Design of Beam-Columns

IRA HOOPER

THE DESIGN OF BEAM-COLUMNS in the 1963 AISC Specification[1] recognizes two important features of structural behavior—frame stability and moment amplification due to lateral deflection.

Frame stability may be defined as the ability of a structure to resist sidesway when vertical load is applied. It is not usually a problem for simply connected structures braced against sidesway. Old buildings have heavy masonry walls that provide bracing although no calculations were made for the bracing effect. Modern buildings can no longer afford the cost or wasted space of massive walls. Also, rigidly connected frames have become common. Rigidly connected frames offer several advantages: reduced beam deflections, greater economy and elimination of floor cracks.

When rigidly connected frames are not braced against sidesway, the columns and beams must provide the stiffness needed to maintain frame stability in addition to their load resisting function.

The interaction formula in use prior to the 1963 AISC Specification was intended to apply to members with pinned ends in braced frames. It did not account for the effects of lateral deflections in the columns.

The new formulas include the effects of restrained ends and of lateral deflection. The range of design includes rigidly connected frames with or without sidesway bracing. The new sections of the AISC Specification for beam-columns deal with (a) effective length and (b) interaction formulas (6), (7a) and (7b).

EFFECTIVE LENGTH

Effective length is a geometrical property of the frame incorporating the column being considered. Effective length does not depend upon the application of moments to the column, nor will it be influenced by the presence or absence of lateral loads applied to the frame. Effective length is a useful comparison between actual columns in a frame and the isolated column of theory.

Ira Hooper is Associate, Seelye, Stevenson, Value and Knecht, New York, N. Y., and is a Professional Member of AISC.

Columns Without Sidesway—The classical derivation of the Euler formula for the critical load of a concentrically loaded column applies to a vertical member of uniform cross-section pinned at each end (Fig. 1); lateral displacement is prevented at each end, but vertical displacement is permitted at the top. At critical load, the column deflects laterally in a smooth curve between the ends or nodes.

All other column types are referred to this classical case, so for convenience its actual length is defined as its effective length, or $K = 1.0$. Such columns occur in actual practice. The AISC rules permit the use of $K = 1.0$ for columns in frames with sidesway prevented by vertical bracing, by shear walls, or by attachment to other braced structures. The rules would be theoretically correct for frames with hinged connections between beams and columns; the rules are conservative for rigidly connected braced frames.

In comparing other column types with the classic case it is necessary to find the distance between nodes to establish the effective length. In some cases it may be difficult to locate two nodes, and another concept may be helpful. Each half of the classic column may be considered to be a "flagpole" (Fig. 2). A "flagpole" is fixed in

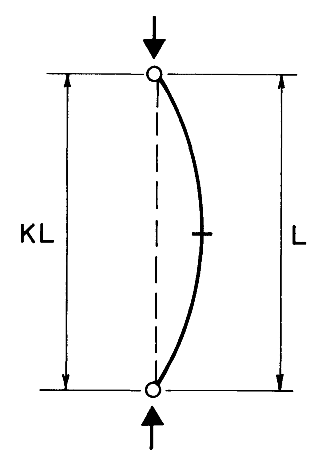


Fig. 1. Classic case—2 ends pinned, no sidesway. $K = 1.0$

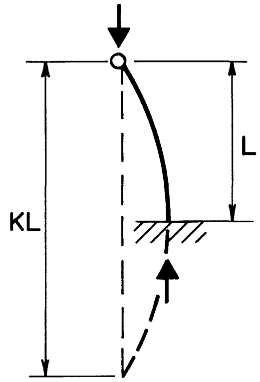


Fig. 2. "Flagpole". K = 2.0

vertical position at its base and supports the critical vertical load at its free end. Two "flagpoles" with joined bases form the classic column; therefore, the effective length of each "flagpole" is twice its actual length.

Figure 3 shows a column braced against sidesway, but with ends fully restrained against rotation. At critical load, the deflection curve includes four "flagpoles" of equal length and two inflections at the quarter points. The critical load for the restrained column is equal to the critical load for a pinned-ended column of half the length. Therefore, the effective length of the restrained column is half the actual length. Such columns rarely occur in actual practice, since it is not possible to provide full restraint against end rotation.

The two cases with ends pinned and ends fully restrained are the limiting conditions for columns with sidesway prevented, and the K-values cannot exceed 1.0 nor be less than 0.5. It is conservative to use K equal to 1.0 for all columns with sidesway prevented; for columns of usual proportions the extra weight will be small. For tall slender columns with rigid beam connections, the true K-values will result in worthwhile savings.

Columns With Sidesway—When sidesway is not prevented, it is possible to set only a lower bound to the effective length of a column. This will occur when both ends are fully restrained against rotation (Fig. 4). Under critical load, the column will assume a reverse curve consisting of two "flagpoles" with tips joined at mid-height. The effective length is, therefore, equal to the actual unsupported length.

Full restraint against rotation is a convenient concept that can only be approached in practice (Fig. 5). The beams will permit rotation at the ends of a real column, so the curves above and below the inflection point will be only parts of a "flagpole." The curves must be extended beyond the actual column ends to reach the points where the tangents would become vertical. The effective length will always be larger than the actual length and will increase as the restraint against rotation is reduced.

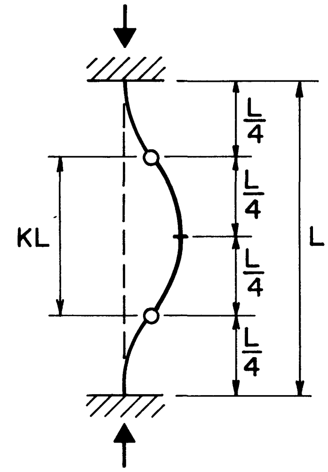


Fig. 3. Fixed ends, no end rotation, no sidesway. K = 0.5

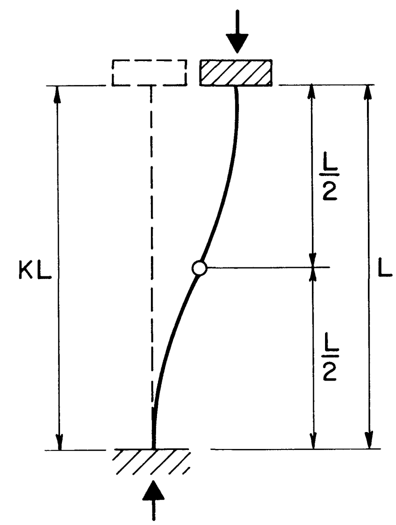


Fig. 4. Fixed ends, free to sway. K = 1.0

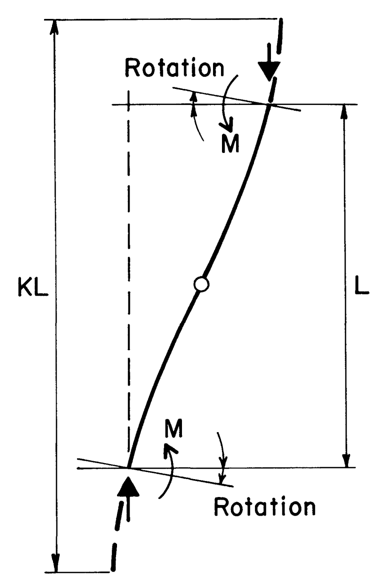


Fig. 5. Semi-fixed ends, free to sway. K > 1.0

Design Methods for Effective Length—Two methods of evaluating effective length are offered by the AISC Specification. Table C1.8.2 in the Commentary[2] (Fig. 6) shows diagrams for all the cases discussed above with theoretical and recommended values of K. This table may be used for preliminary selection of K, but it is better to use the second method for final K-values, especially when sidesway is not prevented by bracing.

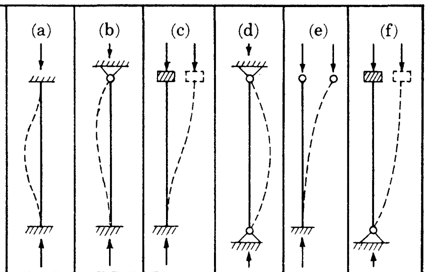
	(a)	(b)	(c)	(d)	(e)	(f)
Buckled shape of column is shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2 0	2.0
Recommended design value when ideal conditions are approximated	0.65	0.80	1.2	1.0	2.10	2 0
End condition code		Rotation fixed and translation fixed Rotation free and translation fixed Rotation fixed and translation free Rotation free and translation free				

Fig. 6. Table C1.8.2 of Commentary on AISC Specification

The second method uses an alignment chart based on stiffnesses of a column and of the beams rigidly connected to each end of the column. The AISC Commentary shows only one alignment chart for side-sway uninhibited, but another chart for side-sway prevented is available in the Column Research Council Guide[3] (Fig. 7). The charts are intended to be used for frames with rectangular panels.

To enter the alignment charts, it is necessary to calculate the stiffness ratios at the top and at the bottom of the column under investigation (Fig. 8). The stiffness ratio is equal to the sum of the column stiffnesses at a joint, divided by the sum of the restraining girder stiffnesses at the joint:

$$G = \frac{\Sigma I_c/L_c}{\Sigma I_g/L_g}$$

In general, one or two columns will be present at a joint, restrained by one or two girders. For column ends bearing on a footing but not rigidly connected, the stiffness ratio is theoretically infinite, but may be taken equal to 10 unless actually designed as a true friction-free pin. If the column is rigidly attached to the footing, the stiffness ratio may be taken as 1.0.

The appropriate chart is entered with the stiffness ratios at top and bottom to find the K-value.

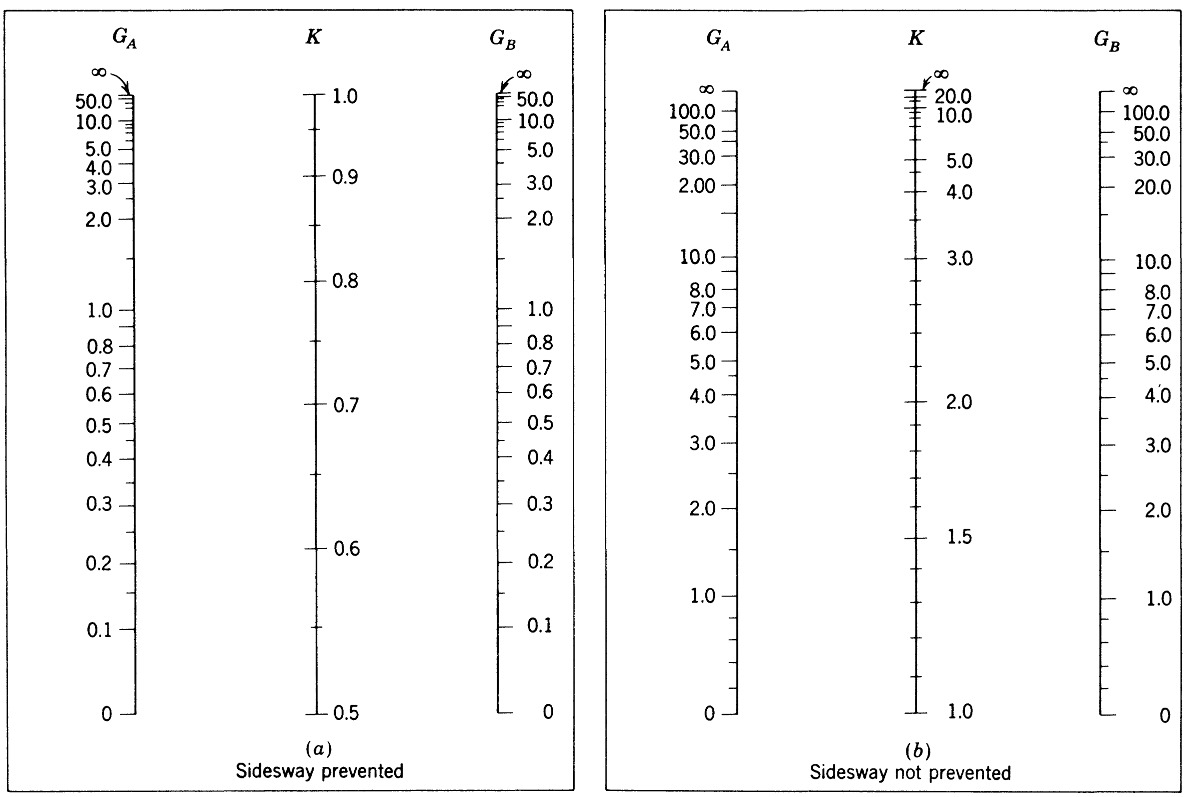


Fig. 7. Alignment charts for effective length of column in continuous frames (from Column Research Council Guide[3])

The alignment chart is not a direct design method; it analyzes assumed sizes. The girder sizes are easily estimated (Fig. 9) by assuming the maximum moment will not exceed the fixed end moment due to uniformly distributed load. The assumption covers a wide range of end-fixities.

Estimates of column sizes are more difficult. To find K, one must choose a column section for strength, but the choice depends on the effective length. This vicious circle can be broken by the following procedure:

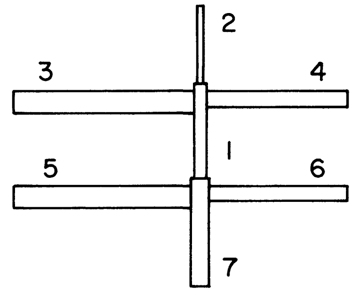
1. Assume a trial K-value. Design aids are described below.
2. Choose a column section for strength based on the trial K-value.
3. Calculate K and compare with procedure 1 above; adjust the column size if required.

Design Aids for Trial K-Values—Two design aids are available for assistance in assuming the trial K-value. Table C1.8.2 of the AISC Commentary has already been mentioned. A more detailed table is included in the Bethlehem Steel Corp. publication, *Beam-Column Tables for Structural Shapes*[4] (Fig. 10).

The best design aid for estimating K-values is knowledge of previous values used elsewhere in a structure. The greatest difficulty lies in guessing the values for the first beam-column. In multi-story frames, the K-values vary little between stories. After the calculations have been made for the top story, close estimates can be made for the succeeding stories below.

In similar fashion, calculations for the first few bays of a large single-story building give excellent clues for estimating the K-values for the remaining bays.

The alignment charts were originally developed for use with rigidly-connected frames. It is possible to use the charts with partially-rigid connections. A method has been developed to reduce the girder stiffness to account for the moment-rotation characteristics of the connections.[5] The method has been elaborated and practical design information has been made available. The method is particularly useful for one- or two-story buildings with ordinary framed connections and without sway bracing; in a low building, the column effective length with framed connections has been shown to be only slightly greater than with rigid connections.[6]



$$k = \frac{I}{L}\ ; \quad G = \frac{\Sigma k_{col}}{\Sigma k_{gird}}$$

$$G_{top} = \frac{k_1 + k_2}{k_3 + k_4}$$

$$G_{bot} = \frac{k_1 + k_7}{k_5 + k_6}$$

Figure 8

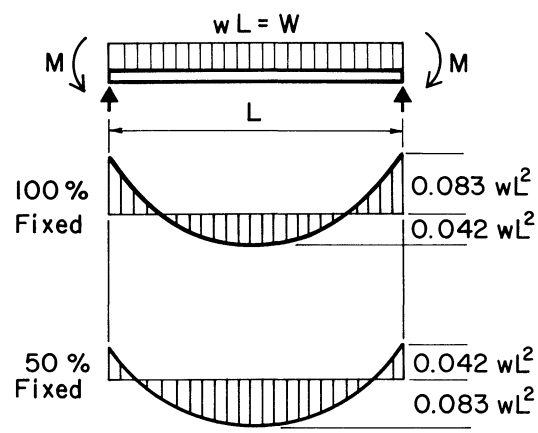


Figure 9

Case	Bracing Condition	Connections		Single Story		Multistory		Bottom Story of Multistory	
		Top	Bottom	G_B	K_y & K_x	K_y	K_x	K_y	K_x
A	Braced	NR	NR		1.00	1.00	1.00	1.00	1.00
B	Braced	NR	R	1.00	.87	.95	.95	.87	.87
C	Braced	R	NR	10.0	.80	.95	.95	.90	.90
D	Braced	R	R	1.00	.75	.90	.90	.80	.80
E	Unbraced	R	R	1.00	1.25	1.80	2.25	1.60	1.60
F	Unbraced	R	NR	10.0	1.80	—	—	2.0	2.50

Legend NR—Non-rigid connection free to rotate under load (i.e., simply supported girders, hinged column connections, etc.)

R—Rigid connection capable of holding the original angle between members virtually unchanged.

Underlined items are final values and need not be modified for the actual member stiffnesses.

Fig. 10. Suggested first trial values of K (from Bethlehem Beam-Column Tables[4])

INTERACTION FORMULAS

After the effective length has been established or assumed, beam-column design is controlled by Formulas (6), (7a) and (7b) of the AISC Specification.

When $f_a/F_a \leq 0.15$:

$$\frac{f_a}{F_a} + \frac{f_b}{F_b} = 1 \qquad \text{Formula (6)}$$

When $f_a/F_a > 0.15$:

$$\frac{f_a}{F_a} + \frac{C_m f_b}{(1 - f_a/F_e')F_b} = 1 \qquad \text{Formula (7a)}$$

$$\frac{f_a}{0.6F_y} + \frac{f_b}{F_b} = 1 \qquad \text{Formula (7b)}$$

Formula (6) is the familiar unity-ratio formula of previous AISC rules, but now applies only when the axial load is less than 15 percent of the member's allowable axial load capacity.

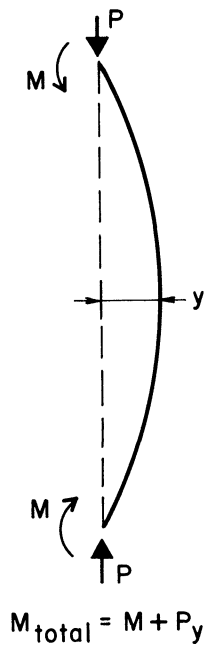


Figure 11

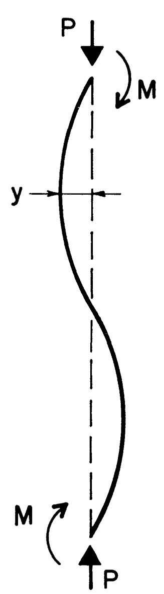


Figure 12

When the axial load exceeds 15 percent of the allowable axial load capacity, Formula (7a) or (7b) may govern the design.

The middle of a column may deflect horizontally when end-moments are applied, creating eccentricity of the compressive load and additional moment (Fig. 11). Such amplified moment is accounted for by Formula (7a) and it is understandable that Formula (7a) should be affected by the column slenderness.

When the end moments are applied in the same sense (Fig. 12), the beam-column will be bent into a reverse curve with smaller moments due to deflection. Under these conditions, Formula (7a) may not govern and Formula (7b) will tend to establish the design. Since no eccentric effects due to deflection can occur at the ends, Formula (7b) is independent of the column slenderness.

Interaction Curves—Formulas (7a) and (7b) are modifications of Formula (6). For understanding of the formulas it will be helpful to plot them as graphs. In Fig. 13, the vertical coordinate is the ratio of computed axial stresses to $0.6F_y$ and the horizontal coordinate is the ratio of actual to allowable bending stresses.

With these coordinates, Formula (7b) is a straight line between values +1.0 on the horizontal axis and +1.0 on the vertical axis.

Formula (6) is a series of straight lines from various intercepts on the vertical axis converging to +1.0 on the horizontal axis. The *Y*-axis intercepts are determined by the column slenderness ratio; note that Formula (6) for the case of a column of zero length is identical with Formula (7b).

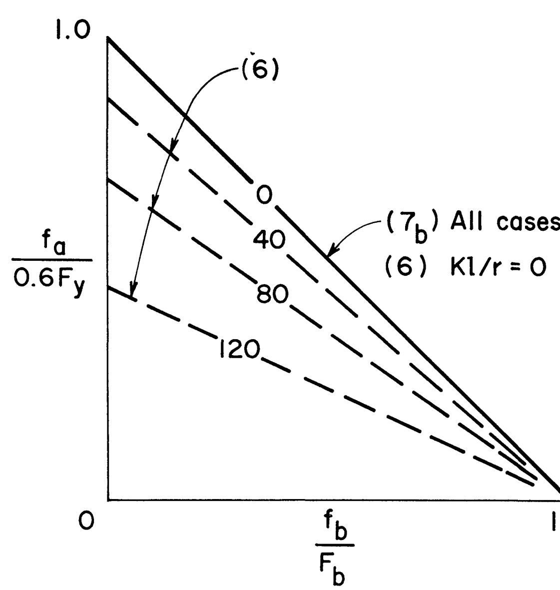


Fig. 13. Curves for Formulas (6) and (7b) at indicated Kl/r values

Figure 14 shows the graphs for Formula (7a). Formula (7a) is similar to formula (6) except that the second term is multiplied by a fraction. The denominator of the entire fraction is always less than 1; it accounts for the added moment due to lateral deflection of the beam-column. The reciprocal of the denominator is called the "amplification factor" and gives correct results when the applied moment neglecting lateral deflection is uniform for the full height of the column. The graphical significance of the amplification factor is shown in Fig. 14. For a given value of slenderness ratio and axial stress ratio, the distances a and b represent the bending stress ratios in the second terms of Formulas (6) and (7a) when $C_m = 1$. The ratio a/b is equal to the amplification factor:

$$\frac{a}{b} = \frac{1}{1 - f_a/F_e'}$$

Also,

$$b = \left(1 - \frac{f_a}{F_e'}\right) a$$

and

$$c = a - b = \left(\frac{f_a}{F_e'}\right) a$$

The AISC Manual[7] includes tables for calculating the amplification factor. The Bethlehem Beam-Column Tables[4] yield quick results when calculating the amplification factor.

In Fig. 14, the value of the numerator, C_m, is 1.0. Any other arrangement of moments causes less lateral deflection, resulting in less amplified moment, so that the value of C_m becomes less than 1.0; hence, it is called the "reduction factor". The AISC Specification gives rules for evaluating the reduction factor. For beam-columns in braced frames, C_m may vary between 0.4 and 1.0 (Fig. 15); for unbraced frames, C_m is specified as 0.85.

When the C_m-value is 1.0, the intercepts on the two axes for Formula (7a) are the same as for Formula (6). Between the intercepts for any given column slenderness, the amplification factor depresses the curve for Formula (7a) below the straight line for Formula (6).

When the C_m-value is less than 1.0 (Fig. 16), the intercepts on the vertical axis are the same as for Formula (6), but the point of convergence on the horizontal axis shifts to the right to a value equal to the reciprocal of C_m. The shift causes the curves for Formula (7a) to cross the line for Formula (7b); where this occurs, Formula (7a) no longer governs and (7b) will determine the design.

The lowest value of C_m permitted by AISC rules is 0.4; this value applies to beam-columns in braced frames with moments applied at the ends causing a reverse curve. In Fig. 16 the curves for Formula (7a) lie above

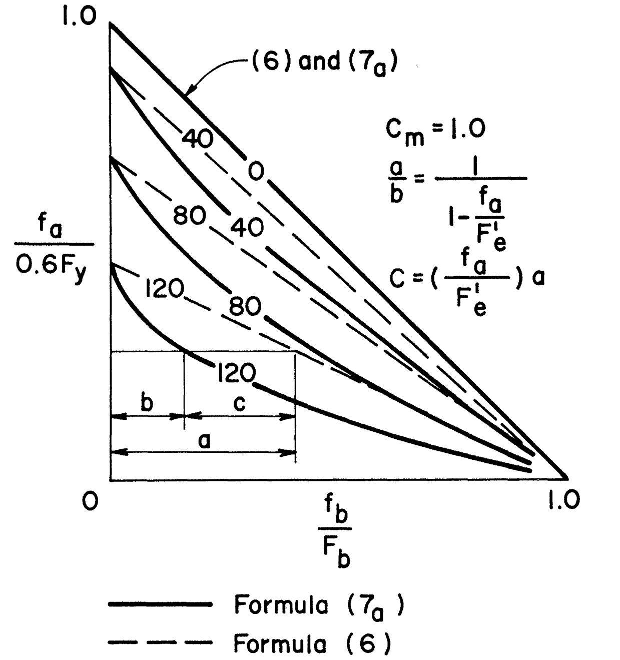


Fig. 14. Curves for Formulas (6) and (7a) at indicated Kl/r values

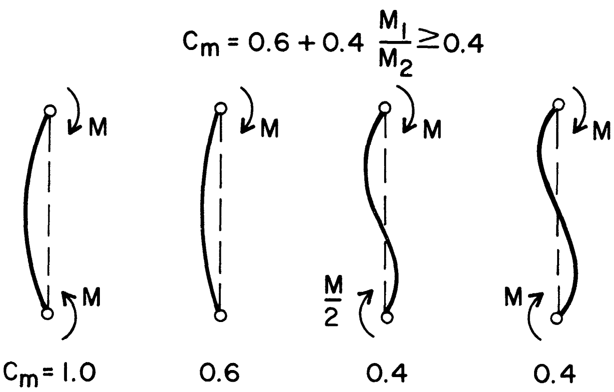


Fig. 15. Reduction factor, C_m, for columns in braced frames

Formula (7b) for most of the range, so Formula (7b) governs as would be expected. Note that slender columns have a region where Formula (7a) will govern; in this region the axial stress is high and bending stress is low.

An important value of C_m is 0.85, which is specified by AISC to be used with unbraced frames. Figure 17 shows that Formula (7a) will govern for slenderness ratios over 80, except for a small region of very high moment. For slenderness ratios less than 60, each of the three formulas controls a part of the range. The combined curve is very complicated, but the straight line for Formula (6) is a close approximation, generally within 5 percent.

Columns with slenderness ratio less than 60 occur frequently. For practical purposes, it is suggested that in designing A36 columns with sidesway permitted, Formula (6) will give satisfactory results when the slenderness ratio is less than 60, and that Formula (7a) will govern when the ratio is above 80.

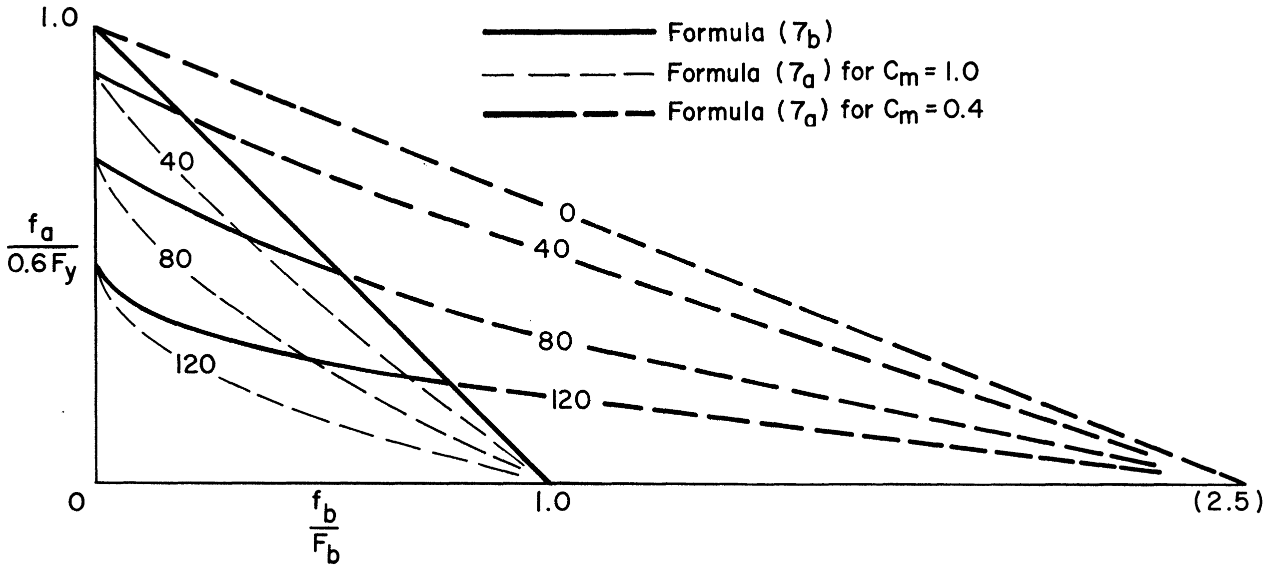


Fig. 16. Effect of reduced C_m value on Formula (7a)

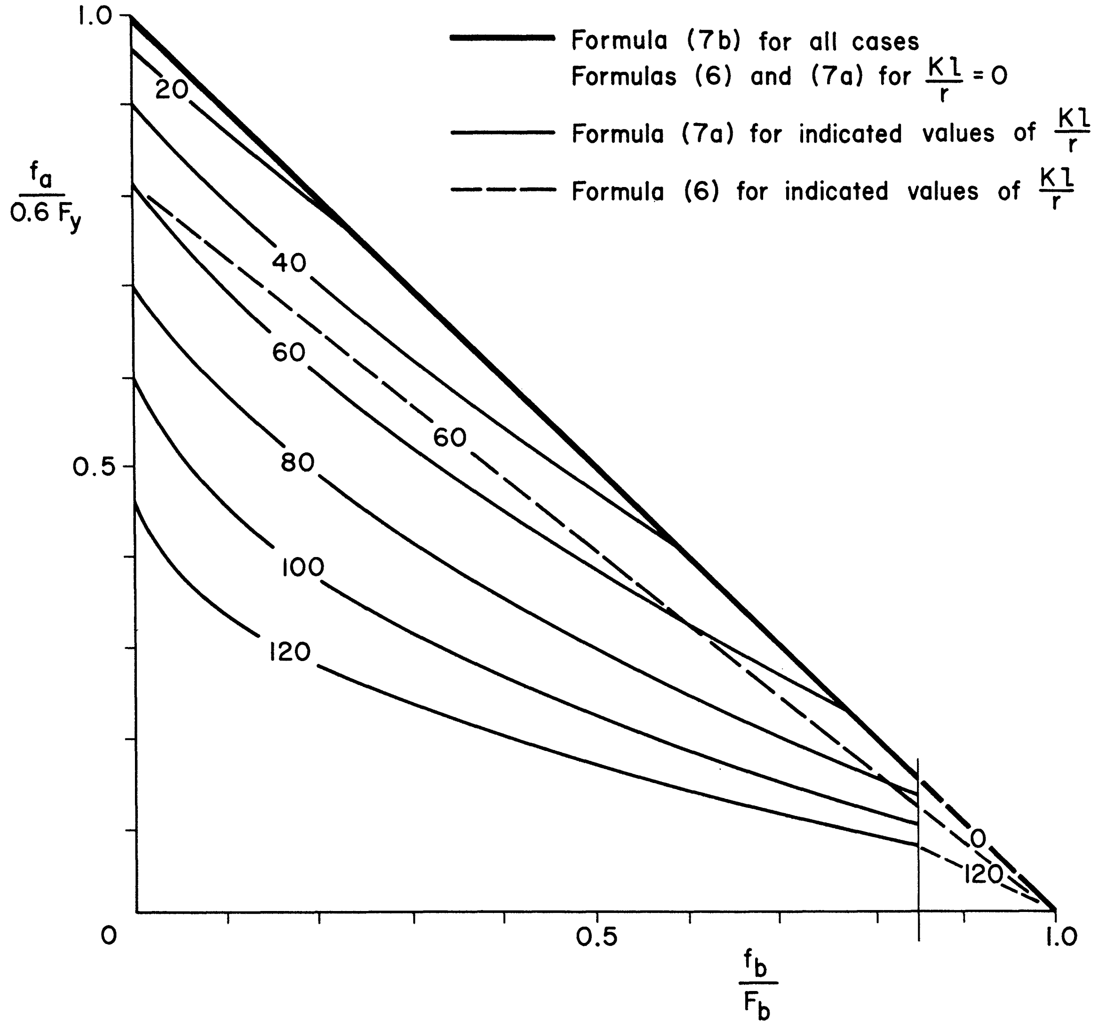


Fig. 17. Interaction curves for C_m = 0.85, F_y = 36 ksi

GENERAL INTERACTION CHART

It is possible to construct a design chart of the interaction formulas that includes all values of C_m (Fig. 18). The coordinates are the same as for the previous interaction charts, except that the horizontal coordinate is multiplied by C_m.

In Fig. 18, the curves for Formula (7a) remain the same as in Fig. 14 with C_m equal to 1.0. Formula (7b) is now represented by a series of straight lines intersecting the vertical axis at 1.0 and the horizontal axis at values equal to C_m. Formula (6) can be plotted as a series of lines intersecting the Y-axis at the usual points, converging to an X-axis intercept at the C_m-value.

The slope of a line through the origin is:

$$s = \frac{f_a/0.6F_y}{C_m f_b/F_b}$$

Substituting

$$f_a = \frac{P}{A} \;;\; f_b = \frac{M}{S} \;;\; e = \frac{M}{P} \;;\; B = \frac{A}{S}$$

$$s = \frac{1}{C_m eB}\left(\frac{F_b}{0.6F_y}\right)$$

This expression holds the key to rapid trial design, since all factors can be readily evaluated:

1. C_m is determined by AISC rules.
2. e is the eccentricity obtained by dividing the known moment by the known axial load.
3. B is the bending factor listed in the AISC Manual. It varies little for column sections of equal depth.
4. The quotient $F_b/0.6F_y$ can be estimated equal to 1.0 for almost all cases. Examination of the column design tables in the AISC Manual will show that most members with slenderness ratio less than 120 have an allowable bending stress of $0.6F_y$; columns with greater slenderness ratios are neither usual nor economical. Compact sections with slenderness ratios less than 50 may use $0.66F_y$, in which case the use of $0.6F_y$ is slightly conservative.

The intersection of the s-line with the appropriate Formulas (7a) and (7b) represent solutions. Values of B and r must be known in order to find the intersections. Fortunately, B and r are almost constant for steel members of the same dimension. Recommended average values are given in the table on the general interaction chart (Fig. 18).

The vertical coordinate of the point of intersection is used to find the required weight per foot of the column section:

$$\text{Column weight} = 3.4 \times A = \frac{3.4P}{f_a}$$

Dividing the numerator and the denominator by $0.6F_y$:

$$\text{Column weight} = \frac{3.4/0.6F_y}{f_a/0.6F_y} \times P$$

For $F_y = 36$ ksi:

$$\text{Column weight (lbs per ft)} = \frac{0.157}{f_a/0.6F_y} \times P \text{ (kips)}$$

When the slope, s, is greater than 1.0, it is convenient to use the reciprocal for interpolation; these values are indicated on the general interaction chart.

In addition to giving a close estimate of column weight, the general chart shows which formula governs. Furthermore, the chart shows the effect of changing any variable, which is often of interest to the designer but is not evident from the formulas or from tables.

Use of the General Interaction Chart—

1. Known items: F_y, P, M_{top}, M_{bot}, sway bracing condition, C_m, effective length.
2. Choose overall column dimensions; find corresponding B and r in the table on the chart.
3. Evaluate Kl/r; locate the corresponding curve for Formula (7a).
4. Draw Formula (7b) as straight line between 1.0 on the vertical axis and C_m on the horizontal axis.
5. Evaluate $C_m eB$ or $1/C_m eB$; draw the corresponding line through the origin; find points of intersection with the curves for Formulas (7a) and (7b).
6. Using the vertical coordinate of the lower intersection point, calculate the estimated section weight and choose a trial column size.
7. Check the trial column size. A check is necessary because average values of B and r were used. The check will be simple, since the governing formula has been determined.

Design Aids for Checking the Trial Column Size—The AISC Manual can be used to find the ratios of axial stresses in the interaction formulas, but it requires much calculation to find the ratio of bending stresses and amplification.

The Beam-Column Tables published by Bethlehem Steel Corporation,[3] available for F_y values of 36 and 50 ksi, are very helpful for all parts of the final check. The Tables show data for every rolled section. Each pair of facing pages deals with the same sections; the left page gives the allowable axial load for all practical lengths as well as the Euler load, P_e', to be used in the amplification factor. The right page shows allowable moments for all practical lengths, including allowances for flanges unbraced laterally.

A quick but less accurate check of the results from the general interaction chart can be made by inspecting

d nom	B_x av	r_x av	b nom	B_y av	r_y av
			16	0.47	4.0
14	0.185	6.5	$14\frac{1}{2}$	0.52	3.7
12	0.22	5.5	12	0.64	3.0
10	0.265	4.5	10	0.77	2.5
8	0.33	3.5	8	1.00	2.0
			$6\frac{1}{2}$	1.25	1.6
$\text{COL. WT/FT} = \dfrac{0.157}{f_a/0.6F_y} \times P_{kip}$					

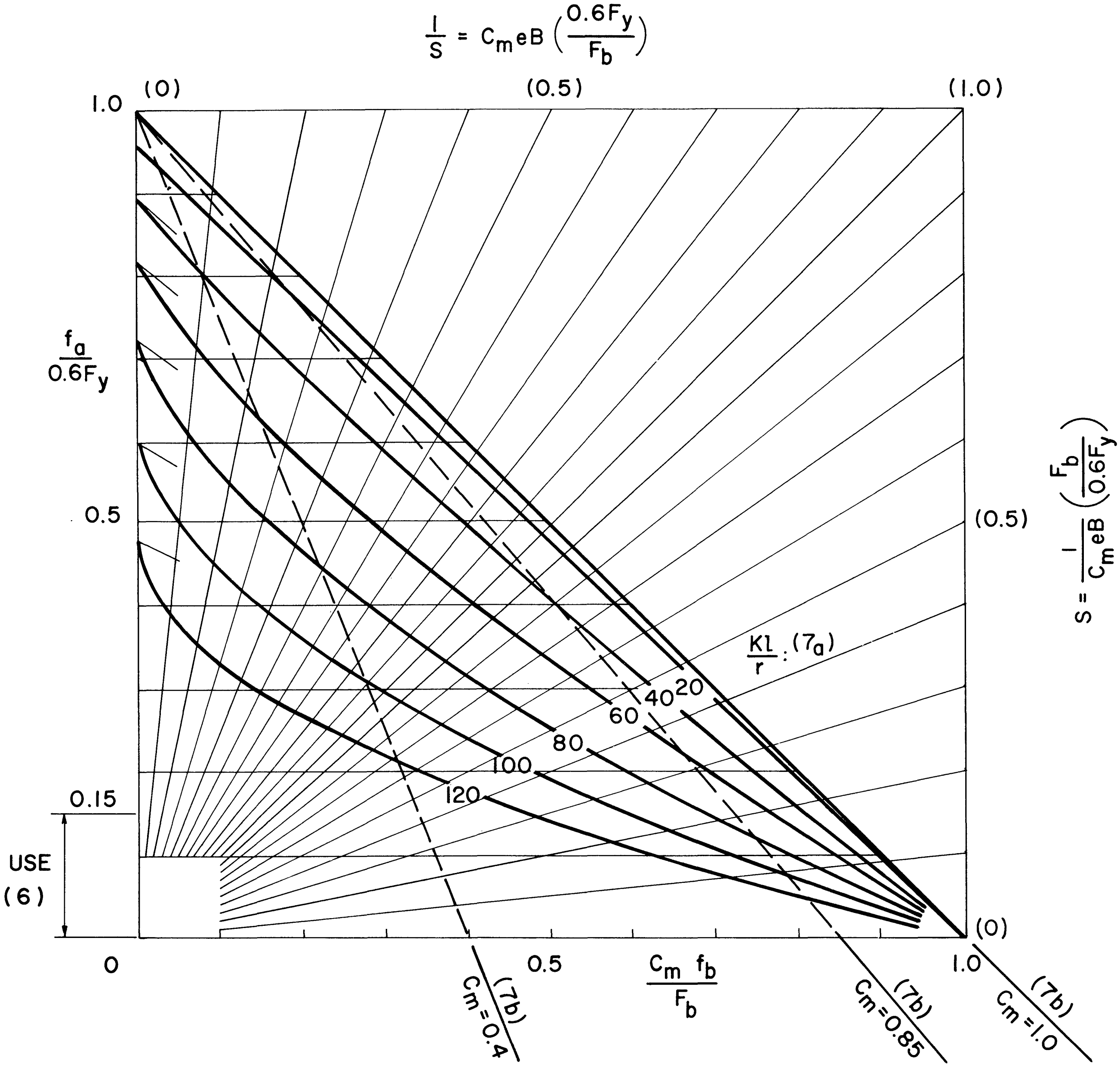


Fig. 18. General column interaction chart, F_y = 36 ksi

the difference between the average values assumed and the actual values of the steel section selected, then adjusting the size if required.

Bending About the Non-critical Axis—The discussion to this point has applied to columns being bent about the axis critical for concentric load. The general interaction chart can be used for size estimates when bending occurs in the non-critical direction. It is also possible to use the chart to estimate sections with bending in both directions.

Formula (7b) is not affected by bending about the non-critical axis, but Formula (7a) requires adjustment. To understand the effect on Formula (7a), visualize a column with the same slenderness ratio in both directions, resisting moment in only one direction. The interaction curve for Formula (7a) in the general chart will apply for the column described. The lowest curve in Fig. 19 represents the interaction equation for such a column.

Allow the slenderness ratio resisting the moment to decrease; the amplification due to lateral deflection will also decrease and the curve for Formula (7a) in Fig. 19 will tend to straighten. When the slenderness ratio has reduced to zero, no amplification will take place and the interaction curve will be a straight line between the intersection on the vertical axis and 1.0 on the horizontal axis. The intersection on the vertical axis of the chart is determined by the slenderness ratio in the direction critical for concentric load; it is not affected by changes in slenderness ratio for the other direction.

Applying to Fig. 19 the expressions derived in explaining the graphical significance of the amplification factor in Fig. 14:

$$\text{For} \quad \frac{Kl}{r} = 80: \qquad c_{80} = \left(\frac{f_a}{F'_{e80}}\right)a$$

$$\text{For} \quad \frac{Kl}{r} = 60: \qquad c_{60} = \left(\frac{f_a}{F'_{e60}}\right)a$$

$$F_e' = \frac{149{,}000{,}000}{(Kl/r)^2} \quad \text{(as defined in AISC Specification)}$$

$$\frac{c_{60}}{c_{80}} = \frac{F'_{e80}}{F'_{e60}} = \left(\frac{60}{80}\right)^2$$

These equations show that the c-distances measured horizontally at a given value of $f_a/0.6F_y$ will vary directly as the square of the slenderness ratio of the non-critical axis.

For each value of critical slenderness ratio, a "family" of curves similar to Fig. 19 could be drawn for values of slenderness ratio in the non-critical direction. Each "family" of curves would have the same intercepts on the chart axes and would be closely spaced for low values of non-critical slenderness so that straight-line interpolation between the limiting curves of a "family" will be conservative. For legibility, the additional curves have not been drawn on the general chart. The curves in Fig. 18 are the lower "family" limits for the indicated slenderness ratio; the upper "family" limits are easily obtained by placing a straightedge between the intercepts on the two axes.

Solutions of Formula (7a) for bending about the non-critical axis are obtained by finding the intersection of the s-line through the origin with the appropriate interpolated member curve of a "family" (Fig. 20).

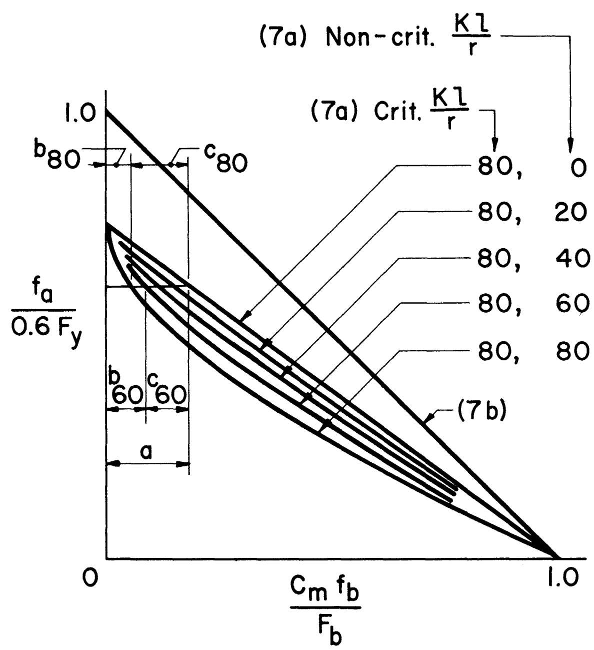


Figure 19

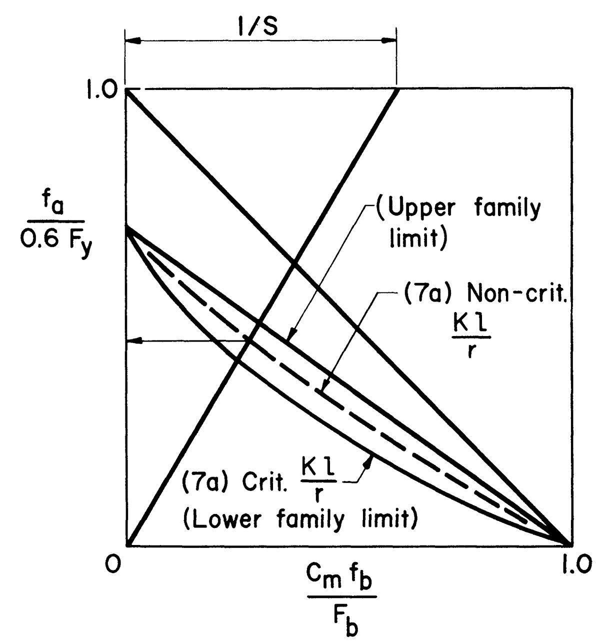


Figure 20

Wind and Seismic Stresses—The AISC Specification allows a 33 percent increase in allowable stresses for members carrying wind or seismic loads, acting alone or in combination with dead and live loads.

Designers usually reduce the combined loads by one-quarter and design at the usual allowable stresses. This procedure applies directly to general chart solutions for Formula (7b); f_a and f_b and both terms of the formula are reduced by the same proportion.

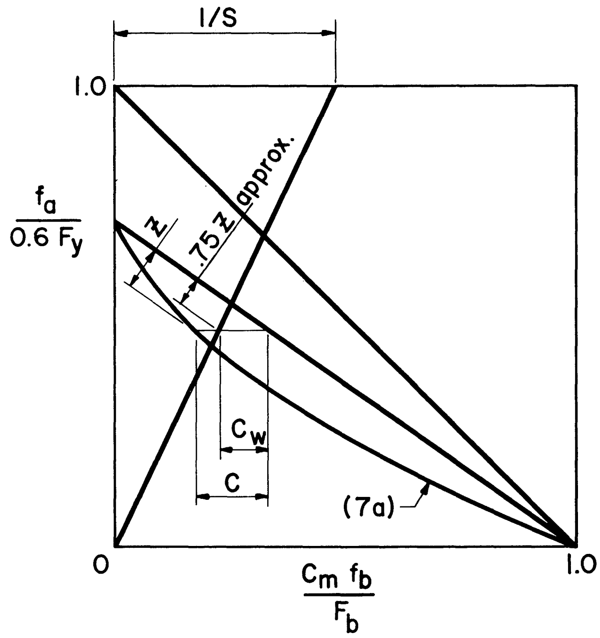


Figure 21

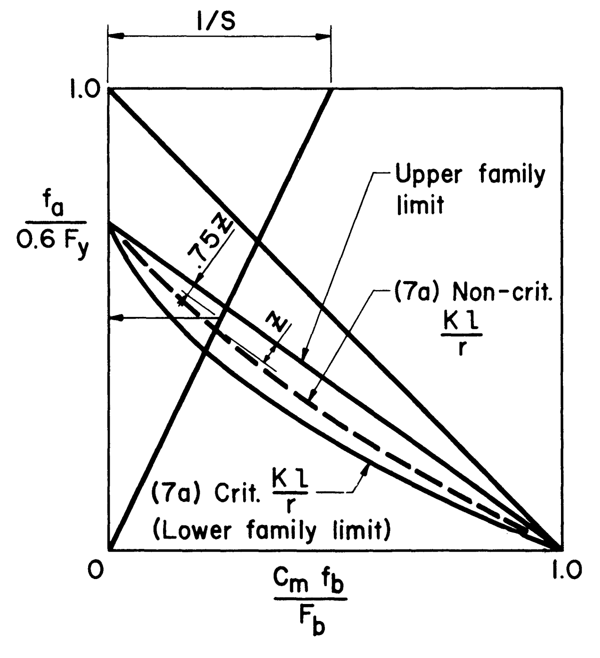


Figure 22

When f_a and f_b are reduced one-quarter in Formula (7a), the two terms are not reduced in the same proportion, since f_a appears in the denominator of the second term. The graphical significance of f_a/F_e' was explained for Fig. 14; the reduction of f_a by one-quarter will reduce the c-distance also by one-quarter.

Figure 21 illustrates the recommended procedure for chart solution of Formula (7a) with wind loads and bending about the axis critical for concentric load:

1. Reduce the axial load, P, and the bending moment, M, by 25 percent.
2. Plot the s-line.
3. Locate the curve for Formula (7a).
4. Find the point on the s-line where c_w equals $0.75e$ (or interpolate $0.75z$ as shown in Fig. 21 as a good approximation).
5. Calculate the column weight and check.

Figure 22 illustrates the same procedure for solving Formula (7a) when bending occurs about the non-critical axis.

Biaxial Bending—The general interaction chart can be of greatest help in estimating the size of a beam-column with bending in both directions.

To understand the use of the chart for Formula (7a), visualize a column with equal slenderness ratios about both axes, supporting an axial load and moments about both axes. The eccentricities can be computed and the s-lines through the origin can be plotted on the chart (Fig. 23). The bending in each direction must use only a portion of the strength available for single axis bending; the solution must lie below points **g** and **h**. The solution may be found by adding $1/s_x$ to $1/s_y$ to obtain the "slope-reciprocal", $1/s_x + 1/s_y$, of a line through the origin that intersects the curve for Formula (7a) at point **j**. The proof follows:

$$\frac{f_a}{F_a} + \frac{C_{mx} f_{bx}}{(1 - f_a/F_{ex}')F_{bx}} + \frac{C_{my} f_{by}}{(1 - f_a/F_{ey}')F_{by}} = 1$$

Formula (7a)

but

$$\frac{K_x l_x}{r_x} = \frac{K_y l_y}{r_y}$$

therefore,

$$F_{ex}' = F_{ey}' = F_e'$$

Factoring,

$$\frac{f_a}{F_a} + \left(\frac{C_{mx} f_{bx}}{F_{bx}} + \frac{C_{my} f_{by}}{F_{by}}\right) \frac{1}{1 - f_a/F_e'} = 1$$

From Fig. 23,

$$\frac{C_{mx}f_{bx}}{F_{bx}} = b_x \text{ and } \frac{C_{my}f_{by}}{F_{by}} = b_y$$

$$b = b_x + b_y$$

Substituting,

$$\frac{f_a}{F_a} + (b)\frac{1}{1 - f_a/F_e'} = 1$$

The last equation is Formula (7a) for single-axis bending, using terms explained in Fig. 14; it shows that biaxial bending of a column with the same slenderness in both directions can be reduced to an equivalent form of Formula (7a) for single-axis bending and that point **j** represents the solution.

The assumption of equal slenderness in two directions is not intended for practical use. Columns usually have different slenderness ratios for the two directions; the use of the higher ratio for both directions will yield too conservative an estimate.

The recommended method is to add the two "slope-reciprocals" to find the line through the origin as before, but to interpolate as shown in Fig. 24. It has been found that straight-line interpolation yields good estimates of the size required by Formula (7a).

Formula (7b) for biaxial bending is independent of slenderness ratios. In Fig. 25, the *s*-lines have been shown for both axes and for the sum. The solution lies between

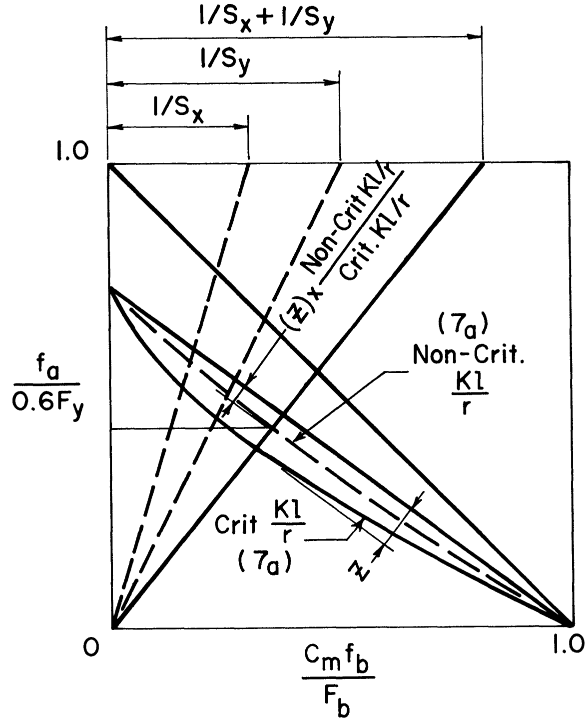


Figure 24

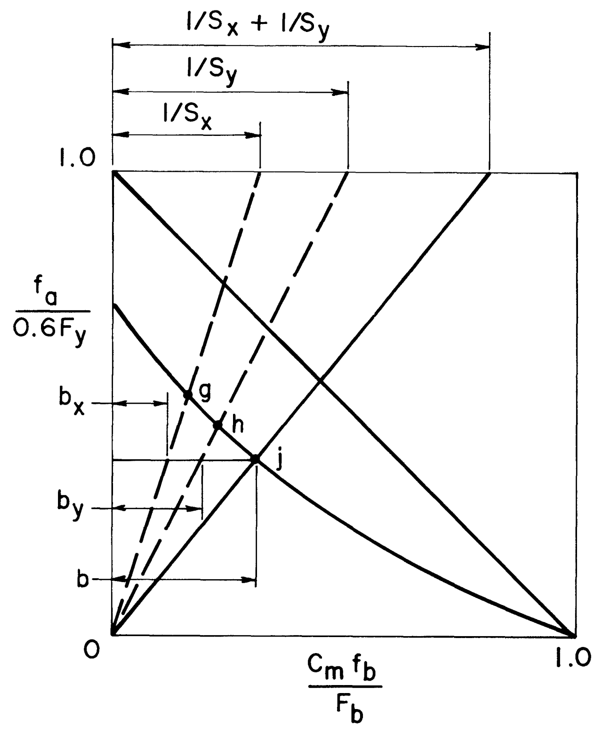


Figure 23

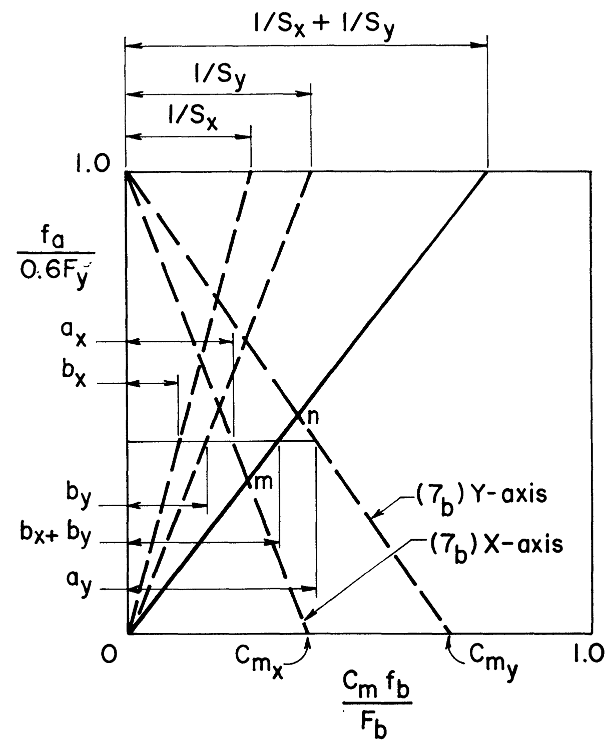


Figure 25

points **m** and **n**; point **m** would give too conservative a solution; point **n** would be unconservative. When **m** and **n** are close together, interpolation can be done by inspection. When **m** and **n** are not close together, the following procedure should be used when $C_{my} > C_{mx}$ (Fig. 26):

1. Find the intersection point **p** of the s_x-line with the graph of Formula (7b) for the X-axis.
2. Find the horizontal distance q' between **p** and the graph of Formula (7b) for the Y-axis.
3. Lay off distance q' to the right of point **u**; draw line **vo**. Intersection of line **vo** with the graph of Formula (7b) for the Y-axis indicates the solution at point **t**.

When $C_{mx} > C_{my}$ the same procedure may be used for interpolation except the X and Y terms are interchanged.

The proof of the construction follows:

$$\frac{f_a}{0.6F_y} + \frac{f_{bx}}{F_{bx}} + \frac{f_{by}}{F_{by}} = 1 \qquad \text{Formula (7b)}$$

By transposing and dividing,

$$\frac{f_{bx}}{(1 - f_a/0.6F_y)F_{bx}} + \frac{f_{by}}{(1 - f_a/0.6F_y)F_{by}} = 1$$

In Fig. 25,

$$b_x = \frac{C_{mx}f_{bx}}{F_{bx}} \quad \text{and} \quad b_y = \frac{C_{my}f_{by}}{F_{by}}$$

$$a_x = C_{mx}\left(1 - \frac{f_a}{0.6F_y}\right) \quad \text{and} \quad a_y = C_{my}\left(1 - \frac{f_a}{0.6F_y}\right)$$

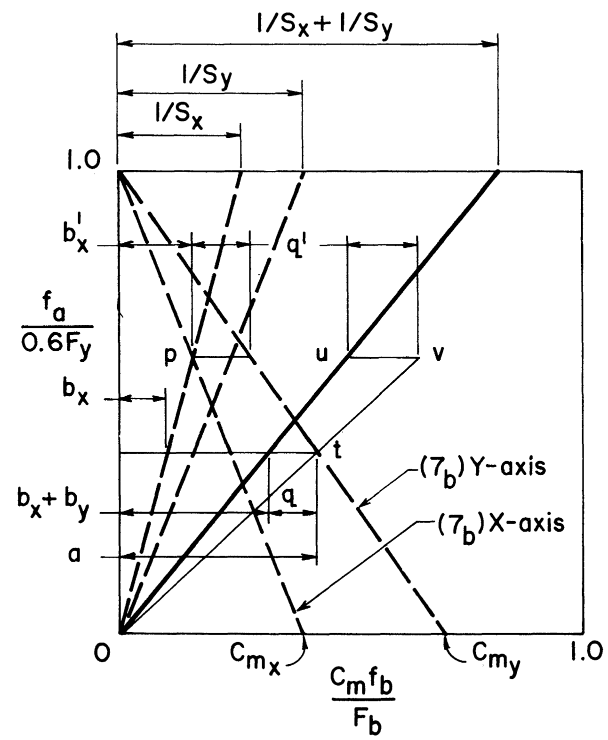


Figure 26

Then,

$$\frac{b_x}{a_x} = \frac{f_{bx}}{(1 - f_a/0.6F_y)F_{bx}}$$

and

$$\frac{b_y}{a_y} = \frac{f_{by}}{(1 - f_a/0.6F_y)F_{by}}$$

Therefore,

$$\frac{b_x}{a_x} + \frac{b_y}{a_y} = 1$$

It is now necessary to find a geometrical construction to locate the horizontal line at the appropriate level of $f_a/0.6F_y$ to satisfy the equation above:

$$a_x = \frac{C_{mx}}{C_{my}}(a_y)$$

Substituting in the equation above,

$$\frac{(C_{my}/C_{mx})(b_x)}{a_y} + \frac{b_y}{a_y} = 1$$

$$a_y = \frac{C_{my}}{C_{mx}}(b_x) + b_y = \left(\frac{C_{my}}{C_{mx}} - 1\right)(b_x) + b_x + b_y$$

At point **p** in Fig. 26, a convenient relationship exists:

$$q' = b_x'\left(\frac{C_{my}}{C_{mx}}\right) - b_x' = \left(\frac{C_{my}}{C_{mx}} - 1\right)b_x'$$

If distance q' is drawn to the right of point **u** and line **vo** is drawn, then at point **t**, by proportionality,

$$q = \left(\frac{C_{my}}{C_{mx}} - 1\right)b_x$$

$$a = q + (b_x + b_y)$$

$$a = \left(\frac{C_{my}}{C_{mx}} - 1\right)b_x + b_x + b_y = a_y$$

which indicates the solution.

In using the chart for biaxial bending, inspection of the intersection points of the $(1/s_x + 1/s_y)$-line with the graphs for Formulas (7a) and (7b) will usually indicate which formula governs the design.

DESIGN EXAMPLES

Six design examples are provided in the Appendix to this paper. These are arranged in three groups:

Examples 1 and 2—Bending about the axis critical for concentric load

Examples 3 and 4—Bending about the axis not critical for concentric load

Examples 5 and 6—Biaxial bending

ACKNOWLEDGMENTS

This paper contains material originally presented by the author to the Steel Structures Symposium at the University of Illinois, Urbana Campus, October 25–26, 1966. The symposium was sponsored by the Mississippi Valley Structural Steel Company, the American Institute of Steel Construction and the University of Illinois.

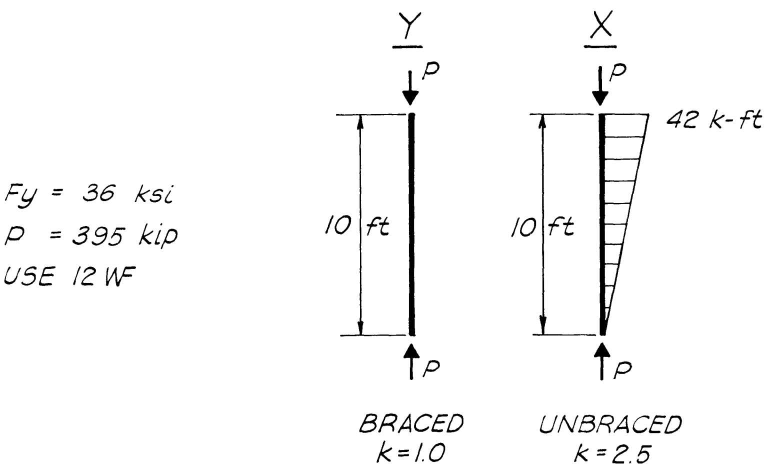
The work leading to this paper was sponsored by Bethlehem Steel Corporation. The author wishes to thank Dr. William C. Hansell and Mr. William A. Milek for their review and helpful suggestions.

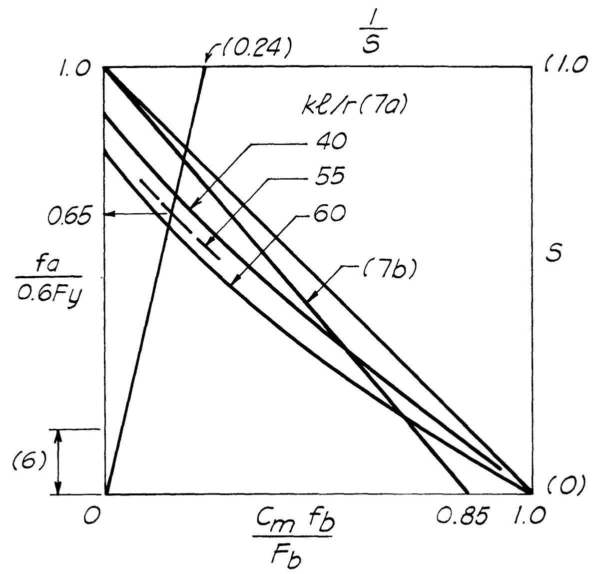
REFERENCES

1. Specification for the Design, Fabrication and Erection of Structural Steel for Buildings, *American Institute of Steel Construction, New York, N. Y., 1963.*
2. Commentary on the Specification for the Design, Fabrication and Erection of Structural Steel for Buildings, *American Institute of Steel Construction, New York, N. Y., 1963.*
3. Guide to Design Criteria for Metal Compression Members, *Column Research Council of the Engineering Foundation, New York, N. Y.*
4. Beam-Column Tables for Structural Shapes, *Bethlehem Steel Corporation, Bethlehem, Pa.*
5. *Tall, Lambert, Editor-in-Chief* Structural Steel Design, *Ronald Press, New York, N. Y., 1964, p. 720.*
6. *De Falco, Fred and Marino, Frank J.* Column Stability in Type 2 Construction, *AISC Engineering Journal, Vol. 3, No. 2, Apr., 1966.*
7. Manual of Steel Construction, *American Institute of Steel Construction, New York, N. Y., 1963.*

APPENDIX

Example 1—Bottom story of multi-story building.





Given:

Item	**Unit**	***Y***	***X***	**Remarks**
F_y	ksi	36		—
P	kips	395		—
Size	—	12WF		—
Sway	—	Br.	Unbr.	—
L	ft	10	10	—
K	—	1.0	2.5	(Fig. 10, Case F for X-axis)
M_{top}	k-ft	0	42	—
M_{bot}	k-ft	0	0	—

Solution:

r_{av}	in.	3.0	5.5	Fig. 18
Kl/r	—	(40)	55	X-axis governs
F_b	ksi	36		Assumed
C_m	—	—	0.85	Unbraced
e	in.	—	1.28	$M_{top} \times 12/P = 42 \times 12/395$
B_{av}	in.$^{-1}$	—	0.22	Fig. 18
$1/s$	—	—	0.24	$C_m e B_{av}(0.6\ F_y/F_b) = 0.85 \times 1.28 \times 0.22 \times (22/22)$
$f_a/0.6F_y$	—	—	0.65	Fig. 18, chart intersection (7a)
Col. wt.	lbs/ft	—	95.5	$0.157P/(f_a/0.6F_y) = 0.157 \times 395/0.65$

Try 12W99: Chart (Fig. 18) shows Formula (7a) governs.

Item	Actual	Assumed	Remarks
r_x	5.43	5.5	AISC Manual p. 1-15
B_x	0.216	0.22	AISC Manual p. 3-20
F_b	24	22	Compact, $L < L_c$; AISC Manual p. 3-20
Kl/r_x	55	55	—

$1/s = C_m e B(0.6F_y/F_b) = 0.85 \times 1.28 \times 0.216 \times (22/24) = 0.22$ (was 0.24)
$f_a/0.6F_y$ from Fig. 18 $= 0.67$ (was 0.65)

This is a small, conservative change; therefore, continue check for 12W99.

Check 12W99 using AISC Manual, Formula (7a):

Item	Unit	Value	Remarks
A	in.^{2}	29.09	—
r_x	in.	5.43	—
S_x	in.^{3}	134.7	—
Kl/r_x	—	55	—
F_a	ksi	17.9	Table 1-36, p. 5-68
F_e'	ksi	49.3	Table 2-36, p. 5-69
F_b	ksi	24	—
f_a	ksi	13.55	$P/A = 395/29.09$
f_b	ksi	3.74	$M_{top}/S = (42 \times 12)/134.7$

$$\frac{f_a}{F_a} + \frac{C_m f_b}{\left(1 - \frac{f_a}{F_e'}\right)F_b} \leq 1.0 \quad \text{Formula (7a)}$$

$$\frac{13.55}{17.9} + \frac{0.85 \times 3.74}{(1 - 13.55/49.3) \times 24} = 0.76 + 0.18 = 0.94 < 1.0 \quad \textbf{OK}$$

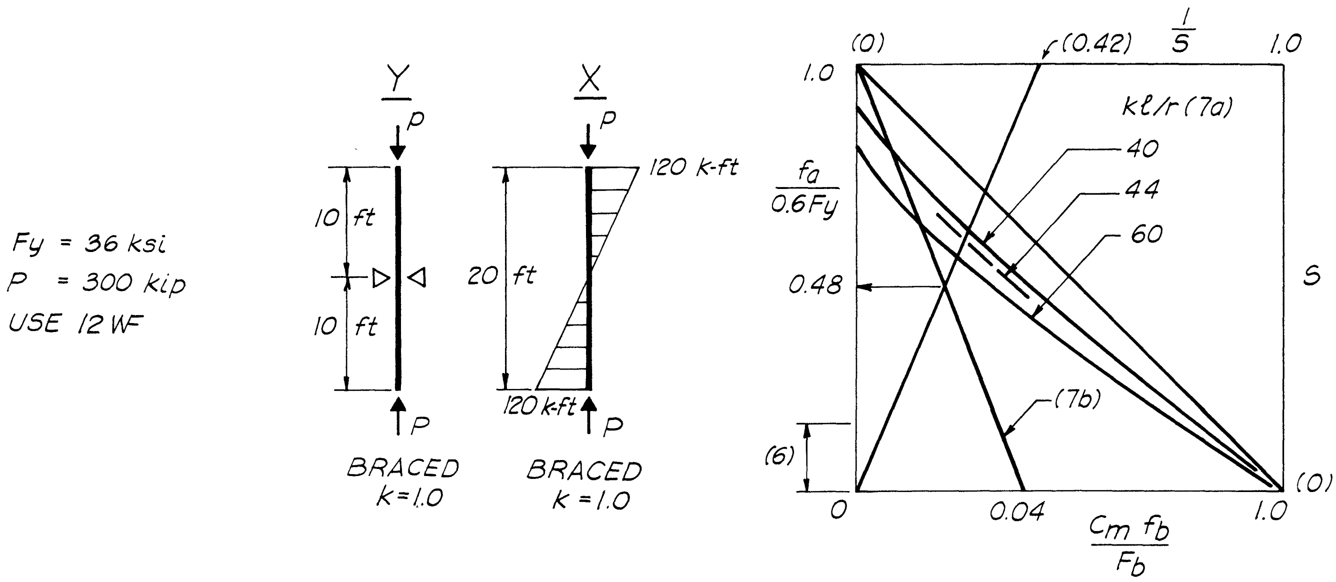
Check 12W99 using Bethlehem Beam-Column Tables, Formula (7a):

r_x/r_y	—	1.76	p. 106 ($1.76 < K = 2.50$)
Eff. K_yL_y	ft	14.2	$2.5 \times 10/1.76$
P_a	kips	520	p. 106, for $K_yL_y = 14.2$
P_e'	kips	1,420	p. 106, for $K_xL_x = 25$
M_a	k-ft	267	p. 107, for $L = 10$

$$\frac{P}{P_a} + \frac{C_m M}{(1 - P/P_e')M_a} \leq 1.0 \quad \text{(Modified Formula (7a), see p. 5)}$$

$$\frac{395}{520} + \frac{0.85 \times 42}{(1 - 395/1{,}420) \times 267} = 0.76 + 0.185 = 0.945 < 1.0 \quad \textbf{OK}$$

Example 2—Multi-story building; weak axis braced at mid-height.



Given:

Item	**Unit**	***Y***	***X***	**Remarks**
F_y	ksi	36		—
P	kips	300		—
Size	—	12WF		—
Sway	—	Br.	Br.	—
L	ft	10	20	—
K	—	1.0	1.0	—
M_{top}	k-ft	0	+120	Clockwise
M_{bot}	k-ft	0	+120	Clockwise

Solution:

r_{av}	in.	3.0	5.5	Fig. 18
Kl/r	—	(40)	44	X-axis governs
C_m	—	—	0.40	0.6 − 0.4 (120/120) = 0.2 (use 0.4)
e	in.	—	4.8	120 × 12/300
B_{av}	in.$^{-1}$	—	0.22	Fig. 18
$1/s$	—	—	0.42	0.40 × 4.8 × 0.22
$f_a/0.6F_y$	—	—	0.48	Fig. 18, chart intersection (7b)
Col. wt.	lbs/ft	—	98	0.157 × 300/0.48

Try 12WF99:

Chart (Fig. 18) shows Formula (7b) governs.

Item	Actual	Assumed	Remarks
B_x	0.216	0.22	AISC Manual p. 3-20
F_b	22	22	$L_c < 20 < L_u$; AISC Manual p. 3-20
r_x	—	—	Not required for (7b)

Close agreement between actual and assumed. Continue check of 12WF99.

Check 12WF99 using AISC Manual, Formula (7b):

Item	Unit	Value	Remarks
A	in.^{2}	29.09	—
S_x	in.^{3}	134.7	—
f_a	ksi	10.3	$P/A = 300/29.09$
f_b	ksi	10.7	$M/S = (120 \times 12)/134.7$
F_b	ksi	22	—

$$\frac{f_a}{0.6F_y} + \frac{f_b}{F_b} \leq 1.0 \quad \text{Formula (7b)}$$

$$\frac{10.3}{22} + \frac{10.7}{22} = 0.47 + 0.49 = 0.96 < 1.0 \quad \textbf{OK}$$

Check 12WF99 using Bethlehem Beam-Column Tables, Formula (7b):

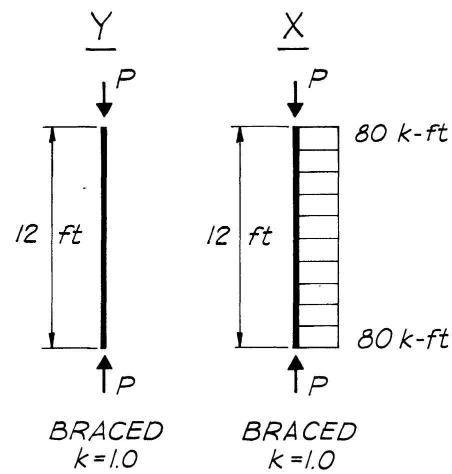
P_0	kips	628	p. 106, for $K_yL_y = 0$
M_a	k-ft	242	p. 107, for $K_yL_y = 20$

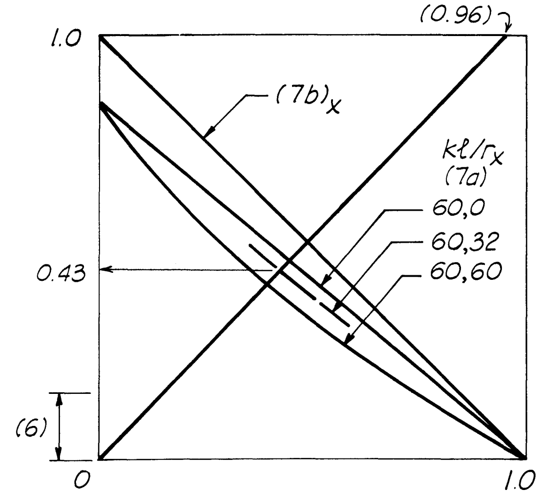
$$\frac{P}{P_0} + \frac{M}{M_a} \leq 1.0 \quad \text{(Modified Formula (7b), see p. 5)}$$

$$\frac{300}{62.8} + \frac{120}{242} = 0.48 + 0.49 = 0.97 < 1.0 \quad \textbf{OK}$$

Example 3—Multi-story building; column offset in *X*-direction from stories above and below.

Fy = 36 ksi
P = 265 kip
USE 10 WF





Given:

Item	**Unit**	**Y**	**X**	**Remarks**
F_y	ksi	36		—
p	kip	265		—
Size	—	10WF		—
Sway	—	Br.	Br.	—
L	ft	12	12	—
K	—	1.0	1.0	—
M_{top}	k-ft	0	80	Clockwise
M_{bot}	k-ft	0	80	Counterclockwise

Solution:

r_{av}	in.	2.5	4.5	Fig. 18
Kl/r	—	58	32	*Y*-axis governs for axial
C_m	—	—	1.0	$0.6 + 0.4\ (80/80)$
e	in.	—	3.63	$M/P = 80 \times 12/265$
B_{av}	in.$^{-1}$	—	0.265	Fig. 18
$1/s$	—	—	0.96	$1.0 \times 3.63 \times 0.265 \times (22/22)$
$f_a/0.6F_y$	—	—	0.43	Fig. 18, chart intersection (7a)
Col. wt	lbs/ft	—	97	$0.157 \times 265/0.43$

Try 10WF100:
Chart (Fig. 18) shows Formula (7a) governs.

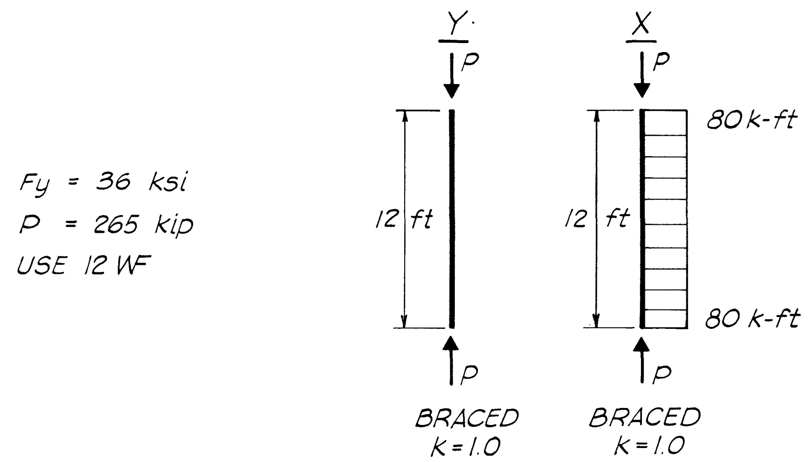
Item	**Actual**	**Assumed**	**Remarks**
r_x	4.61	4.5	—
r_y	2.65	2.5	—
B_x	0.262	0.265	AISC Manual, p. 3-21
F_b	22	22	$L_c < L < L_u$; AISC Manual p. 3-21

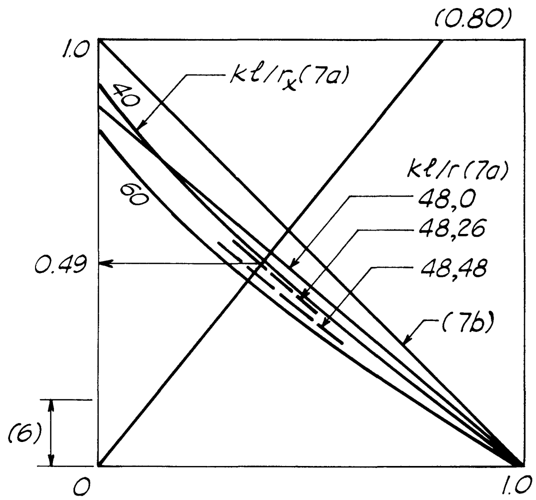
Close agreement. Continue check of 10WF100.

Check 10WF100 by Bethlehem Beam-Column Tables, Formula (7a):

$$\frac{P}{P_a} + \frac{C_m M}{(1 - P/P_e')M_a} = \frac{265}{529} + \frac{1.0 \times 80}{(1 - 265/4{,}500)202} = 0.50 + 0.42 = 0.92 < 1.0 \quad \textbf{OK}$$

Example 4—Same as Example 3, except use 12WF and note any weight saved.





Solution:

Item	**Unit**	**Y**	**X**	**Remarks**
r_{av}	in.	3.0	5.5	—
Kl/r	—	48	26	—
C_m	—	—	1.0	See Example 3
e	in.	—	3.63	See Example 3
B_{av}	in.$^{-1}$	—	0.22	—
$1/s$	—	—	0.80	$1.0 \times 3.63 \times 0.22$
$f_a/0.6F_y$	—	—	0.49	Fig. 18, chart intersection (7a)
Col. wt.	lbs/ft	—	85	$0.157 \times 265/0.49$

Try 12WF85, Formula (7a):

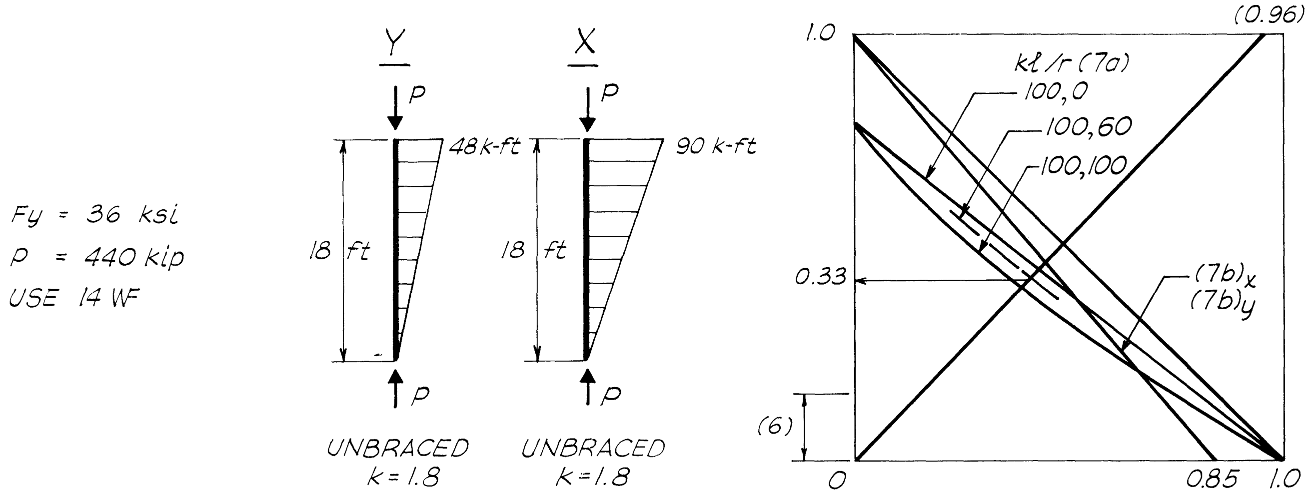
Item	**Actual**	**Assumed**	**Remarks**
r_y	3.07	3.0	—
r_x	5.38	5.5	—
B_x	0.216	0.22	—
F_b	24	22	Compact, $L < L_c$; AISC Manual p. 3-20

Check 12WF85 by Bethlehem Beam-Column Tables, Formula (7a):

$$\frac{P}{P_a} + \frac{C_m M}{(1 - P/P_e')M_a} = \frac{265}{465} + \frac{1.0 \times 80}{(1 - 265/5{,}200)229} = 0.57 + 0.37 = 0.94 < 1.0 \quad \textbf{OK}$$

Next smaller column (12WF79) is too small: (79 lbs/ft/85 lbs/ft) < 0.94
Note that the 12WF85 carries the same load and moment as the 10WF100 of Example 3.

Example 5—Single story building.



Given:

Item	**Unit**	***Y***	***X***	**Remarks**
F_y	ksi	36		—
P	kips	440		—
Size	—	14WF		—
Sway	—	Unbr.	Unbr.	—
L	ft	19	18	—
K	—	1.8	1.8	(Fig. 10, Case F)
M_{top}	k-ft	48	90	—
M_{bot}	k-ft	0	0	—

Solution:

r_{av}	in.	3.7	6.5	—
Kl/r	—	105	60	—
C_m	—	0.85	0.85	Unbraced
e	in.	1.30	2.45	M/P
B_{av}	in.$^{-1}$	0.52	0.185	—
$1/s$	—	0.575	0.385	$C_m e B$
$\Sigma 1/s$	—	0.96		0.575 + 0.385
$f_a/0.6F_y$	—	0.33		Fig. 18, chart intersection (7a)
Col. wt.	lbs/ft	210		0.157 × 440/0.33

Try 14WF158. Assumptions for r, B and F_b fairly close.

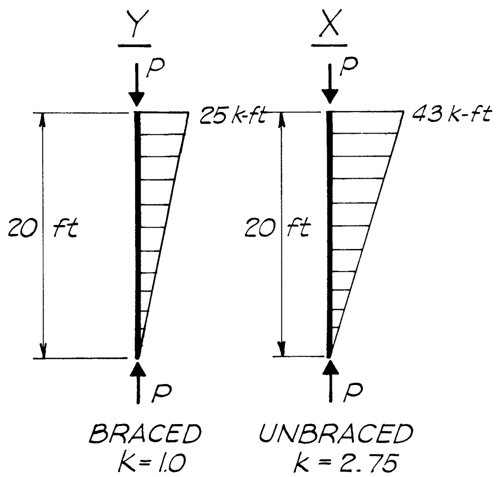
Check by Bethlehem Beam-Column Tables, Formula (7a):

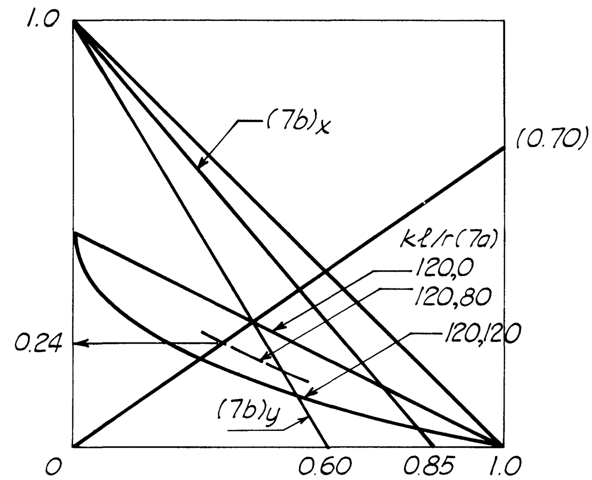
$$\frac{P}{P_a} + \frac{C_{mx}M_x}{(1 - P/P_{ex}')M_{ax}} + \frac{C_{my}M_y}{(1 - P/P_{ey}')M_{ay}} \leq 1.0 \text{ (Modified Formula (7a))}$$

$$\frac{440}{838} + \frac{0.85 \times 90}{(1 - 440/2{,}620)611} + \frac{0.85 \times 48}{(1 - 440/1{,}020)258} = 0.53 + 0.15 + 0.28 = 0.96 < 1.0 \quad \textbf{OK}$$

Example 6—Bottom story of multi-story building.

Fy = 36 ksi
P = 147 kip
USE 12 WF





Given:

Item	**Unit**	***Y***	***X***	**Remarks**
F_y	ksi	36		—
P	kips	147		—
Size	—	12WF		—
Sway	—	Br.	Unbr.	—
L	ft	20	20	—
K	—	1.0	2.75	(Fig. 10 shows 2.5 for Case F, X-axis)
M_{top}	k-ft	25	43	Both clockwise
M_{bot}	k-ft	25	0	Clockwise

Solution:

r_{av}	in.	3.0	5.5	—
Kl/r	—	80	120	—
C_m	—	0.6	0.85	—
e	in.	2.05	3.5	M/P
B_{av}	in.$^{-1}$	0.64	0.22	—
$1/s$	—	0.79	0.65	$C_m e B_{av}$
$\Sigma 1/s$	—	1.44		0.79 + 0.65
s	—	0.70		1/1.44
$f_a/0.6F_y$	—	0.24		Fig. 18, chart intersection (7a)
Col. wt.	lbs/ft	96		0.157 × 147/0.24

Try 12WF99. Assumptions for r_x, r_y, B, and F_b are close.

Check by Bethlehem Beam-Column Tables, Formula (7a):

$$\frac{P}{P_a} + \frac{C_{mx}M_x}{(1 - P/P_{ex}')M_{ax}} + \frac{C_{my}M_y}{(1 - P/P_{ey}')M_{ay}} \leq 1.0 \quad \text{(Modified Formula (7a))}$$

$$\frac{147}{298} + \frac{0.85 \times 43}{(1 - 147/302)242} + \frac{0.6 \times 25}{(1 - 147/730)90} = 0.49 + 0.30 + 0.21 = 1.00 \quad \textbf{OK}$$