquad (2d)$$

$$\begin{aligned} Y'_1 - 2Y'_2 + Y'_3 &= (1.5)^2\,(83)/EI \\ &= 186.75/EI \end{aligned} \qquad (2e)$$

$$\begin{aligned} Y'_2 - 2Y'_3 + Y_3 &= (1.5)^2\,(77.75)/EI \\ &= 174.9735/EI \end{aligned} \qquad (2f)$$

With boundary conditions of $Y_2 = -1/EI$ (3432) and $Y_3 = -1/EI$ (2940), and substituting into Equation (4) again,

$$\begin{vmatrix} Y'_1 \\ Y'_2 \\ Y'_3 \end{vmatrix} = -\frac{1}{4EI}\begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix}\begin{vmatrix} 3620.4375 \\ 185.75 \\ 3114.9375 \end{vmatrix} =$$

$$\frac{1}{EI}\begin{vmatrix} -3587.4375 \\ -3554.3750 \\ -3334.6875 \end{vmatrix}$$

The maximum deflection obviously shows at the point Y'_1, whose numerical value is computed as:

$$Y'_1 = \frac{3587.4375 \times 1000 \times 1728}{29 \times 10^6 \times 270} = 0.791 \text{ in.} \downarrow$$

In comparing with the maximum value of 0.775 in. obtained by the conventional conjugate beam method, the difference is about 2 percent, well within the tolerance of engineering design computation.

Numerous extensions of the technique are feasible. Among the extensions are provisions for shears, moments, and stresses for two and three dimensional structural elements. Several other topics presently under development may add to the technique.

APPENDIX

When a beam is subjected to transverse distributed loads, the moments can be expressed by a differential equation as:

$$\frac{d^2M_n}{dx^2} = q \qquad \text{(A-1)}$$

where q = distributed load, negative sign, when acting downward. Following the same pattern of derivation in deflection analysis, Equation (A-1) can be expanded by finite difference expression as:

$$M_{n-1} - 2M_n + M_{n+1} = h^2q \qquad \text{(A-2)}$$

In matrix form,

$$\begin{aligned} [A]\{M\} &= \{K\} \\ \{M\} &= [A]^{-1}\{K\} \end{aligned} \qquad \text{(A-3)}$$

where $\{M\}$ and $\{K\}$ are the moment and constant matrices, respectively. From Equation (A-3),

$$\begin{vmatrix} M_{n-1} \\ M_n \\ M_{n+1} \end{vmatrix} = -\frac{1}{4} \begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix} \begin{vmatrix} K_{n-1} \\ K_n \\ K_{n+1} \end{vmatrix} \qquad \text{(A-4)}$$

where $K_{n-1} = h^2q_{n-1} - M_{n-2}$, $K_n = h^2q_n$, and $K_{n+1} = h^2q_{n+1} - M_{n+2}$

Referring to Fig. 1, the three simultaneous moment equations, one for each pivotal point, are:

$$M_0 - 2M_1 + M_2 = 6(-10) = -60 \qquad \text{(A-2a)}$$

$$M_1 - 2M_2 + M_3 = 6[-4 + 3\,x(-2)] = -60 \qquad \text{(A-2b)}$$

$$M_2 - 2M_3 + M_4 = 6[6\,x(-2)] = -72 \qquad \text{(A-2c)}$$

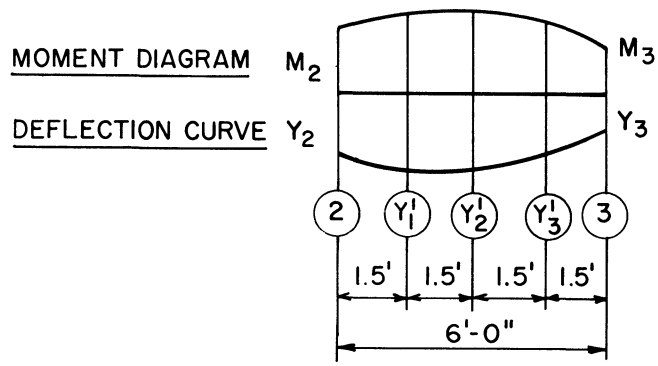

Figure 2

The applied end moments constituting the boundary conditions are $M_0 = -76$ kip-ft and $M_4 = -16$ kip-ft. Substituting the values first into Equations (A-2a) and (A-2c), and then applying Equation (A-4), the solutions in matrix format are:

$$\begin{vmatrix} M_1 \\ M_2 \\ M_3 \end{vmatrix} = -\frac{1}{4} \begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix} \begin{vmatrix} +16 \\ -60 \\ -56 \end{vmatrix} = \begin{vmatrix} 32 \\ 80 \\ 68 \end{vmatrix}$$

Referring to Fig. 2, the following moment equations are obtained:

$$M_2 - 2M'_1 + M'_2 = 1.5(-2 \times 1.5) = -4.5 \qquad \text{(A-2d)}$$

$$M'_1 - 2M'_2 + M'_3 = 1.5(-2 \times 1.5) = -4.5 \qquad \text{(A-2e)}$$

$$M'_2 - 2M'_3 + M_3 = 1.5(-2 \times 1.5) = -4.5 \qquad \text{(A-2f)}$$

With boundary values of $M_2 = +80$ kip-ft and $M_3 = +68$ kip-ft, substituting into Equations (A-2d) and (A-2f), and later applying Equation (A-4), the solutions are as follows:

$$\begin{vmatrix} M'_1 \\ M'_2 \\ M'_3 \end{vmatrix} = -\frac{1}{4} \begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix} \begin{vmatrix} -84.5 \\ -4.5 \\ -72.5 \end{vmatrix} = \begin{vmatrix} 83.75 \\ 83.00 \\ 77.75 \end{vmatrix}$$

REFERENCES

1. *Pei, Ming L.* Matrix Solution of Beams with Variable Moment of Inertia, *ASCE Structural Journal, Oct. 1959.*
2. *Salvadori, M. G. and Baron, M. L.* Numerical Methods in Engineering, *Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961.*
3. Specification for the Design, Fabrication and Erection of. Structural Steel for Buildings, *American Institute of Steel Construction, New York, N. Y., 1963.*