Historical Note on K-Factor Equations

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It is the intent of this informal note to provide some historical background on the formulae giving effective-length factors, or K-factors, in the simple cases where such formulae are applicable.

It has been known for a long time that, within a frame, a column may have an effective length larger or smaller than its actual length, that is, a K-factor smaller or greater than one. This is a subject of much concern to structural engineers, and as such, it has received a lot of attention. However, it is difficult enough not to have been introduced to the AISC Specification until the sixth edition of the Manual (1963). Earlier specifications, as late as the revision of 1949 included in the fifth edition (1955), did not mention the "KL/r" ratio of a column, but instead its "l/r" (which was not to exceed 120 for primary members). Moment amplification in beam-columns was also not mentioned at that time. Forty years later, Chapter C, "Frames and Other Structures," covers two and a half pages of the LRFD Specification and thirteen pages are devoted to its commentary (AISC, 1994).

1. THE "EXACT" FORMULAE FOR CONTINUOUS FRAMES

Building frames fall into two general classes, "braced frames" in which sidesway is prevented by braces or lateral supports, and "unbraced frames" in which it is resisted by the frame's own bending stiffness.

Two "exact" equations may be derived with the help of not less than nine rather restrictive assumptions. These assumptions are listed in the AISC LRFD Commentary (1994), and are discussed by McGuire (1968), who also gives the derivations of these equations. There is no indication in the literature about the authorship of either of these formulae, which are:

$$\frac{G_A G_B}{4}\left(\frac{\pi}{K}\right)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\dfrac{\pi}{K}}{\tan\dfrac{\pi}{K}}\right) + \frac{\tan\dfrac{\pi}{2K}}{\dfrac{\pi}{2K}} = 1 \tag{1}$$

for braced frames and trusses, and:

$$\frac{G_A G_B\left(\dfrac{\pi}{K}\right)^2 - 36}{6(G_A + G_B)} = \frac{\dfrac{\pi}{K}}{\tan\dfrac{\pi}{K}} \tag{2}$$

for unbraced frames.

The formulae make use of the restraint factors G_A and G_B at the two ends of the column section being considered. An end restraint factor G is defined as

$$G = \frac{\sum(I_c/L_c)}{\sum(I_b/L_b)} \tag{3}$$

Stiff beams, relative to the column, lead to small values of G, and, conversely, flexible beams give large G's.

Solving for K in either one of these equations with a calculator is no easy task. Imagine doing it with a slide rule. Fortunately, Julian and Lawrence developed the two well-known alignment charts found in the AISC Manuals for the past three decades. For a braced frame or a truss, the effective length factor is never larger than unity and using $K = 1$ is always on the safe side.

The problem is much more difficult to solve for an unbraced frame. The Commentary to Chapter C provides guidance in that respect, and the reader is strongly urged to refer to it. However, the alignment chart remains a useful tool within its range of validity.

2. THE FRENCH "CM 66" FORMULAE

Today's structural engineer uses a personal computer to run a variety of programs, and especially spreadsheets. With the spreadsheet application in mind, two simple formulae were presented (Dumonteil, 1992) to serve as the equivalent of the alignment charts in digital calculations. These two equations were originally listed in the French steel code or "CM 66 Rules" approved in December 1966. In the French text, they are given in two different forms: one using factors akin to the G-factors, and another one, also described in the 1992 paper, using flexural springs. The European Recommendations of 1978, the predecessor to Eurocode 3, list essentially the same formulae. Neither the CM 66 Rules nor the European Recommendations give the origin of these two formulae.

For braced frames or trusses, with sidesway effectively prevented, the effective length factor K is given by the

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rather simple approximation:

$$K = \frac{3G_AG_B + 1.4(G_A + G_B) + 0.64}{3G_AG_B + 2(G_A + G_B) + 1.28} \tag{4}$$

This formula underestimates K by not more than 0.5% and overestimates it by less than 1.5%.

For unbraced frames, K is approximated within 2% by the expression:

$$K = \sqrt{\frac{1.6G_AG_B + 4.0(G_A + G_B) + 7.5}{G_A + G_B + 7.5}} \tag{5}$$

3. DONNELL'S APPROXIMATION FOR BRACED FRAMES

While the 1992 paper presenting Formulae (4) and (5) stated that their origin was unknown, it mentioned that Donnell was thought to have published an approximation for braced frames. Although we were not able to obtain a copy of the original article by L. H. Donnell, Rondal (1988) states that it is found in a "Reissner Anniversary Volume," however without giving a date. We found this Anniversary Volume mentioned in unrelated books, and its date of publication seems to be 1949.

After some manipulation of Donnell's formula as reported by Rondal to use the AISC parameters G and K, it takes the following form:

$$K = \sqrt{\frac{G_AG_B + 0.43(G_A + G_B) + 0.17}{G_AG_B + 0.86(G_A + G_B) + 0.68}} \tag{6}$$

Such is the formula presented by Dumonteil and Valley in their discussion (1995) of a paper dealing with effective lengths. Its accuracy ranges from -0.4 to 1.3%.

4. NEWMARK'S WORK ON BRACED FRAMES

Following our mention of Donnell's work, Prof. Jostein Hellesland sent the writer a copy of an article published by N.M. Newmark in 1949. In this paper, Newmark describes work done in 1944 under the sponsorship of the Consolidated Vultee Aircraft Corporation.

Manipulating Newmark's notations to introduce the G-factors, Prof. Hellesland writes Newmark's formula in the following form:

$$K = \sqrt{\frac{(G_A + 4/\pi^2)(G_B + 4/\pi^2)}{(G_A + 8/\pi^2)(G_B + 8/\pi^2)}} \tag{7}$$

While the accuracy of Equation (7) is remarkable, it could still be slightly improved if the term $4/\pi^2 (= 0.405)$ is replaced with 0.41, and Newmark's formula becomes:

$$K = \sqrt{\frac{(G_A + 0.41)(G_B + 0.41)}{(G_A + 0.82)(G_B + 0.82)}} \tag{8}$$

Equation (8) underestimates K by not more than 0.1% and overestimates it by less than 1.5%.

From a physical standpoint, all three Equations (4), (6), and (8) applicable to braced frames and trusses are equally acceptable. Mathematically speaking, the improved Newmark formula is slightly more accurate.

There is no evidence in the literature that either Donnell or Newmark would have published similar solutions for the unbraced frame problem, for which the CM 66 formula is a good approximation.

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