Formulas for Beams with Semi-rigid Connections

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ABSTRACT

This paper presents practical formulas for beams with semi-rigid connections of variable stiffness, and describes certain limitations for application of these formulas. Using the tables of formulas, the engineer is able to rapidly determine the effect of a partially restrained connection. A practical example illustrates the application of the formulas.

INTRODUCTION

Legend:

C = Rotational stiffness of beam-to-column connection (Spring Constant), kips/in.
M = Bending moment applied at beam-to-column connection, kips/in.
φ = Change of angle (distortion) between beam and column at the connection
l = Length of beam from connection to connection, in.
E = Modulus of elasticity for structural steel, 29,000 ksi.
I = Moment of inertia of beam, in.^{4}
μ = Parameter of semi-rigid frame, 1/in.^{2}
i = Unit stiffness of beam, kip/in.

The *Manual of Steel Construction* classifies types of construction as type FR (fully restrained) and type PR (partially restrained).[1] Type FR assumes that connections have sufficient rigidity to maintain the angles between intersecting members, while type PR assumes that connections have insufficient rigidity to maintain the angles between intersecting members. The Manual also provides formulas for one particular case: uniformly loaded beam with both ends partially restrained.

The majority of research work for semi-rigid connections concentrates on the matrix form presentation.[2] This paper is based on the previous research work by the author and V. Soloviev-Holmogorov and provides simplified formulas for broader cases of beams with various loading conditions and support restraints. The calculation example demonstrates how these formulas can be used in the practical analysis of beams, and shows that even for the commonly assumed fully restrained connection, the reduction in the moments at the supports could be worth noting. On the other hand, underestimating the stiffness can lead to overload of fasteners or welds under combined gravity and lateral loads if the connections do not have sufficient rotation capacity.

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Figure 1 depicts simplified characteristics of basic types of frames. In semi-rigid frames the angle between the beam and the column distorts (changes from the initiai preloaded condition). This distortion depends on the stiffness of the beam/column connection. Assuming elastic behavior, the rotational stiffness of the connection is equal to the bending moment required to produce a distortion of a 1 radian angle from the initial position:

$$C = \frac{M}{\varphi}$$

The value of C can be represented as a constant rotational spring restraint at the support. Tables 1 and 2 show various beam diagrams with different types of loadings and support restraints. The rotational stiffness of the spring is equal to C. In cases where both beam ends have rotational stiffness, the values of stiffness C in Tables 1 and 2 are assumed to be identical for both beam ends. Furthermore, the formulas are applicable only when the connections at both beam ends are absolutely elastic and identical for both directions of moments (loading and unloading). Because of the nonlinear momenlt-rotation behavior of most connections, similar connections (even with identical geometries) may have different

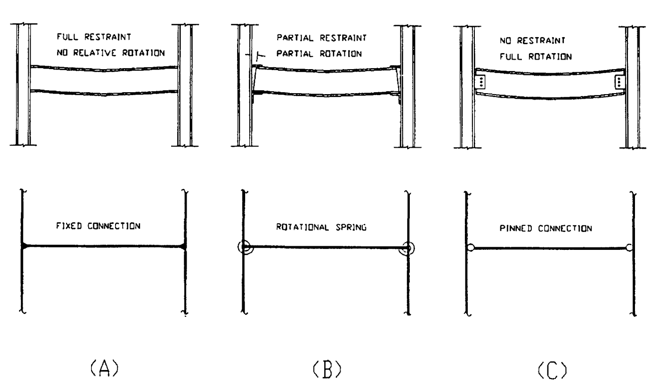


Fig. 1. (a) FR moment connection behavior & static model, (b) PR moment connection behavior & static model, (c) no restraint or simple connection & static model.

stiffness as a result of their loading and unloading characteristics. The stiffness of a connection undergoing loading will be different from that of a connection undergoing unloading. Connection loading and unloading occur when the frame is subject to sidesway movement. The formulas in Tables 1 and 2 were derived from common static principals.

DESIGN FORMULAS

Table 1 depicts formulas for beam moment and shear at the supports resulting from a unit rotation of the support $\varphi = 1$ or a unit vertical displacement of the support $\Delta = 1$.

In these formulas

$$i = \frac{EI}{l};\ \mu = \frac{i}{C}$$

For example, for semi-rigid connections at both beam supports, the moments and the reactions due to the unit rotation of $\varphi_A = 1$ at the support A, yield the following values:

$$M_A = 4i\frac{3\mu + 1}{12\mu^2 + 8\mu + 1}$$

$$M_B = 2i\frac{1}{12\mu^2 + 8\mu + 1}$$

$$R_A = \frac{6i}{l}\frac{1}{6\mu + 1}$$

$$R_B = -R_A$$

For the same beam diagram and for absolutely rigid end connections ($C = \infty$ and $\mu = 0$) the above formulas transform into the well known formulas for beams with fixed ends:

$$M_A = 4i$$

$$M_B = 2i$$

$$R_A = \frac{6i}{l}$$

Table 2 depicts formulas for beam moment at the supports for various loading conditions. For example, for a uniformly distributed load on the beam with both supports partially restrained (semi-rigid connection) the bending moment at the support is equal to:

$$M_A = M_B = \frac{ql^2}{12}\frac{1}{2\mu + 1}$$

For absolutely rigid connections ($C = \infty$ and $\mu = 0$, full fixity at the support) the value of moments becomes the well known fixed beam formula

$$M_A = M_B = \frac{ql^2}{12}$$

On the other hand, for a very "weak" spring or unrestrained support condition ($C = 0$ and $\mu = \infty$, pinned connection) the end moments become $M_A = M_B = 0$.

EXAMPLE

Consider a bolted flange-plated moment connection for a W18×50 beam to W14×99 column-flange connection (Figure 2). Assume that the beam is located in the middle of a multiple span frame. Each span is 20 ft. A uniform vertical load is applied to all spans, so that beam-column joints do not rotate at this middle span. Determine the effect of the beam-to-column connection on the value of the bending moments.

Solution

1. Determine the stiffness of the beam-to-column connection. The angle of rotation

$$\varphi = \frac{\Delta_f}{h/2}$$

where

h = distance between plates, 18 inches
Δ_f = axial deformation of the flange under the moment M.

The force in the beam flange can be expressed as

$$F_f = \frac{\Delta_f E A_p}{L_p}$$

where

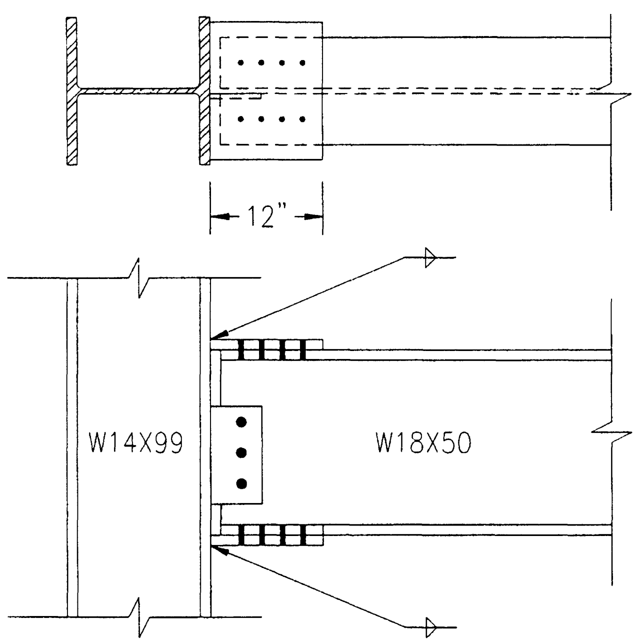


Fig. 2. Column flange support, bolted flange plates.

A_p = gross area of one flange plate, 4.275 in.^{2}
L_p = effective length of the bolted plate that can be considered fully loaded, 6 in.

Dr. Astanesh-Asl suggested that the length be equal to ½ of the total length of the flange plate.[4]

Bending moment at the support:

$$M = F_f h$$

Stiffness of beam-to-column connection:

$$C = \frac{M}{\varphi}$$

$$= \frac{F_f h^2}{2\Delta_f}$$

$$= \frac{h^2}{2}\frac{EA_p}{L_p}$$

$$= \frac{18^2}{2}\frac{29{,}000(4.275)}{6}$$

$$= 3{,}347{,}325 \text{ kips-in.}$$

2. Determine parameter of semi-rigid frame.

$$I = 800 \text{ in.}^4$$

$$l = 20 \times 12 \text{ in.} = 240 \text{ in.}$$

$$i = EI / l = 29{,}000 \times 800 / 240 = 96{,}667 \text{ kip/in.}$$

$$\mu = i / C = 96{,}667 / 3{,}347{,}325 = 0.0288 \text{ [1/in.}^2\text{]}$$

3. Determine bending moments at the support. From Table 2.

$$M_A = M_B = \frac{ql^2}{12}\frac{1}{2\mu + 1} = 0.945\left(\frac{ql^2}{12}\right)$$

CONCLUSION

With certain limitations presented formulas are useful for the rapid evaluation of the effect of partial fixity at the beam-to-column connections. The above example demonstrates that there could be a reduction in the moment at the support even for connections commonly assumed to be absolutely rigid. In the presented example, the reduction of the moment could be in the order of 5.5 percent compared with the conventional rigid frame analysis. It should be noted, however, that the shear slippage of the bolts will further reduce both the stiffness of the connection and the moment at the support. Further studies are needed for the evaluation of the stiffness characteristics for different types of beam-to-column connections.

REFERENCES

1. American Institute of Steel Construction, *Manual of Steel Construction, Load & Resistance Factor Design*, 2nd Ed., Chicago, IL 1993.
2. Bhatti, M. Asghar and Hingtgen, James D., "Effects of Connection Stiffness and Plasticity on the Service Load Behavior of Unbraced Steel Frames," AISC, *Engineering Journal*, Vol. 32, No. 1, 1995.
3. Kotlyar, N. and Soloviev-Holmogorov V., "Deformations of framework connections," *Trudi MNIITP*, Moscow, Russia, 1970, pp. 171–192.
4. Astanesh-Asl, A. "Seismic Design of Bolted Steel Moment-Resisting Frames," *Technical Report*, University of California, Berkeley, 1995.

Table 1.
Formulas for Moment and Shear from Unit Support Displacement

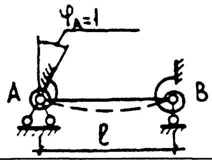
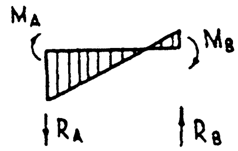
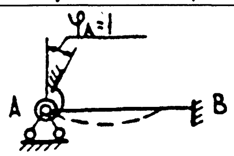
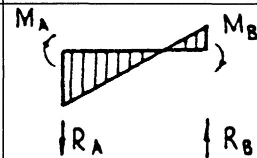
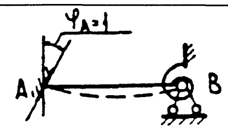
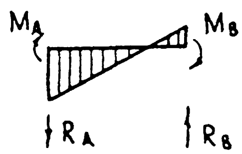
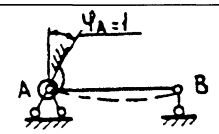
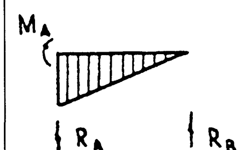
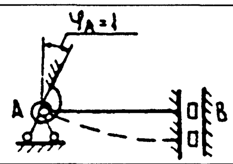
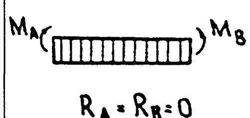
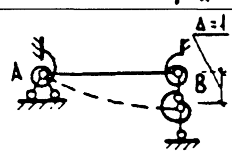
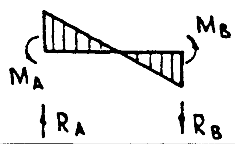
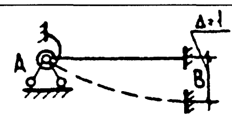
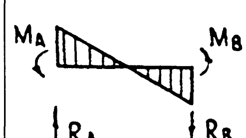
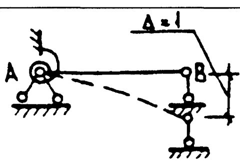
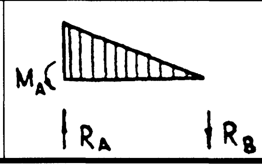
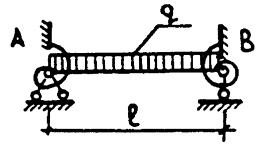
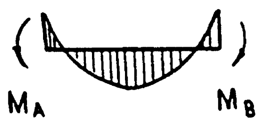
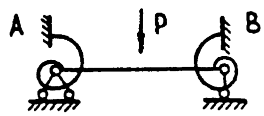
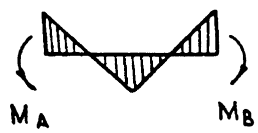
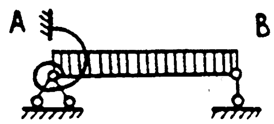
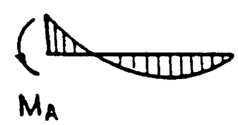
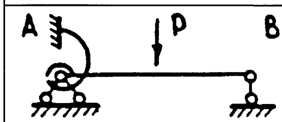
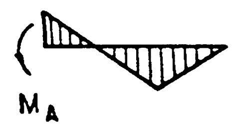
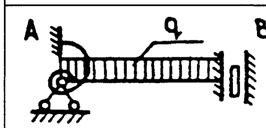
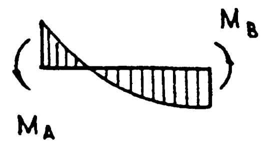
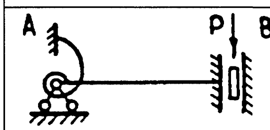
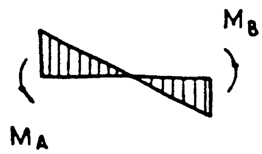
Beam Diagram	Moment Diagram	Formulas for Moment	Formulas for Shear
		$M_A = 4i \times \frac{3\mu + 1}{12\mu^2 + 8\mu + 1}$ $M_B = 2i \times \frac{1}{12\mu^2 + 8\mu + 1}$	$R_A = \frac{6i}{l} \frac{1}{6\mu + 1}$ $R_B = -R_A$
		$M_A = 4i \times \frac{1}{4\mu + 1}$ $M_B = 2i \times \frac{1}{4\mu + 1}$	$R_A = \frac{6i}{l} \frac{1}{4\mu + 1}$ $R_B = -R_A$
		$M_A = 4i \times \frac{3\mu + 1}{4\mu + 1}$ $M_B = 2i \times \frac{1}{4\mu + 1}$	$R_A = \frac{6i}{l} \frac{2\mu + 1}{4\mu + 1}$ $R_B = -R_A$
		$M_A = 3i \times \frac{1}{3\mu + 1}$	$R_A = \frac{3i}{l} \frac{1}{3\mu + 1}$ $R_B = -R_A$
		$M_A = \frac{i}{\mu + 1}$	$R_A = R_B = 0$
		$M_A = \frac{6i}{l} \times \frac{1}{6\mu + 1}$ $M_A = M_B$	$R_A = \frac{12i}{l^2} \times \frac{1}{6\mu + 1}$ $R_B = -R_A$
		$M_A = \frac{6i}{l} \times \frac{1}{4\mu + 1}$ $M_B = \frac{6i}{l} \times \frac{2\mu + 1}{4\mu + 1}$	$R_A = \frac{12i}{l^2} \times \frac{\mu + 1}{4\mu + 1}$ $R_B = -R_A$
		$M_A = \frac{3i}{l} \times \frac{1}{3\mu + 1}$	$R_A = \frac{3i}{l^2} \times \frac{1}{3\mu + 1}$ $R_B = -R_A$

Table 2.
Formulas for Moment for Various Loading Conditions

Beam Diagram	Moment Diagram	Formulas for Moment
		$M_A = M_B = \frac{ql^2}{12} \times \frac{1}{2\mu + 1}$
		$M_A = M_B = \frac{Pl}{8} \times \frac{1}{2\mu + 1}$
		$M_A = \frac{ql^2}{8} \times \frac{1}{3\mu + 1}$
		$M_A = \frac{3Pl}{16} \times \frac{1}{3\mu + 1}$
		$M_A = \frac{ql^2}{3} \times \frac{1}{\mu + 1}$ $M_B = \frac{ql^2}{6} \times \frac{3\mu + 1}{\mu + 1}$
		$M_A = \frac{Pl}{2} \times \frac{1}{\mu + 1}$ $M_B = \frac{Pl}{2} \times \frac{2\mu + 1}{\mu + 1}$