

The Accuracy of Column Stability Calculations in Unbraced Frames and the Influence of Columns with Effective Length Factors Less Than One

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ABSTRACT

Until 1993, AISC design specifications did not permit the use of effective length (K) factors less than one in unbraced frames. However, it is well known that, in certain practical situations, a column which is in an unbraced moment-resisting frame may exhibit a K factor less than one. While there are a number of equations that have been suggested for the calculation of effective length, to the authors' knowledge the accuracy of these different equations has not been thoroughly studied and compared for unbraced frames in which the K factor of one or more of the columns is less than one. This paper investigates and attempts to elucidate the stability behavior associated with columns in unbraced frameworks which possess this effective length characteristic. Based on an isolated column subassembly and a set of small frames that are felt to be representative of a wide range of situations encountered in practice, the accuracy of various methods for calculating effective length factors is identified. Although a variety of methods may be used to calculate the effective length factors in such frames, some of the formulas appear to be consistently more accurate than others. In addition, and perhaps more importantly, it is demonstrated that, for any of the effective length approaches considered in this paper, the accuracy of the effective length computations is identical for every column in a given story, regardless of whether these values are less than or greater than one. Thus, the conclusions from this paper are in many respects relevant to all unbraced frames, regardless of whether some of the frame's members have an effective length factor less than one.

1. INTRODUCTION

The Second Edition of the AISC LRFD Specification (AISC 1993) includes revised provisions for stability design which, among other benefits, do not restrict effective length (K) factors to be greater than one for the design of columns in

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unbraced frames. This allows greater economy and design flexibility in certain practical cases.

An effective length factor for a column may be less than one when the column participates in the lateral resistance of an unbraced frame, but is essentially "braced" by the other members of the lateral system of the frame. This situation can occur for columns turned in weak-axis bending which are part of the lateral resistance system, or for columns which have a large amount of distributed plasticity due to high axial loads and residual stress effects. For example, this behavior is exhibited by column b shown in Figure 1 (the effective length factors of this frame were recently studied by LeMessurier (LeMessurier 1991)). The "exact" K factor for weak-axis flexural buckling of column b is less than one if the ratio $\gamma = I_{xa} / I_{yb}$ is greater than 1.0 (all parameters are defined in Appendix A).

When certain columns of a story are significantly restrained against sidesway by the flexural stiffness of other members in the story, an appropriate engineering analysis must be conducted to determine the "correct" effective length factors of these restrained columns and of the columns which provide this restraint. Specifically, the calculation of the effective length must account sufficiently for all significant factors that influence the stability of the structure. For example, the computation should incorporate the effects of gravity ("leaner") columns* on story stability. Also, in certain cases,

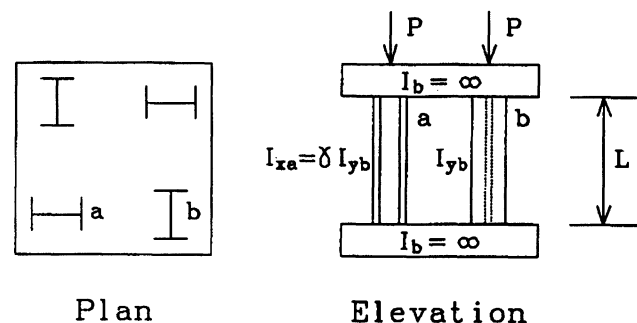


Fig. 1. Example of unbraced frame with braced column.

it may be important to consider the effects of inelasticity in the columns at incipient buckling of the structural system.

There are a number of formulas which might be used for this calculation (see Section 2). Each of these formulas considers the buckling of a given story in a building as an isolated unit in an unbraced, sidesway mode of failure. However, as one may find from a study of their derivations, these "story-based" effective length formulas do not account for an individual column failure in a braced (or sidesway restrained) mode of failure, nor do their derivations account explicitly for any possible interaction between sidesway and no-sidesway modes of buckling. It is possible that significant interaction of this nature may occur in columns of the lateral system whose effective length is less than one. This paper investigates the accuracy of several contemporary effective length formulas for stories with these types of structural characteristics.

While a column whose calculated value of K is less than one may participate in a sidesway mode of buckling, that column is predominantly braced by the other members of the structural system. Therefore, one practice which has been suggested is to calculate the effective length factors both from formulas based upon sidesway buckling and no-sidesway buckling. The larger of the two K factors would be used for design. For example, in Figure 1, a braced mode of buckling controls (i.e., the braced K factor of 0.5, obtained from the sidesway inhibited nomograph (AISC 1993), is larger than the unbraced K factor) when γ is greater than 7.0. For these cases, according to the above approach, the braced K factor would be used for design. As will be seen in Section 2, however, the LRFD Commentary (AISC 1993) imposes a somewhat stricter limit on the effective length, thus insuring not only that a braced mode of buckling is accounted for, but also that these effective length calculations are accurate or conservative for all unbraced frames.

The investigations in this paper pertain to the use of the sidesway buckling formulas presented in Section 2 combined with the use of a minimum limit on these values where appropriate. The paper is divided into six sections. Section 2 provides a brief overview of several formulas, based on story sidesway buckling, which may be used to calculate effective length factors. Section 3 summarizes the results from a study of an isolated column supported by rotational and translational springs, comparing exact K factors computed using the transcendental equation for the system to the K factor computed using one of the approaches summarized in Section 2. Section 4 presents the results from a series of studies performed on a variety of portal frames, comparing the various K factor approaches to a K factor computed from an exact system buckling analysis. Section 5 investigates the relationship between the errors in the effective length calculation of

every rigidly-connected (i.e., non-leaner) column in a story of an unbraced frame and illustrates that, as long as K is greater than a limiting value specified in the Commentary (AISC 1993) for all columns in the story, the accuracies of these calculations are acceptable. Furthermore, it is shown that the errors in effective length are identical for every rigidly-connected column in the story regardless of whether K is greater than or less than one. Conclusions are drawn in the final section.

2. OVERVIEW OF PROCEDURES FOR CALCULATING EFFECTIVE LENGTH

LeMessurier has proposed two related story-based approaches for computing an accurate K factor (LeMessurier 1976, LeMessurier 1977). These are identified here as $K_{\Delta i}$ and $K_{\beta i}$, respectively, where, for column i in a story:

$$K_{\Delta i} = \sqrt{\frac{I_i \pi^2 E \sum_{i_{all}} P_{ui} + \sum_{i_{rigid}} C_{Li} P_{ui}}{P_{ui} L_i^2 (\sum_{i_{rigid}} H_i L_i) / \Delta_{oh}}} \quad (1a)^*$$

and

$$K_{\beta i} = \sqrt{\frac{I_i \pi^2 \sum_{i_{all}} P_{ui} + \sum_{i_{rigid}} C_{Li} P_{ui}}{P_{ui} \sum_{i_{rigid}} \beta_i I_i}} \quad (2a)$$

where $\sum_{i_{all}}$ is the summation of all columns in the story, $\sum_{i_{rigid}}$ is the summation of all rigidly-connected columns in the story, and

$$\beta_i = \frac{6(G_{top} + G_{bottom}) + 36}{2(G_{top} + G_{bottom}) + G_{top} G_{bottom} + 3} \quad (3)$$

for columns which are restrained at both ends,**

$$C_{Li} = \frac{\beta_i K_{ni}^2}{\pi^2} - 1 \quad (4)$$

and

$$G = \frac{\sum I_c / L_c}{\sum I_b / L_b} \quad (5a)$$

The G factors and K_{ni} , the K factor obtained from the

*For the the purpose of this discussion, a leaner column is defined as any column which is idealized as having zero rotational restraint at its ends.

*A number of equations presented throughout the paper are closely related in their derivation. These equations are numbered "a," "b," "c," etc. to represent their association.

**Values of β for conditions in which one of the column ends is pinned are given in (LeMessurier 1977).

sideway uninhibited nomograph (Figure C-C2.2 of the LRFD Specification), are calculated as described in LRFD Commentary Section C2. The factor C_{Li} accounts for the reduction in stiffness of column i due to the axial load acting through the curvature of the member (the $P-\delta$ effect). The C_{Li} and β_i factors are equal to zero for the leaned columns of a story. It should be emphasized that the derivations of Equations 1 through 4 presume a story sideway mode of buckling (LeMessurier 1977). Also, in the derivation of Equations 1a and 2a, it is assumed that the lengths of all columns in the story are equal.

The derivations of both K_{ni} and β_i presume that, as the unbraced story buckles in a sideway mode, the beams restraining the story's columns are in reverse curvature with an inflection point at midspan (AISC 1993, LeMessurier 1977). The LRFD Commentary (AISC 1993) suggests that this limiting assumption may be alleviated to some extent by using modified G factors in lieu of Equation 5a as follows:

$$G_{\text{mod}} = \frac{\sum I_c / L_c}{\sum I_b / L_b [2 - M_{\text{far}} / M_{\text{near}}]} \quad (5b)$$

M_{near} and M_{far} are moments at the near and far ends, respectively, of the restraining beams. They are obtained from a first-order lateral load analysis in which gravity loads are customarily not applied. Note that these G factors may in fact be negative (AISC 1993, Bridge 1977, Bridge 1987).

When using Equation 2a, inelastic G factors may be used to calculate K_{ni} and β_i to incorporate the effects of material nonlinearity (see (Yura 1971, AISC 1993) for the details of this approach). For Equation 1a, Δ_{oh} is obtained from a first-order lateral load analysis (i.e., an analysis similar to that which would be used to obtain M_{near} and M_{far}). Therefore, $K_{\Delta i}$ has the advantage of accounting for the relative stiffnesses of the beams and columns as a by-product of an ordinary structural analysis (see (LeMessurier 1977, Liew 1991) for a more comprehensive discussion of the use of Equations 1 through 5). The time-consuming calculation of G factors need not be performed for this approach if a value (e.g., a conservative value of 0.216 (LeMessurier 1977)) for C_{Li} is assumed for the columns in the story. However, Equation 1a cannot produce an inelastic K factor, unless, for example, as an alternative to performing a purely linear elastic analysis, the Young's modulus of each column of the lateral system is individually reduced during the analysis by a tangent modulus factor which accounts for the distribution of inelasticity in the columns in the presence of axial force (Baker 1991, Abdelrazaq 1993).

The LRFD Commentary also includes variations on the

following similar K factor formulas

$$K_{R,i} = \sqrt{\frac{I_i \pi^2 E}{P_{ui} L_i^2} \frac{\sum_{i_{\text{all}}} P_{ui}}{\sum_{i_{\text{rigid}}} H L_i} \Delta_{oh} (0.85 + 0.15 R_L)} \quad (6a)$$

where R_L is the ratio of the load on the leaned columns in the story to the load on all the columns in the story:

$$R_L = \frac{\sum_{i_{\text{leaner}}} P_{ui}}{\sum_{i_{\text{all}}} P_{ui}}$$

and

$$K_{K,i} = \sqrt{\frac{I_i}{P_{ui}} \frac{\sum_{i_{\text{all}}} P_{ui}}{\sum_{i_{\text{rigid}}} \frac{I_i}{K_{ni}^2}}} \quad (7a)$$

Note that Equation 6a is actually expressed in the Commentary in the following form:

$$P_{eR,i} = \frac{P_{ui}}{\sum_{i_{\text{all}}} P_{ui}} \frac{\sum_{i_{\text{rigid}}} H L_i}{\Delta_{oh}} (0.85 + 0.15 R_L) \quad (6b)$$

where $P_{eR,i}$ is the buckling load of column i , as predicted by this approach (LeMessurier 1993, LeMessurier 1994, Squarzini 1993, Hajjar 1995). This column buckling load may also be expressed for each column i as:

$$P_{eR,i} = \lambda_{R,i} P_{ui} = \frac{\pi^2 E I_i}{K_{R,i}^2 L_i^2} \quad (6c)$$

from which Equation 6a may be derived. See Section 5 for a discussion of the significance of the story buckling parameter, $\lambda_{R,i}$, of this approach and of the relation of Equation 6b to Equation 6c.

LeMessurier (LeMessurier 1993, LeMessurier 1994) derived Equation 6a, while Wu (Wu 1985) and Baker (Baker 1987), among others, derived Equation 7a, Equation 6a is similar to Equation 1a except that a relatively conservative value of C_{Li} (0.176) is assumed for all of the rigidly-connected columns. Note that in this equation the $P-\delta$ effect correctly decreases towards zero as the percentage of the story's gravity load that is taken by the story's leaned columns increases (i.e.,

as R_L goes to one, $0.85 + 0.15R_L$ goes to one, thus causing the debilitating effect of $P-\delta$ on the story stability to diminish towards zero, as it should). Equation 7a, in turn, is similar to Equation 2a, although the relationship between $K_{\beta i}$ and K_{K_i} is not as direct as that between $K_{\Delta i}$ and K_{R_i} . The derivation of K_{K_i} is based upon the assumption that the sum of the axial loads which cause sway buckling of a story is equal to the sum of the individual column buckling resistances of the story calculated using K_{ni} . Equation 7a accounts for the $P-\delta$ effect implicitly through each of the individual values of K_{ni} rather than through the use of C_{Li} . Of course, K_{ni} provides an accurate effective length for a given column, and thus an accurate representation of an individual column's buckling resistance, only if all of the assumptions of the nomograph are met (as stated in LRFD Commentary Section C2). The equations governing the derivation of Equation 7a are outlined in Section 5. Equation 7a is derived fully in Liew 1991. As with Equation 2a, inelastic G factors may be used with Equation 7a to calculate K_{ni} such that the effects of inelasticity may be incorporated. Also, the derivations of Equations 1a, 2a, 6a, 6b, and 7a presume that all of the columns in a story are of equal length.

K_{R_i} will generally be greater than or equal to $K_{\Delta i}$, and thus more conservative, since it essentially assumes a relatively conservative value of C_{Li} for all rigidly-connected columns in the story. K_{R_i} will tend to be unconservative only for frames in which a majority of the columns exhibit a large $P-\delta$ effect at incipient buckling (LeMessurier 1993, Squarzini 1993). While the values of K_{K_i} are not strictly bounded by $K_{\beta i}$, they are almost always smaller, and thus K_{K_i} is generally less conservative than $K_{\beta i}$.

Also, one may consider Equations 1a and 2a with values of C_L of zero:

$$K_{\Delta 0i} = \sqrt{\frac{I_i \pi^2 E}{P_{ui} L_i^2} \frac{\sum_{i_{all}} P_{ui}}{(\sum_{i_{rigid}} H_i L_i) / \Delta_{oh}}} \quad (8a)$$

$$K_{\beta 0i} = \sqrt{\frac{I_i \pi^2 E}{P_{ui} L_i^2} \frac{\sum_{i_{all}} P_{ui}}{\sum_{i_{rigid}} \beta_i I_i}} \quad (9a)$$

$K_{\Delta 0i}$ will always be less than or equal to $K_{\Delta i}$, and thus potentially less conservative. There is a similar relation between $K_{\beta 0i}$ and $K_{\beta i}$. The effects of neglecting C_{Li} on the accuracy of the effective length computations will be discussed in Sections 4, 5, and 6.

Finally, both elastic and inelastic K factors may be calculated using the results of an eigenvalue buckling analysis. Although in many cases the effective length factors based on

a story subassembly (Equations 1 through 9) may be more appropriate for design, the eigenvalue buckling K factor may be considered to be the exact K factor for a column (Equations 1 through 9 may often be more appropriate in part due to the differences in the buckling load parameters for the different stories of multi-story frames; see Liew 1991 for a discussion of the advantages and disadvantages of these various approaches). For single-story frames, an eigenvalue buckling analysis undoubtedly provides the most representative (i.e., the analytical or exact) effective length factor for design. Once a buckling analysis has been performed, a K factor for each column i may be computed based on the analysis results as:

$$K_{bi} = \sqrt{\frac{\pi^2 E I_i}{\lambda_b P_{ui} L_i^2}} \quad (10)$$

A column will have a K factor less than one if it is largely braced by other columns in the story (e.g., if the axial force, $\lambda_b P_{ui}$, at incipient buckling of the system is larger than $P_{ei} = \pi^2 E I_i / L_i^2$). Of the approaches to calculating K mentioned above, eigenvalue analysis with the use of Equation 10 is the only one that explicitly captures any potential interaction between unbraced and braced modes of failure for a given column in the structural system. Equations 1 through 9 all assume only a sidesway mode of buckling in their formulation.

It should be noted that the direct use of K_{ni} with unmodified G factors (Equation 5a) as the effective length factor for design is inappropriate for frames of general configuration primarily since it does not consider the fact that certain members may be dominant in destabilizing the structural system while other members may be dominant in resisting these destabilizing actions (e.g., leaner column effects are not considered in the calculation of K_{ni}), and because K_{ni} is based on the assumption that the stiffness parameter $L_i \sqrt{P_{ui}} / E I_i$ of every rigidly-connected column in the story must be equal. The use of K_{ni} with modified G factors (Equation 5b) improves the accuracy of the nomograph effective lengths. However, modified G factors remain somewhat inaccurate for frames exhibiting asymmetry (Squarzini 1993, Hajjar 1994a), and, in the authors' opinion, these factors may be quite cumbersome to calculate relative to the computation required for K_{R_i} .

3. STUDY OF A COLUMN SUPPORTED BY SPRINGS

Over a decade ago, Bridge and Trahair (Bridge 1977) studied the behavior of a single column supported at its top and bottom by rotational and translational springs to discern the behavior of the column when subjected to arbitrary end restraints. To illustrate the relative stiffness required to cause a K factor to be less than one for a column in an unbraced frame, a similar column is studied here. As shown in Figure 2, this column is supported by springs at one end and hinged

at the other. The parameters of this subassembly may be varied to represent a wide range of characteristics in members of an unbraced moment-resisting frame. The column is supported at its top by a rotational spring with stiffness K_r and a translational spring with stiffness K_t . The rotational spring represents the rotational restraint provided by adjacent beam and column members and their connections to the column under consideration. The translational spring represents the lateral stiffness in the direction of the spring provided by the remainder of the structure's lateral resistance system at this story. A gravity load, αP , is applied to the remainder of the story (see the figure for a clarification of what this term represents). The transcendental equation describing the equilibrium state of the buckled configuration takes the form:

$$\frac{\pi}{K_e} \cot\left(\frac{\pi}{K_e}\right) = \frac{\frac{1}{R_t} \left[\left(\frac{\pi}{K_e}\right)^2 - R_t + \alpha \left(\frac{\pi}{K_e}\right)^2 \right] - \frac{R_t}{(\pi/K_e)^2} + \alpha}{1 - \frac{R_t}{(\pi/K_e)^2} + \alpha} \quad (11)$$

where $R_t = K_t L / EI$. This equation may be solved iteratively to obtain K_e , the exact factor for this column.

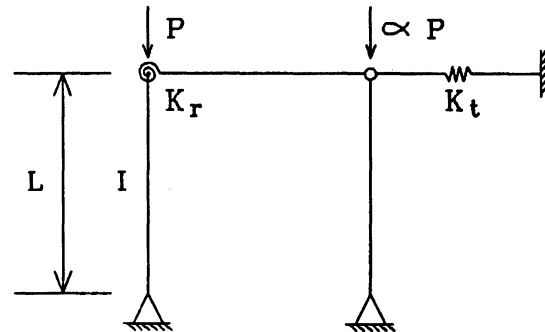
Similarly, K_{Δ_i} (Equation 1a) may be derived in terms of R_r , R_t , and α (note that K_{β_i} is not presented here since it is not possible to express $\sum_{i_{rigid}} \beta_i I_i$ in terms of R_t):

$$K_{\Delta_i} = \sqrt{\frac{\pi^2 \left\{ 1 + \alpha + \left[\frac{0.216}{(1 + 3/R_t)^2} \right] \right\}}{R_t + \frac{1}{\frac{1}{3} + \frac{1}{R_t}}}} \quad (12)$$

In their studies, the authors varied the spring stiffnesses from zero to infinity, such that a comprehensive set of configurations could be analyzed. These relative stiffnesses may be thought of as encompassing the effects of either *elastic* or *inelastic* K factor computations (e.g., one may consider EI of the columns as representing the effective elastic portion of a partially plastic cross-section). Thus, it is appropriate to consider Δ_{oh} as being from a first-order analysis with the column EI values based on an effective elastic moment of inertia (or tangent modulus) that is a function of the column axial load. The leaner load factor α was varied from zero to eight (although a leaner load of zero is an extreme, and is unrealistic if K_t is at all greater than zero). Several representative results are shown in Table 1. In addition to the exact effective length factor, K_e , and the factor K_{Δ_i} computed from Equation 12, the effective length factor for the sidesway restrained condition, K_{braced} , is also shown (the G factor at the top of the column equals $6/R_t$). The maximum of K_{Δ_i} and K_{braced} and the percent errors of this maximum value with respect to K_e are indicated

as well. The last column shows the percent error of K_{Δ_i} when K_{braced} is larger than K_{Δ_i} . A negative percent error in the table indicates that the larger of K_{Δ_i} or K_{braced} (or simply K_{Δ_i} for the last column) is smaller than K_e , thus indicating that these values are unconservative. The following trends are illustrated in Table 1 and were observed in the full range of cases studied:

1. K_{Δ_i} is highly deficient as the column becomes infinitely braced. That is, it degenerates to zero as R_t becomes infinite, as shown in the last row of the table. The exact effective length factor must degenerate to a value of K_{braced} between 0.7 and 1.0, depending on the rotational stiffness at the top of the column. Obviously, it is necessary to compute the effective length by some other means if K_{Δ_i} is less than K_{braced} . As previously noted, one of the goals of this paper is to investigate the accuracy associated with simply using K_{braced} , or some related limit (see Section 4), as a lower limit on values such as K_{Δ_i} . The trends exhibited by the structure in Figure 2 regarding this aspect are discussed under item 3 below.
2. K_{Δ_i} is highly accurate when the column being considered participates in resisting the sidesway buckling of the system and buckles dominantly in a sidesway mode. This is opposed to the cases cited in item 1 above, in which the column is primarily "leaning" on the other components of the system, with these other components being the ones which provide the sidesway resistance.



$$K_r = \frac{R_r E I}{L}$$

$$K_t = \frac{R_t E I}{L^3}$$

$$\alpha P = \sum P + \sum C_L P$$

Summation is for all columns in the story except the column on the left

Fig. 2. Column supported by springs.

Table 1.
Accuracy of $K_{\Delta I}$ for Column Supported by Springs

Row	R_t	R_r	α	K_e	$K_{\Delta I}$	K_{braced}	maximum ($K_{\Delta h}$ K_{braced})	percent error maximum ($K_{\Delta h}$ K_{braced})	percent error in $K_{\Delta I}$ when $K_{braced} > K_{\Delta I}$
1	8.0	1.0	0.0	1.08	1.07	0.922	1.07	-0.926	—
2	16.0	1.0	0.0	0.933	0.773	0.922	0.922	-1.18	-17.15
3	32.0	1.0	0.0	0.926	0.553	0.922	0.922	-0.432	-40.28
4	8.0	8.0	0.0	1.07	1.04	0.773	1.04	-2.80	—
5	16.0	8.0	0.0	0.869	0.778	0.773	0.778	-10.5	—
6	32.0	8.0	0.0	0.798	0.568	0.773	0.773	-3.13	-28.82
7	8.0	1.0	1.0	1.51	1.51	0.922	1.51	-0.136	—
8	16.0	1.0	1.0	1.10	1.09	0.922	1.09	-0.909	—
9	32.0	1.0	1.0	0.928	0.779	0.922	0.922	-0.647	-16.06
10	8.0	8.0	1.0	1.44	1.43	0.773	1.43	-0.694	—
11	16.0	8.0	1.0	1.09	1.07	0.773	1.07	-1.84	—
12	32.0	8.0	1.0	0.847	0.782	0.773	0.782	-7.67	—
13	8.0	1.0	2.0	1.84	1.84	0.922	1.84	-0.042	—
14	16.0	1.0	2.0	1.33	1.33	0.922	1.33	-0.150	—
15	32.0	1.0	2.0	0.969	0.953	0.922	0.953	-1.65	—
16	8.0	8.0	2.0	1.74	1.74	0.773	1.74	-0.147	—
17	16.0	8.0	2.0	1.31	1.30	0.773	1.30	-0.561	—
18	32.0	8.0	2.0	0.968	0.949	0.773	0.949	-1.96	—
19	64.0	64.0	8.0	1.17	1.16	0.710	1.16	-0.855	—
20	∞	0.0	any α	1.00	0.00	1.00	1.00	0.00	-100.0

Of course, for the idealized example being considered in this section, whether the column behaves according to either of the above extremes depends on a combination of the factors α , R_r , and R_t . For small values of α (e.g., $\alpha = 0.0$ or 1.0 in Table 1), $K_{\Delta I}$ is accurate compared to K_e only when R_t is sufficiently small, so that the column is providing a significant fraction of the side-sway resistance. Also, for small values of α and as R_t rises, the column in Figure 2 becomes increasingly braced, the $K_{\Delta I}$ values drop below one, and the error in $K_{\Delta I}$ increases. As α increases, the accuracy of $K_{\Delta I}$ improves as the system sidesway stability becomes dominant over the “braced” mode of buckling of the column. For large values of α (e.g., α greater than or equal to 2), a large R_t is required to generate any inaccuracy in $K_{\Delta I}$. That is, if the other components of the idealized story are loaded heavily, they must have a large sidesway stiffness to provide substantial bracing to the column on the left in Figure 2. The values of $K_{\Delta I}$ generally become smaller as R_t increases, but no direct correlation between the error in $K_{\Delta I}$ and the magnitude of R_t is evident.

3. If K_{braced} is utilized whenever it has a larger value than

$K_{\Delta I}$, the unconservative error in the resulting effective length factor for the range of parameters considered (α varying from zero to eight and R_r and R_t varying from zero to infinity) is never greater than 10.5 percent. This error corresponds to the case shown in row 5 of the table.

4. The largest error in the K factor calculation based on $K_{\Delta I}$ and K_{braced} occurs at the conditions where these two values are equal. If R_t is increased from its value corresponding to the conditions associated with $K_{\Delta I} = K_{braced}$, the K_{braced} value controls and is increasingly accurate. Conversely, if R_t is decreased from its value corresponding to this state, $K_{\Delta I}$ controls and is increasingly accurate as R_t becomes smaller.

The significance of having a maximum possible unconservative error of 10.5 percent in the calculation of the effective length depends of course on the slenderness of the column being considered. Nevertheless, it should be noted that the larger errors tend to occur for large values of R_t and R_r combined with small values of α (i.e., conditions leading to $K_{\Delta I} = K_{braced}$)

The model in Figure 2 may be made more comprehensive

by restraining the rotations at the base of the column by a second pair of springs. However, while the values of spring stiffnesses at which the column essentially becomes braced would vary from the current model, the pattern of behavior of the system would not change (i.e., at some level of stiffness some of the assumptions embedded in the effective length calculation of Equation 12 would break down, and some interaction effects between braced and unbraced failure modes would be evident at the point that $K_{\Delta i} = K_{braced}$). The accuracy of the K factor calculations for a pin based column versus a column with restraints at both ends is considered in the next section.

The above results offer a glimpse at both the reliability of $K_{\Delta i}$ and its potential deficiencies. For example, Table 1, row 12, indicates that when R_i is 32 and R_r is 8, both $K_{\Delta i}$ and K_{braced} are moderately inaccurate (about 8 percent) when α equals one. As will be explained below, the LRFD Commentary has sought to alleviate these sorts of errors by establishing a stricter minimum limit on the unbraced effective length factor than K_{braced} .

4. STUDY OF PLANAR BENTS

In general it is difficult to judge the ranges of parameters in the above spring study which correspond to realistic design cases. To obtain a better feel for the limits of these K factors and also to judge the accuracy of the other formulas reviewed in Section 2, several planar bents were studied. Four of these bents are shown in Figures 3a and 3b, along with the ranges of properties that were investigated for each bent. A radius of gyration of 5.0 was assumed for all of the frame members. This value is representative of wide-flange sections subjected to strong-axis bending. The base length L utilized in all the studies was 100 (thus $L/r = 20$). All members were assumed to have the same elastic modulus.

For each configuration, K_{R_i} , K_{K_i} , $K_{\Delta i}$, $K_{\Delta 0i}$, $K_{\beta i}$, $K_{\beta 0i}$, and K_{braced} were calculated. For the calculation of K_{R_i} , $K_{\Delta i}$, and $K_{\Delta 0i}$, Δ_{oh} was determined solely based on an applied lateral load to the bent (these frames exhibit sidesway under gravity load alone).* The K factors are compared to K_{bi} , the effective length factor computed from an eigenvalue buckling analysis based solely on application of the gravity loads to the frame. As mentioned in Section 2, this K factor can be considered as exact for these types of frames. The displacement Δ_{oh} , as well as the moments M_{near} and M_{far} required for Equation 5b, are calculated, for consistency, using the same member and material properties that are used to calculate K_{bi} .

In similar fashion to the cases investigated in Section 3, the studies of these planar bents inherently capture the behavior represented in both *elastic* and *inelastic* K factor computations, since the values of EI used in the model could represent

either elastic or inelastic stiffnesses at incipient buckling. Also, since frames having a wide range of values of G and $L_i\sqrt{P_{ui}}/EI_i$ have been investigated here, the conclusions drawn from these isolated frame studies apply equally to individual stories within a multistory unbraced frame.

Selected results are shown in Figures 4 through 9. For each study, the graphs plot the values of K_{bi} , K_{R_i} , K_{K_i} and K_{braced} versus the member property that is being varied for that particular study. Of the six unbraced effective length factors that were computed (Equations 1a, 2a, 6a, 7a, 8a, and 9a), only K_{R_i} and K_{K_i} are shown for the purposes of brevity and clarity since these are the formulas which are included in the LRFD Commentary. Note that the first plot, (a), within each figure uses unmodified G factors (Equation 5a) to compute K_{K_i} , while the second plot, (b), within each figure uses modified G factors (Equation 5b) to compute this effective

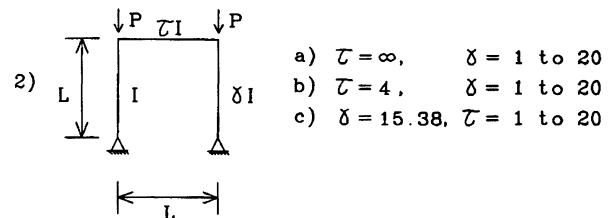
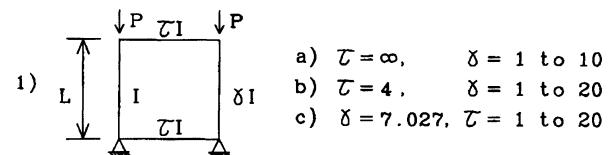


Fig. 3a. Planar bents used to study K factors.

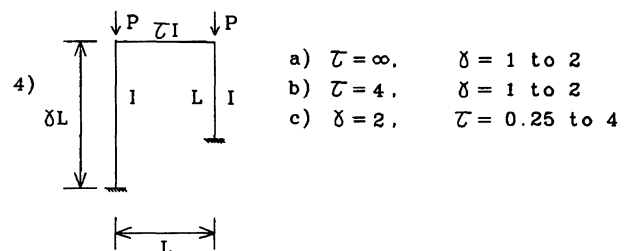
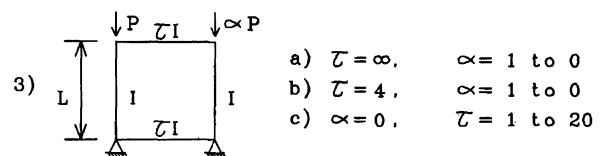


Fig. 3b. Planar bents used to study K factors (cont.).

*Note that in a multistory frame, the engineer may use judgment to determine the distribution of lateral load applied to the frame when determining Δ_{oh} ; however, LeMessurier (LeMessurier 1993) has recommended that the total gravity load of each story be applied as the lateral load to obtain the most appropriate interstory drift values.

length factor. K_{braced} is computed using unmodified G factors in all of the figures; the modified G factors of Equation 5b apply only to effective length computations for which side-sway is uninhibited (see Bridge (Bridge 1987) for a discussion of an approach for computing modified G factors for the case of side-sway inhibited). K_{bi} and K_{R_i} are of course independent of G factors.

All of the results are shown only for the column on the *left side* in the bents of Figure 3 (i.e., the column for which the effective length may be less than one). The percent error versus K_{bi} is shown for K_{R_i} and K_{K_i} at various points along their curves. As will be explained in Section 5, even though the effective length factor of the right column is often *greater* than one, its percent error in K is identical to the left column's for all the examples as long as both columns' effective length factors are greater than K_{braced} as well as other specified limits (see the discussion below). As in Table 1, a negative percent error indicates that the unbraced K factor is smaller than K_{bi} , thus indicating that the unbraced K factor is unconservative. The plots of K_{R_i} and K_{K_i} are extended below the plot of K_{braced} to show their trends, although the percent error is not shown for this portion of the curves, since K_{braced} controls there. The reported values are of course affected by the precision of the numerical solution for K_{bi} using a finite element model. The values for K_{bi} are believed to be accurate within ± 1 percent. Also, the use of realistic areas (i.e., $r = 5$) for all members in the frames has a small effect on the computed effective length values. If infinite axial stiffness is assumed for all members, the results are slightly changed.

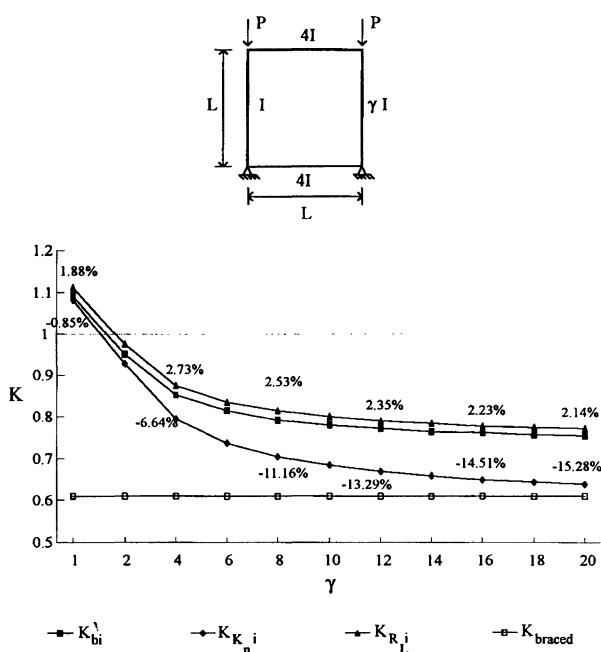


Fig. 4a. Results of study 1b (τ equals 4) using unmodified G factors.

Infinite moments of inertia were represented in the computer analyses by moments of inertia of 1×10^8 .

Figures 4a and 4b show the results of study 1b, in which the moments of inertia of the beams were held constant at a value of four times the moment of inertia of the left column, while the moment of inertia of the right column was increased to twenty times that of the left column. In both plots, K_{R_i} remains quite accurate and is conservative. $K_{\Delta i}$ and $K_{\Delta o i}$ (not shown in the figures) are accurate to a similar degree, with the former having an unconservative error of 2.4 percent when $\gamma = 20$, and the latter an unconservative error of 5 percent. However, Figure 4a indicates that when unmodified G factors are used, the accuracy of K_{K_i} deteriorates rapidly with increasing values of γ . Reasonably practical values of γ such as $\gamma = 8$, cause unconservative errors in the range of 11 percent for K_{K_i} . Using unmodified G factors, $K_{\beta o i}$, which neglects $C_{L i}$ completely, and $K_{\beta i}$ each have plots (not shown in the figure) with a similar shape to the plot shown for K_{K_i} , but they have errors at $\gamma = 20$ of 17 percent and 14 percent, respectively.

Figure 4b indicates that modified G factors have the potential of greatly increasing the accuracy of K_{K_i} . The complete set of results (not included in the figures) show that the accuracy of $K_{\beta i}$ and $K_{\beta o i}$ is improved as well, resulting in errors of 2 percent and 7 percent, respectively, at $\gamma = 20$.

For the second frame (Figures 5 through 7), similar behavior is seen, although, as may be expected, a larger bracing stiffness is required to achieve a K less than one since the frame is more flexible due to the lack of rotational restraint

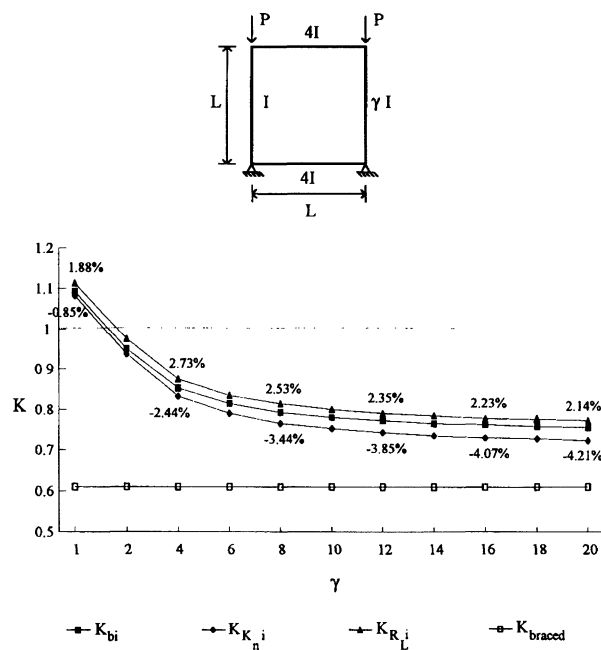


Fig. 4b. Results of study 1b (τ equals 4) using modified G factors.

at its base. For the case shown in Figure 5, in which the girder has infinite stiffness, K_{K_i} has a maximum error of 10 percent (since the girder has infinite stiffness, the modified and unmodified G factors produce identical results). When γ increases from this point of maximum error, K_{braced} becomes larger than K_{K_i} and has a more accurate value. For this case, $K_{\beta_{0i}}$ has a plot (not shown) with a shape that is similar to the plot shown for K_{K_i} , but it crosses the curve of K_{braced} near $\gamma = 12$ with an error of 15.5 percent. Alternately, K_{R_i} , while generally more accurate than K_{K_i} , has an unconservative error of approximately 7 percent before its value drops below K_{braced} .

One may observe from Figure 5 that Equation 6a may exhibit unconservative errors. This is due to the fact that the $P-\delta$ effect (i.e., the value of C_{Li} for the columns of the story) may be larger than the effect embedded within the assumption used in the derivation of the equation that C_{Li} equals 0.176 for all rigidly-connected columns in the story.

Figures 4 and 5 indicate that establishing a lower limit of K_{braced} may not be sufficient to insure an accurate calculation of effective length by these story-based procedures. For asymmetric frames in which one column's buckling mode is approaching a braced condition, the behavioral assumptions embedded in the computation of the sidesway inhibited nomograph effective length factor are sufficiently different from the actual behavior of the frame that even the K_{braced} limit is not adequate. To rectify this situation, LeMessurier (LeMessurier 1993, LeMessurier 1994) has proposed alternative limits which are included in the LRFD Commentary. For

K_{R_i} the Commentary expresses the limit in terms of a maximum permissible buckling load that may be predicted by this approach:

$$P_{eR_i} \leq \frac{1.7HL_i}{\Delta_{oh}}$$

where P_{eR_i} is expressed in Equation 6b. The limit on the use of K_{R_i} may then be derived by using Equation 6c:

$$K_{R_i} = \sqrt{\frac{\pi^2 EI_i}{L_i^2} \frac{\Delta_{oh}}{1.7HL_i}} \quad (6d)$$

The limit on the use of K_{K_i} , as expressed in the LRFD Commentary, is:

$$K_{K_i} \geq \left(\sqrt{\frac{5}{8}} \right) K_{ni} \quad (7b)$$

A detailed derivation of these limits is beyond the scope of this paper (see (LeMessurier 1993, LeMessurier 1994, Squarzini 1993, Hajjar 1995) for further detail). However, a very wide range of problems have been tested using these two approaches, and it may be shown (LeMessurier 1993, Squarzini 1993) that these limits not only insure that a column's transition from an unbraced to a braced mode of buckling is accounted for in a manner that is more comprehensive than simply computing K_{braced} , but also that Equations 6a and 7a are never more than 5 percent unconservative.

These limits are plotted in Figures 5 through 9. In Figure 5, one can see that the limit of Equation 6d is breached at a value of γ of approximately 3, while Equation 7b is breached at a value of γ of approximately 2. In both of these cases, the resulting unconservative errors of Equations 6a and 7a remain below 2 percent. On the other hand, the use of the limits of Equations 6d and 7b (i.e., taking K_{R_i} as being equal to the value of Equation 6d, and taking K_{K_i} as being equal to the value of Equation 7b) may be seen to be quite conservative as γ increases. Of course, if the limits of Equations 6d and 7b are breached by an excessive amount, the member that is breaching the limit should almost always be redesigned (Squarzini 1993, Hajjar 1994a).

In Figure 6a, note that K_{K_i} is inaccurate even when its value is larger than one, if the limit of Equation 7b is not invoked. This is due to the fact that, as the assumptions embedded in the sidesway uninhibited nomograph break down K_{K_i} is unable to retain its accuracy. As Figure 6b shows, using modified G factors noticeably increases the accuracy of K_{K_i} . Section 5 will elaborate further on errors in effective length factors greater than one. If the limit of Equation 7b is used, K_{K_i} is approximately 3 percent unconservative before the limit controls, after which the effective length value used for design becomes conservative. Alternately, note that the combination

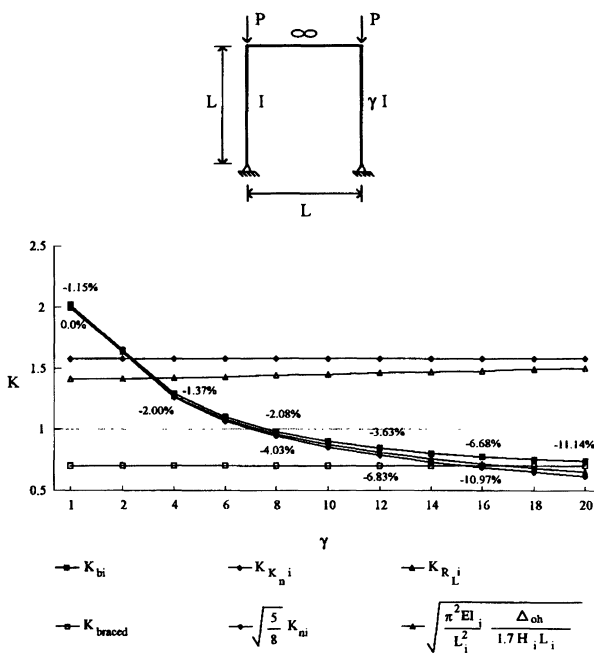


Fig. 5. Results of study 2a (τ equals ∞) using either unmodified or modified G factors.

of Equations 6a and 6d retain excellent accuracy for all values of γ .

For the study shown in Figures 7a and 7b, γ was held constant at 15.38. This is the value at which the braced K factor controls if the beam has infinite stiffness. K_{K_i} exhibits a maximum error of 19 percent for this case if unmodified G factors are used, unless Equation 7b is invoked to limit the value of K_{K_i} . In this case, the limit controls for all values of τ if unmodified G factors are used, and for most values of τ if modified G factors are used, and thus the use of K_{K_i} becomes quite conservative unless the member is properly redesigned. K_{R_i} is accurate or conservative for all values of τ .

Figures 8a and 8b indicate that both K_{R_i} and K_{K_i} are relatively accurate when the moments of inertia of the two columns are the same. For this study, case 3c (Figure 3b), the applied gravity load on the right column is zero. Note that, because of the manner in which the limit of Equation 6d is derived (LeMessurier 1993, LeMessurier 1994, Squarzini 1993), this limit lies directly on top of K_{R_i} for these cases. Although K_{R_i} and K_{K_i} are relatively accurate, when $\tau = 20$, $K_{\beta_{0i}}$ has an unconservative error of 10 percent and $K_{\Delta_{0i}}$ has an unconservative error of 9 percent (these are not shown in the figure). These errors in $K_{\beta_{0i}}$ and $K_{\Delta_{0i}}$ become small (less than a few percent) only when C_{L_i} approaches zero (e.g., as the beam weakens) and thus the $P-\delta$ effects become less significant. Of course, the limits of Equations 6d and 7b may be invoked when using K_{β_i} , $K_{\beta_{0i}}$, K_{Δ_i} , and $K_{\Delta_{0i}}$, with a resulting

decrease in the unconservative error exhibited by these approaches.

None of the equations presented thus far in this paper account for stories having columns with unequal lengths, such as for the case studied in Figures 9a and 9b. For example, K_{K_i} , K_{β_i} , $K_{\beta_{0i}}$ have errors ranging up to 62 percent, 62 percent, and 49 percent, respectively, if their values are calculated according to Equations 2a, 7a, and 9a using unmodified G factors (and using no limit on the value of these effective lengths other than K_{braced}). This is because, in the derivations of these formulas, it is assumed that the lengths of all columns in the story are equal. The accuracy of these equations quickly deteriorates as the lengths of the columns become unequal. The accuracy of K_{R_i} , K_{Δ_i} , and $K_{\Delta_{0i}}$ deteriorates similarly. However, if the individual column lengths are taken into account, the K factor formulas are as follows (Hajjar 1994):

$$K_{R_i} = \sqrt{\frac{I_i \pi^2 E}{P_{ui} L_i^2} \frac{\sum_{i_{all}} \frac{P_{ui}}{L_i}}{\sum_{i_{rigid}} H_i} \frac{\Delta_{oh}}{1.7 H_i L_i} (0.85 + 0.15 R_L)} \quad (6e)$$

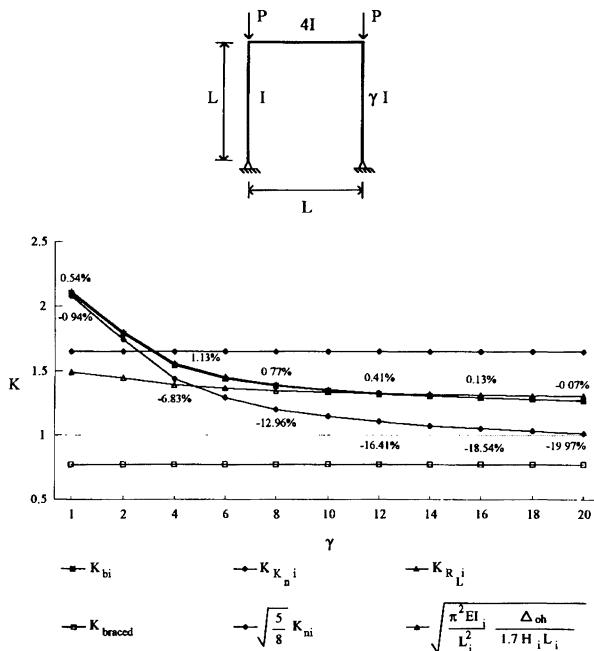


Fig. 6a. Results of study 2b (τ equals 4) using unmodified G factors.

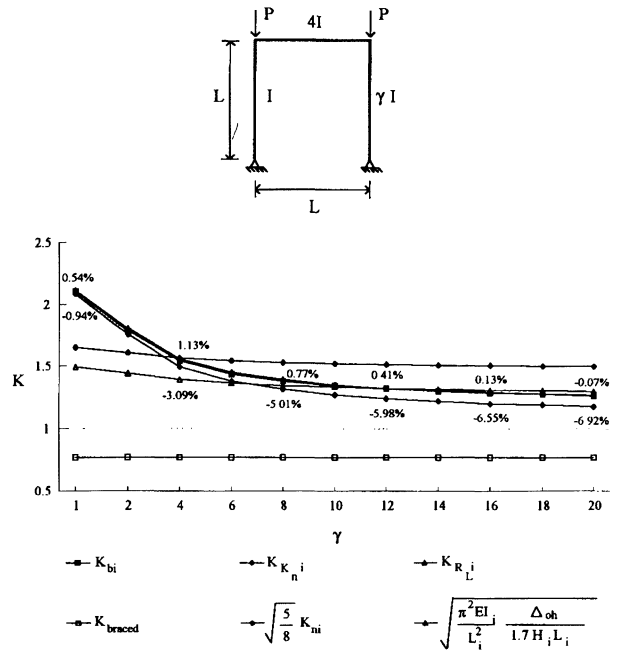


Fig. 6b. Results of study 2b (τ equals 4) using modified G factors.

$$K_{K_i} = \sqrt{\frac{I_i}{P_{ui} L_i^2} \frac{\sum_{i_{all}} \frac{P_{ui}}{L_i}}{\sum_{i_{rigid}} \frac{I_i}{K_{ni}^2 L_i^3}}} \quad (7c)$$

$$K_{\beta_{0i}} = \sqrt{\frac{I_i}{P_{ui} L_i^2} \pi^2 \frac{\sum_{i_{all}} \frac{P_{ui}}{L_i}}{\sum_{i_{rigid}} \frac{\beta_i I_i}{L_i^3}}} \quad (9b)$$

$$K_{\Delta_i} = \sqrt{\frac{I_i}{P_{ui} L_i^2} \frac{\pi^2 E \sum_{i_{all}} \frac{P_{ui}}{L_i} + \sum_{i_{rigid}} \frac{C_{Li} P_{ui}}{L_i}}{(\sum H_i) / \Delta_{oh}}} \quad (1b)$$

$$K_{\Delta_{0i}} = \sqrt{\frac{I_i}{P_{ui} L_i^2} \frac{\pi^2 E \sum_{i_{all}} \frac{P_{ui}}{L_i}}{(\sum H_i) / \Delta_{oh}}} \quad (8b)$$

$$K_{\beta_i} = \sqrt{\frac{I_i}{P_{ui} L_i^2} \pi^2 \frac{\sum_{i_{all}} \frac{P_{ui}}{L_i} + \sum_{i_{rigid}} \frac{C_{Li} P_{ui}}{L_i}}{\sum_{i_{rigid}} \frac{\beta_i I_i}{L_i^3}}} \quad (2b)$$

The values of K_{R_i} and K_{K_i} plotted in Figures 9a and 9b are calculated using Equations 6e and 7c. Using these equations, the errors of K_{R_i} , K_{K_i} , K_{Δ_i} , and K_{β_i} are less than 5 percent unconservative even if no limit besides K_{braced} is invoked, or they become conservative once the limits of Equations 6d (for K_{R_i} , and similarly for K_{Δ_i}) and 7b (for K_{K_i} , and similarly for K_{β_i}) are used. $K_{\Delta_{0i}}$ and $K_{\beta_{0i}}$ are 8 percent and 11 percent unconservative, respectively, at $\gamma = 2$, although they too become conservative if the limits of Equations 6d and 7b are invoked.

5. ANALYSIS OF THE ACCURACY OF EFFECTIVE LENGTH FORMULAS

The manner in which Equations 1 through 9 are derived is critical to the accuracy of the resulting effective length factors and to the proper prediction of the buckling load associated with a given story. In this section we consider these derivations further. First, the error associated with K_{R_i} is investigated. The equations and conclusions for K_{Δ_i} and $K_{\Delta_{0i}}$ are similar. Next the equivalent relationships are derived for K_{K_i} .

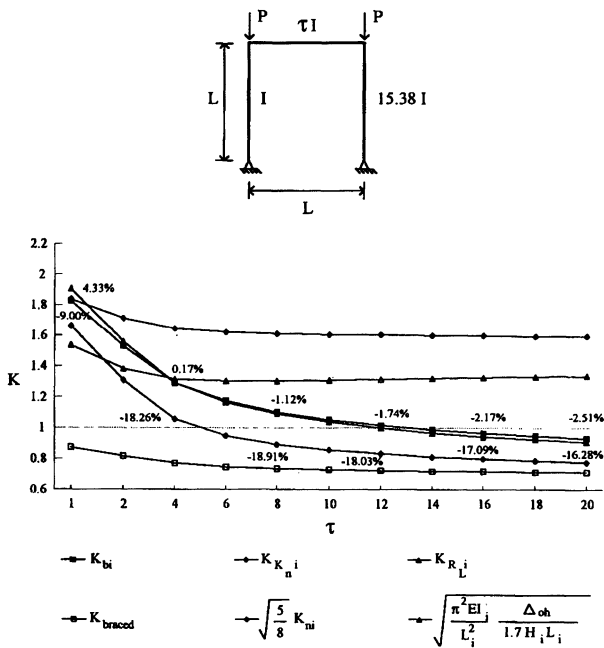


Fig. 7a. Results of study 2c (γ equals 15.38) using unmodified G factors.

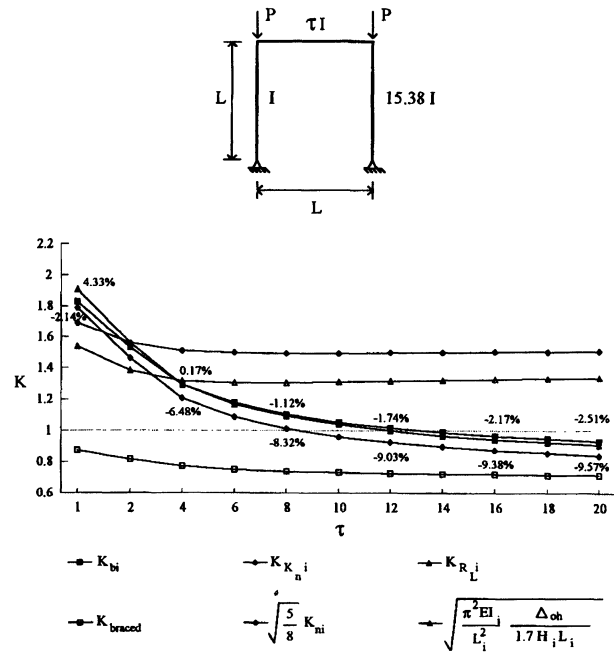


Fig. 7b. Results of study 2c (γ equals 15.38) using modified G factors.

5.1 Accuracy Associated with $K_{R,i}$

The derivation for K_{bi} from Equation 10, is based on the following equation (Liew 1991):

$$\lambda_b P_{ui} = \frac{\pi^2 EI_i}{L_i^2 K_{bi}^2} = P_{ebi} \quad (13)$$

where λ_b is the first mode eigenvalue from a buckling analysis in which the column under consideration has a reference force P_{ui} . Upon summing over all columns in the story (and noting that P_{ebi} equals zero for leaner columns), Equation 13 becomes:

$$\lambda_b \sum_{i_{all}} P_{ui} = \sum_{i_{rigid}} P_{ebi} \quad (14)$$

For purposes of discussion, Equation 13 may be rewritten as:

$$\lambda_b K_{bi}^2 = \frac{\pi^2 EI_i}{L_i^2 P_{ui}} \quad (15)$$

$K_{R,i}$ in turn, is derived based upon the following constraint (LeMessurier 1993, LeMessurier 1994, Squarzini 1993, Hajjar 1994):

$$\lambda_{R,i} \sum_{i_{all}} P_{ui} = \frac{\sum H_i L_i}{\Delta_{oh}} (0.85 + 0.15 R_L) = \sum_{i_{rigid}} P_{eR,i} \quad (16)$$

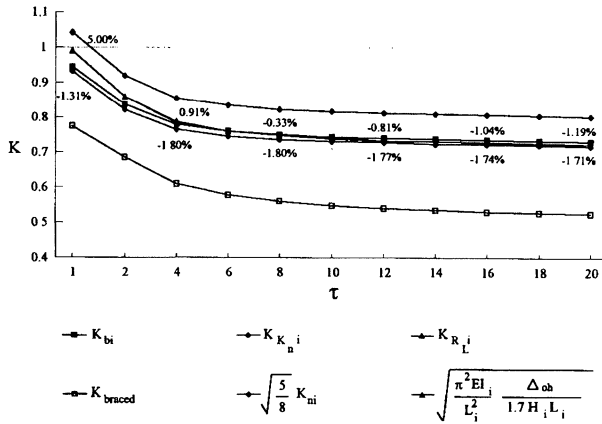
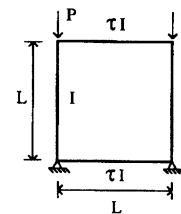
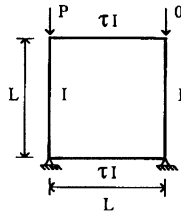


Fig. 8a. Results of study 3c (α equals zero) using unmodified G factors.

Then, for each individual column in the story, it is presumed that (recall Equation 6c):

$$P_{eR,i} = \lambda_{R,i} P_{ui} = \frac{\pi^2 EI_i}{K_{R,i}^2 L_i^2} \quad (17)$$

and therefore

$$\lambda_{R,i} K_{R,i}^2 = \frac{\pi^2 EI_i}{L_i^2 P_{ui}} \quad (18)$$

It should be noted that the right hand sides of Equations 15 and 18 are equal. Therefore, for any given column, i :

$$\lambda_b K_{bi}^2 = \lambda_{R,i} K_{R,i}^2 \quad (19)$$

or

$$\frac{K_{R,i}^2}{K_{bi}^2} = \frac{\lambda_b}{\lambda_{R,i}} = (1 + \epsilon_{R,i})^2 \quad (20)$$

where $\epsilon_{R,i}$ is the percent error (as computed in Sections 3 and 4) in the calculation of the individual column effective length factors $K_{R,i}$:

$$\epsilon_{R,i} = \frac{K_{R,i} - K_{bi}}{K_{bi}} \quad (21)$$

Therefore, from Equation 20, if $K_{R,i}$ for column i is in error by +5 percent, then the ratio $\lambda_b / \lambda_{R,i}$ equals 1.1025. While an engineer never actually calculates $\lambda_{R,i}$, its value is important:

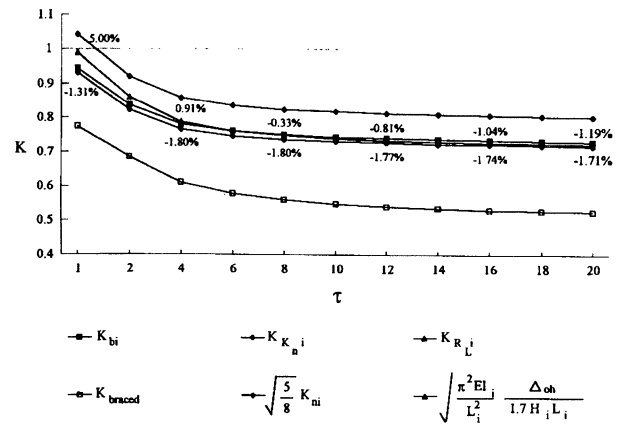


Fig. 8b. Results of study 3c (α equals zero) using modified G factors.

the closer it is to the story's first mode buckling eigenvalue, the more accurate the effective length factors $K_{R,i}$ are for that story, and vice versa. An expression for λ_{R_L} may be derived by dividing both sides of Equation 18 by $K_{R,i}^2$ and substituting in the square of Equation 6e:

$$\lambda_{R_L} = \frac{1}{\sum_{i_{all}} \frac{P_{ui}}{L_i}} \frac{\sum_{i_{rigid}} H_i}{\Delta_{oh}} (0.85 + 0.15R_L) \quad (22)$$

Together, Equations 20 and 22 account for all of the constraints represented in Equations 16 and 17. It is important to note that λ_{R_L} given by Equation 22 is used inherently in the calculation of $K_{R,i}$ for all rigidly-connected columns in the story (Equation 22 also confirms that λ_{R_L} is indeed constant for all columns in the story). Therefore, the error in the effective length factor computed from Equations 6, $\epsilon_{R,i}$, is the same for all rigidly-connected columns in the story. The effective length factors $K_{R,i}$ for all columns of the story are in essence computed based upon a single, presumed story buckling load (just as are the effective length factors $K_{b,i}$). Other columns in the story do not, as might be expected, "compensate" for one column's unconservative effective length factor by having conservative effective length factors. This fact is irrespective of whether any of the effective length factors of a story are less than one or all are greater than one. However, it should be noted that the above conclusion is valid only if no columns in the story breach the additional limiting condi-

tion of Equation 6d. If the limit specified by Equation 6d is breached, the buckling capacity of the column that is being effectively braced may drop substantially below that predicted based upon $K_{R,i}$ (i.e., using Equations 6a or 6e directly) (LeMessurier 1993, LeMessurier 1994, Squarzini 1993, Hajjar 1994a).

5.2 Accuracy Associated with $K_{K,i}$

A similar set of equations may be derived for $K_{K,i}$. The constraints governing the derivation of $K_{K,i}$ are (Liew 1991, Hajjar 1994):

$$\lambda_{K_n} \sum_{i_{all}} P_{ui} = \sum_{i_{rigid}} P_{eK,i} \quad (23)$$

Then, for each individual column in the story, it is presumed that:

$$\lambda_{K_n} P_{ui} = \frac{\pi^2 EI_i}{K_{K,i}^2 L_i^2} = P_{eK,i} \quad (24)$$

As with Equations 17 and 18, this equation may be rewritten as:

$$\lambda_{K_n} K_{K,i}^2 = \frac{\pi^2 EI_i}{L_i^2 P_{ui}} \quad (25)$$

Since the right hand sides of Equations 15 and 25 are equal for any given column i , we may write:

$$\lambda_b K_{b,i}^2 = \lambda_{K_n} K_{K,i}^2 \quad (26)$$

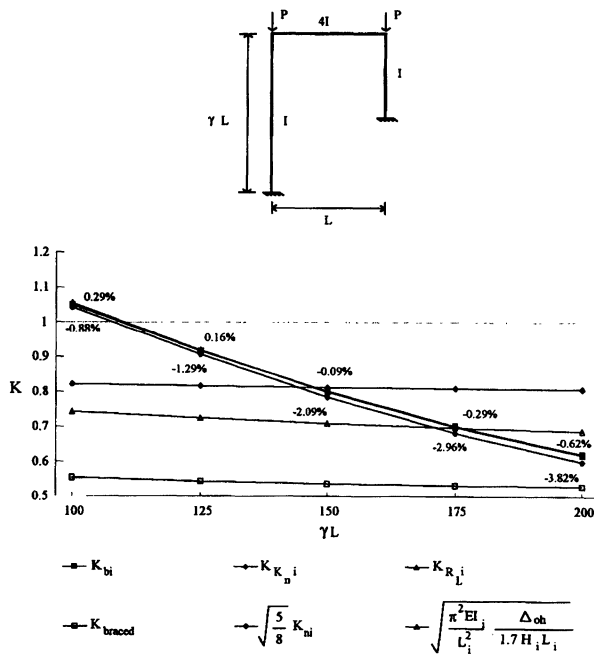


Fig. 9a. Results of study 4b (τ equals 4) using unmodified G factors.

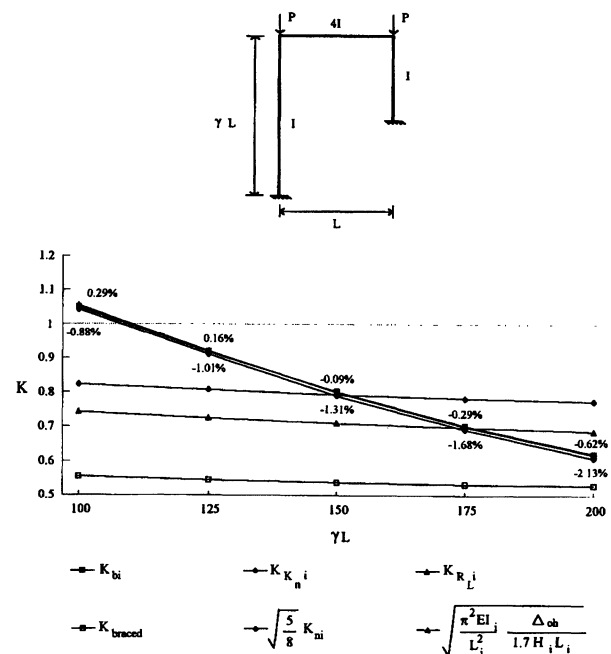


Fig. 9b. Results of study 4b (τ equals 4) using unmodified G factors.

or

$$\frac{K_{K_i}^2}{K_{bi}^2} = \frac{\lambda_b}{\lambda_{K_n}} = (1 + \epsilon_{K_i})^2 \quad (27)$$

where:

$$\epsilon_{K_i} = \frac{K_{K_i} - K_{bi}}{K_{bi}} \quad (28)$$

is the percent error associated with K_{K_i} . A similar set of conclusions may be drawn regarding the uniformity of error of the effective length factors K_{K_i} for all the columns in a story as was drawn for K_{R_i} as long as Equation 7c is used rather than Equation 7a (i.e., unequal length columns require revision of Equation 7a). This is irrespective of whether modified or unmodified G factors are used in the calculations. An expression for λ_{K_n} may be derived by dividing both sides of Equation 25 by $K_{K_i}^2$ and substituting in the square of Equation 7c:

$$\lambda_{R_L} = \pi^2 E \frac{\sum_{i_{rigid}} \frac{I_i}{K_n^2 L_i^3}}{\sum_{i_{all}} \frac{P_{ui}}{L_i}} \quad (29)$$

This value is the same for every column in the story. Similar equations and conclusions may be derived with respect to K_{β_i} and $K_{\beta_{0i}}$.

One may now see, by observing Equations 22 and 29, that all of the story-based effective length procedures are based upon a fundamental set of assumptions which take the following form. On the story level, it is presumed that the summation of the buckling capacities of the rigidly-connected columns of the story (calculated by one of the approaches discussed) equals the sum of the story's applied gravity loads times the story buckling parameter (whose value, as explained above, is implicitly assumed within the formulation). That is:

$$\lambda_{approach} \sum_{i_{all}} \frac{P_{ui}}{L_i} = \sum_{i_{rigid}} \frac{P_{critical(approach) i}}{L_i} \quad (30)$$

Therefore:

$$\lambda_{approach} = \frac{\sum_{i_{rigid}} \frac{P_{critical(approach) i}}{L_i}}{\sum_{i_{all}} \frac{P_{ui}}{L_i}} \quad (31)$$

Then, each individual column is presumed to adhere to the following rule:

$$P_{e(approach) i} = \frac{\pi^2 EI_i}{(K_{(approach) i} L_i)^2} = \lambda_{approach} P_{ui}$$

$$= \frac{\sum_{i_{rigid}} \frac{P_{critical(approach) i}}{L_i}}{\sum_{i_{all}} \frac{P_{ui}}{L_i}} (P_{ui}) \quad (32)$$

Note that Equations 22, 29, 30, 31, and 32 have been expressed in a form suitable for use in a story having columns of unequal length. For equal length columns in the story, Equation 32 may be stated as:

$$P_{e(approach) i} = \frac{\pi^2 EI_i}{(K_{(approach) i} L_i)^2} = \lambda_{approach} P_{ui}$$

$$= \frac{\sum_{i_{rigid}} P_{critical(approach) i}}{\sum_{i_{all}} P_{ui}} (P_{ui}) \quad (33)$$

All of the effective length procedures of Equations 1 through 9, plus other related story-based procedures in the literature (e.g., (Lui 1992)), basically adhere to these assumptions, and thus to the provision that the effective length factors of the rigidly-connected columns in the story are all in error by the same amount (again, so long as no column breaches the limits imposed on these effective length values).

6. CONCLUSIONS

The following conclusions may be drawn from these studies. First, if a story-based effective length factor less than one is to be used in the design of a column contributing to the lateral resistance of an unbraced frame, a minimum limiting value on that factor must be computed for the member. To insure the integrity of any of the unbraced K calculations summarized in Equations 1 through 9, the unbraced effective length factor should not drop below this limit. At a minimum, a braced effective length factor, computed from a sidesway inhibited nomograph (AISC 1993), must be used as the limit. However, to insure that braced buckling is captured comprehensively and that the unbraced effective length procedures are not in error by more than 5 percent (unconservative), the LRFD Commentary specifies that Equations 6d (for K_{R_i} , and similarly for K_{Δ_i}) and 7b (for K_{K_i} , and similarly for K_{β_i}) must be satisfied (AISC 1993). The results presented here illustrate that the effective length values obtained should be acceptable in many cases if the braced K is used whenever its value is greater than the unbraced K , and the effective length values will be either conservative or extremely accurate if the stricter limits of Equations 6d and 7b are adhered to.

The use of the larger effective length factor from sidesway and braced buckling calculations accounts partially for the aspect that a column may buckle in either a braced or an unbraced mode. As the structural characteristics are varied to offer increasing sidesway restraint to the column, the exact

effective length often exhibits a bifurcation behavior in the transition zone from an unbraced to a braced mode of buckling (LeMessurier 1993, Squarzini 1993, Hajjar 1994a). The limits of Equations 6d and 7b account for this transition zone in two ways. First, if a limit is not imposed on the capacity computed by Equations 6e or 7c, then as a weak column's buckling behavior approaches a braced mode but the column continues to buckle in a sidesway mode, the predicted column capacity may be unconservative if the P - δ effects on the story stability exceed the values presumed in these equations (recall that Equation 6e presumes that $C_{Li} = 0.176$ for all rigidly-connected columns in the story, and Equation 7c accounts for the P - δ effect to the extent permitted within the context of the assumptions of the sidesway uninhibited nomograph). Second, these limits insure that the capacity associated with any potential sidesway inhibited buckling mode is predicted conservatively. Nevertheless, any possible interaction between braced and unbraced modes of buckling failure is only approximated by imposing a limiting value on the sidesway effective length. It is difficult to make general conclusions regarding whether the maximum errors associated with approximating this interaction may or may not be significant for practical cases, although as discussed below, several of the approaches to calculating unbraced K factors work quite well for the studies presented in this paper.

A second conclusion is that the limits of Equations 6d and 7b clearly insure that all of the effective length procedures of Equations 1 through 7 are within 5 percent unconservative error (note that the 10.5 percent error of the spring study would have been reduced as well had the limit of Equation 6d been imposed in Section 3). However, these limits can cause the actual capacity of the column to be underestimated by a significant margin (LeMessurier 1993, Squarzini 1993, Hajjar 1994a).

Note that the studies shown in Figure 3, while relatively comprehensive, do not quite encompass problems which would exhibit the most unconservative error in K_{R_i} and K_{K_i} (i.e., 5 percent, including the use of the limits of Equations 6d and 7b). LeMessurier has proposed related one-bay, one-story frames which exhibit these slightly larger errors (LeMessurier 1993, Squarzini 1993, Hajjar 1994a). A particularly effective mechanism which may be used to increase the range of stability parameters ($L_i\sqrt{P_{ui}}/EI_i$) tested in such examples is to vary the axial force in the columns in addition to varying their relative stiffnesses (i.e., essentially to combine study 1 with a case such as study 3 of Figure 3) (LeMessurier 1993, Squarzini 1993, Hajjar 1994a).

A third conclusion is that K_{R_i} and K_{Δ_i} are consistently more accurate than K_{K_i} , K_{β_i} , and $K_{\beta_{oi}}$ if unmodified G factors are used and the limits of Equations 6d and 7b are not imposed. The values of K_{K_i} , K_{β_i} , and $K_{\beta_{oi}}$ using unmodified G factors and no limit can exhibit inaccuracies even when their values are greater than one, although they are often accurate in this range. Although the "theoretical" maximum unconservative

error observed in Section 3 (for a column pinned at its base) was 10.5 percent, even if only the limit of K_{braced} was invoked the unconservative error of both K_{R_i} and K_{Δ_i} never exceeded 7 percent for the planar bents shown in Figure 3 (the conservative error of these K factors never exceeded 6 percent). For these same studies, the unconservative error in $K_{\Delta_{oi}}$ (again, using only K_{braced} as a lower limit) ranged up to 14 percent, but was usually below 10 percent. On the other hand, a definitive range of parameters for which K_{K_i} , K_{β_i} , and particularly $K_{\beta_{oi}}$ are accurate is not evident for the cases studied (unless the limit of Equation 7b is imposed). The errors in these values regularly exceeded 10 percent for practical ranges of stiffness (if no limit was imposed). For example, in studies 1b and 2b, the error of K_{β_i} breaches 10 percent at the approximate values of γ of 8 and 7, respectively. For these same two studies, the error of $K_{\beta_{oi}}$ breaches 10 percent at the approximate values of γ of 4 and 3, respectively (at $\gamma = 20$, its error is 17 percent and 21 percent, respectively). However, the accuracy of K_{K_i} , K_{β_i} , and $K_{\beta_{oi}}$ all increase noticeably if modified G factors are used for their computation, although the errors of $K_{\beta_{oi}}$ still breach 10 percent regularly. Thus these studies also indicate that neglecting C_{Li} completely (i.e., using $K_{\beta_{oi}}$ or $K_{\Delta_{oi}}$) may be unwise.

When calculating any of these effective length factors for members in stories having columns of unequal length, it is critical to use Equations 1b, 2b, 6e, 7c, 8b, and 9b rather than Equations 1a, 2a, 6a, 7a, 8a, and 9a. Note that the LRFD Commentary specifies the formulas only for stories in which all columns are of equal length.

It should be recognized of course that the significance of any of these errors actually depends on the slenderness of the column. For a short column, large errors in the effective length may not be of consequence. In addition, a single column having a K factor less than one in an unbraced frame which contains many columns per story is unlikely to exhibit these larger errors. The spring study of a single column in Section 3 indicated that, as the load (i.e., α) on the other columns in the story is increased while holding their stiffnesses (R_c and R_t) constant, the error in K_{Δ_i} decreases. This trend is most likely true for K_{K_i} , K_{β_i} , and $K_{\beta_{oi}}$ as well. Thus, if the load in the right column of the planar bents of Section 4 is increased as γ is increased, for example, the errors in the K factor computations would decrease correspondingly from the values presented in Section 4. These larger loads on the right column of the two-column bent would most likely represent the relative values of applied loads as compared to stiffness in a typical multi-column unbraced story.

A fourth conclusion is that, as indicated in Section 5, for any given method of calculating effective length, the accuracy in the calculation of the effective length factors of all the columns of a story is the same as long as K for all of the columns in the story is greater than the limits imposed by Equations 6d and 7b. Also, the summation of the buckling loads of the columns computed using these effective length

factors (i.e., the story buckling strength) has corresponding inaccuracy when compared to the first mode eigenvalue obtained from a buckling analysis. Thus, it should be emphasized that, for any of the approaches to calculating effective length represented in Equations 1 through 9, the errors of these effective length factors are irrespective of whether any of the columns in the story have an effective length factor less than one. In fact, by insuring at a minimum that the effective length of a column cannot be below the braced effective length, the effective length factors which are less than one have equal or greater accuracy than the effective length factors which are greater than one in the story. Therefore, if an engineer considers $K_{R,i}$ and/or $K_{K,i}$ to be suitable for use when their values are greater than one, these K factors may also be considered suitable for use when their values are less than one.

K_{β_i} and its derivatives (including $K_{K,i}$) have poorer accuracy than K_{Δ_i} and its derivatives because of certain assumptions regarding the uniform nature of the frame's buckling mode which are made in the formulation of K_{β_i} (LeMessurier 1977). The buckling mode is complex for a column which participates in the lateral resistance of a frame, but whose K factor is less than one. In these cases, the column buckling mode is essentially a local, braced mode of buckling, but this occurs while the story as a whole is buckling in a sidesway mode. Using the results from a first-order analysis to compute Δ_{oh} for the calculation of K_{Δ_i} is not a foolproof or "exact" means of capturing this behavior, but it generally allows the nuances of the story's sway behavior to be incorporated into the K factor calculation, just as they are for the more "exact" eigenvalue buckling analysis (see (Hajjar 1992, Hajjar 1994) for a further discussion of accounting for story stability through the use of a first-order structural analysis). Also, since a value of C_{L_i} is assumed for all the columns of a story, $K_{R,i}$ has the distinct advantage of not requiring the most tedious portion of the nomograph effective length calculation to be performed: that of computing relative beam-to-column stiffnesses for use in Equations 5a and 5b (see (Lui 1992) for a related, alternative approach to calculating effective length). Note that the modified G factors require that the first-order lateral analysis be performed in addition to computing the relative stiffnesses, thus making their use quite inefficient. The primary drawback of using K_{Δ_i} or $K_{R,i}$ for design rather than K_{β_i} or $K_{K,i}$ is that inelastic effective length factors, as currently formulated (Yura 1971), can be incorporated only in conjunction with the G factors of Equation 5 or through the use of a column tangent modulus in the analysis.

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REFERENCES

- Abdelrazaq, A., Baker, W. F., Hajjar, J. F., and Sinn, R. C., "Column Buckling Considerations in High-Rise Buildings with Mega-Bracing," *Proceedings of the Structural Stability Research Council Annual Technical Session*, Milwaukee, Wisconsin, April 6-7, 1993, Structural Stability Research Council, Bethlehem, PA 1993.
- American Institute of Steel Construction, *Load and Resistance Factor Design Specification for Structural Steel Buildings*, Chicago, 1993.
- Baker, W. F., "Design of Steel Buildings for Second Order Effects," ASCE, *Materials and Member Behavior, Proceedings of the American Society of Civil Engineers Structures Congress 87*, Ellifritt, D. S. (ed.), Orlando, FL, August 17-20, 1987, New York, 1987, pp. 534-554.
- Baker, W. F., "Practical Problems in Inelastic Steel Design," ASCE, *Proceedings of the American Society of Civil Engineers Structures Congress 91*, Williamson, T. G. (ed.), Indianapolis, IN, April 29-May 1, 1991, New York, 1991, pp. 572-575.
- Bridge, R. Q. and Trahair, N. S., "Effects of Translational Restraint on Frame Buckling," *Civil Engineering Transactions of the Institute of Engineers, Australia*, 1977, pp. 176-183.
- Bridge, R. Q. and Fraser, D. J., "Improved G-Factor Method for Evaluating Effective Lengths of Columns," ASCE, *Journal of Structural Engineering*, Vol. 113, No. 6, June, 1987, pp. 1341-1356.
- Hajjar, J. F. and White, D. W., "On the Need for the Effective Length Concept in LRFD," ASCE, *Proceedings of the American Society of Civil Engineers Structures Congress 92*, Morgan, J. (ed.), San Antonio, TX, April 13-15, 1992, ASCE, New York, 1992, pp. 334-337.
- Hajjar, J. F., White, D. W., Clarke, M. J., Bridge, R. Q., Lui, E. M., Sheikh, T. M., and Leon, R. T., "Effective Length and Equivalent Imperfection Approaches for Assessing Frame Stability: Implications for Load and Resistance Factor Design," *Report to the American Society of Civil Engineers Technical Committee on Load and Resistance Factor Design*, in preparation (due 1995), 1994.
- Hajjar, J. F. and Squarzini, M. J., *An Evaluation of Techniques for Predicting Column Capacity in LRFD*, in preparation, 1994a.
- LeMessurier, W. J., "A Practical Method for Second-Order Analysis. Part 1: Pin Jointed Systems," AISC, *Engineering Journal*, Vol. 13, No. 4, 4th Qtr., 1976, pp. 89-96.
- LeMessurier, W. J., "A Practical Method for Second-Order Analysis. Part 2: Rigid Frames," AISC, *Engineering Journal*, Vol. 14, No. 2, 2nd Qtr., 1977, pp. 49-67.
- LeMessurier, W. J., personal communication, 1991.

LeMessurier, W. J., presentation made to the American Society of Civil Engineers Technical Committee on Load and Resistance Factor Design, *ASCE Structures Congress 93*, Irvine, CA, April, 1993.

LeMessurier, W. J., "A Practical Method for Second-Order Analysis. Part 3," in preparation, 1994.

Liew, J. Y. R., White, D. W., and Chen, W. F., "Beam-Column Design in Steel Frameworks—Insights on Current Methods and Trends," *Journal of Constructional Steel Research*, Vol. 18, No. 4, 1991, pp. 269–308.

Lui, E. M., "A Novel Approach for K Factor Determination," AISC, *Engineering Journal*, Vol. 29, No. 4, 4th Qtr., 1992, pp. 150–159.

Squarzini, M. J. and Hajjar, J. F., "An Evaluation of Proposed Techniques for Predicting Column Capacity," *Structural Engineering Report No. ST-93-3*, Department of Civil and Mineral Engineering, University of Minnesota, 1993.

Wu, H. B., "Determination of Effective Length of Unbraced Framed Columns," *Proceedings of the Structural Stability Research Council Annual Technical Session*, Cleveland, OH, April 16–17, 1985, Structural Stability Research Council, Bethlehem, PA, 1985, pp. 105–116.

Yura, J. A., "The Effective Length of Columns in Unbraced Frames," AISC, *Engineering Journal*, April, 1971, pp. 37–42.

APPENDIX A—NOTATION

C_{Li} = Stiffness reduction factor for column i
 E = Modulus of elasticity
 G_{bottom} = Ratio of column-to-beam stiffness at bottom of column
 G_{top} = Ratio of column-to-beam stiffness at top of column
 H_i = Horizontal load (i.e., shear) on individual column i in story
 I_b = Moment of inertia of beam
 I_c = Moment of inertia of column
 I_i = Moment of inertia of column i
 I_{xa} = strong-axis moment of inertia of column a
 I_{yb} = Weak-axis moment of inertia of column b
 $K_{(approach) i}$ = Effective length factor for column i computed using a particular approach
 K_{braced} = Sidesway inhibited nomograph effective length factor
 K_{bi} = Effective length factor for column i from eigenvalue buckling analysis
 K_e = Effective length factor from transcendental equation
 K_{K_i} = K factor for column i based upon story buckling load equaling sum of individual column buckling loads calculated using K_{ni}

K_{R_i} = K_{Δ_i} based on a conservative value of $C_L = 0.176$ for all rigidly connected columns
 K_{ni} = Sidesway uninhibited nomograph effective length factor
 K_r = Rotational spring stiffness
 K_t = Translational spring stiffness
 K_{β_i} = Effective length factor of column i computed using β_i
 $K_{\beta_{0i}}$ = K_{β_i} based on C_{Li} equal to zero
 K_{Δ_i} = Effective length factor of column i computed using Δ_{oh}
 $K_{\Delta_{0i}}$ = K_{Δ_i} based on C_{Li} equal to zero
 L_b = Length of beam
 L_c = Length of column
 L_i = Length of column i
 M_{far} = Bending moment at far end of beam
 M_{near} = Bending moment at near end of beam
 $P_{critical (approach) i}$ = Buckling capacity of column i as approximated by a particular approach to computing effective length
 $P_{e (approach) i}$ = Buckling capacity of column i computed using $K_{(approach) i}$
 P_{ebi} = Buckling capacity of column i computed using K_{bi}
 P_{eK_i} = Buckling capacity of column i computed using K_{K_i}
 P_{eR_i} = Buckling capacity of column i computed using K_{R_i}
 P_{ui} = Factored axial force in column i
 R_L = Ratio of gravity load on all leaner columns in a story to gravity load on all columns in the story
 R_r = Rotational spring stiffness factor
 R_t = Translational spring stiffness factor
 r = Radius of gyration of column
 Δ_{oh} = First order story deflection due to $\sum_{i_{rigid}} H_i$ on story
 α = Ratio of load on all the other columns in a story to load on the column being considered
 β_i = First order stiffness factor of column i
 ϵ_{K_i} = Percent error in calculation of K_{K_i}
 ϵ_{R_i} = Percent error in calculation of K_{R_i}
 γ = Stiffness factor of column
 $\lambda_{approach}$ = Buckling factor computed in conjunction with a particular approach to computing effective length
 λ_b = First buckling mode eigenvalue
 λ_{K_i} = Buckling factor computed in conjunction with K_{K_i}
 λ_{R_i} = Buckling factor computed in conjunction with K_{R_i}
 τ = Stiffness factor of beam