A Novel Approach for *K* Factor Determination

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INTRODUCTION

Current specifications[1,2] for the design of steel members and frames in the U.S. make extensive use of the effective length factor, K. The effective length factor is employed in the member interaction equations to facilitate the design of framed members by transforming an end-restrained compressive member to an equivalent pinned-ended member. In frame design, the effective length factor can also be regarded as a parameter which emanates the stability interaction effect among various members of the frame. At present, the effective length factor K for a framed member under compression is determined from a pair of alignment charts. Although the charts provide an easy and a convenient means for designers to evaluate the K factor, the models used in the development of these charts embody a number of assumptions which are not readily realized in actual situations. As a result, the K factor so obtained is often inaccurate. For instance, one assumption used is that all columns of a story reach a state of instability simultaneously. Mathematically, this requires that the quantity $L\sqrt{P / EI}$ (where L is the length, P is the axial force and EI is the flexural rigidity of the member) be equal for all columns of the story. If the alignment charts are employed to evaluate K factors for columns wherein the term $L\sqrt{P / EI}$ varies across the story, significant errors may result. Commonly encountered situations in which the quantity $L\sqrt{P / EI}$ varies include frames with unequal distribution of column axial loads in a story, frames for which the moment of inertia of the columns vary across a story, and frames with leaner columns.

Over the years, various papers[3–11] which address the inadequacies of the alignment charts for determining the effective length factors for framed columns have been published. Modifications to rectify certain deficiencies in the chart solutions were also reported. Nevertheless, all these approaches entail a procedure which continue to make use of the alignment charts for solutions. In some cases, special charts are also required to obtain solutions. In this paper, a simple and straightforward approach for determining the effective length factors for framed compression members which does not rely on the use of the alignment charts nor the use of any special charts is presented. The validity of the proposed approach will be demonstrated for frames with different geometries and loading conditions.

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THE *K* FACTOR

There are various approaches by which the effective length factor K can be determined. An eigenvalue analysis is perhaps the most accurate method to evaluate this K factor. For a frame subjected to a series of compressive forces $\beta_1 P$, $\beta_2 p$, ..., $\beta_n P$ acting on columns 1 to n, respectively, a stiffness equation of the form

$$[S_I + S_G]\,\{u\} = 0 \tag{1}$$

can be written, where S_I is the first-order structure stiffness matrix, S_G is the geometrical structure stiffness matrix and u is the structure displacement vector. In a linear eigenvalue analysis, S_G can be expressed as $-\lambda S'$ where λ is the eigenvalue of the problem linear in P. Thus, Eq. 1 can be written in the form

$$[S_I - \lambda S']\,\{u\} = 0 \tag{2}$$

from which λ can be solved from the equation

$$\det |S_I - \lambda S'| = 0 \tag{3}$$

Once λ is solved, the axial force in each individual column, P_i', can be calculated and the effective length factor for that column can be evaluated from the equation

$$K_i = \sqrt{\frac{\pi^2 EI_i}{P_i' L_i^2}} \tag{4}$$

where K_i is the effective length factor of column i; P_i', I_i, L_i are the axial (compression) force at buckling (i.e., the critical load), the moment of inertia and length of column i, respectively, and E is the modulus of elasticity. Equation 4 is applicable for isolated columns as well as for framed columns in multistory multibay frames. If P_i' equals zero, K_i is indefinite. This is because the effective length factor is defined only for members with finite compressive forces. Members which are subjected to negligible axial forces should be designed as beams which do not require the use of the K factors.

In evaluating the effective length factors for framed columns, it is important for a designer to account for the interaction effect that exists among the various members of the frame. It is a well-known fact that a "strong" column braces a "weak" column at buckling. The result is that the K factor of the stronger column increases and the K factor of the weaker column decreases as the difference in stiffness of the two columns increases. This phenomenon is illustrated numerically in Fig. 1 and Tables 1, 2 and 3.

Each of the three frames (Frames A, B, and C) in Fig. 1 consists of a "strong" column and a "weak" column. For Frame A, the "strong" column is the one with the higher moment of inertia (i.e., αI with $\alpha > 1$). The theoretical K factors of the two columns for different values of α evaluated using an eigenvalue analysis are shown in Table 1. When $\alpha = 1$, the two columns are identical and so their K factors are the same and are equal to 2. As α increases, the right column becomes stronger compared to the left column. The result is that the K factor of the stronger column increases while the K factor of the weaker column decreases. It is worthwhile to note that the effective length factor for the weaker column can have value less than unity even though the frame is buckled in a sway mode. In the context of design, it is possible to design the weaker column using $K < 1$ provided that a larger value of K is used for the design of the stronger column.

For Frame B, the "weak" column is the one which is subjected to a higher axial compressive load (i.e., βP with $\beta > 1$). When $\beta = 1$, both columns are subjected to the same loadings and so $K = 2$ for both columns. However, as β increases, the left column is carrying a higher axial load and becomes the "weak" column. When buckling occurs, the right column, which is the "strong" column, will brace the left column. The result is an increase in K for the right column and a decrease in K for the left column. Table 2 shows the variation of K for the two columns as β increases.

For Frame C, the two columns have different lengths. The "strong" column is the shorter column. As can be seen in Table 3, the phenomenon that the K factor of the stronger column increases and the K factor of the weaker column decreases as γ increases is observed.

Another commonly encountered situation which involves the interaction between a "strong" column and "weak" column is a leaner-column frame shown in Fig. 2. The leaner column, which is the "weak" column, provides no lateral resistance to the frame at buckling. Consequently, only the right column will be effective in resisting the P–Δ overturning moment which develops during buckling. When the load in the "weak" column increases, the K factor of the "strong" column increases as the P–Δ moment intensifies. This phe-

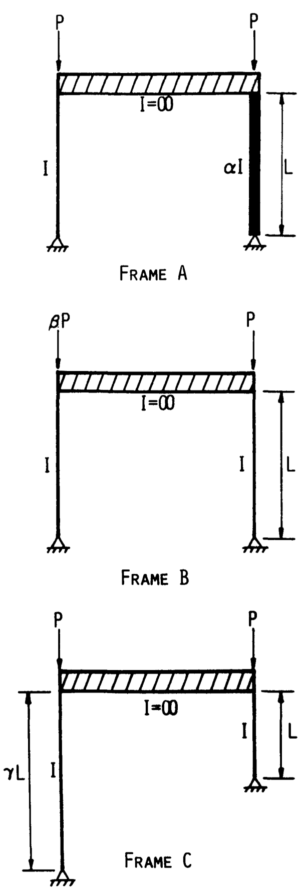


Fig. 1. Demonstration frames.

Table 1.
Theoretical K Factors for Columns of Frame A

$\alpha = I_{right\ column} / I_{left\ column}$	$K_{left\ column}$	$K_{right\ column}$
1	2.00	2.00
2	1.64	2.31
4	1.27	2.54
6	1.08	2.65
8	0.96	2.72
10	0.88	2.78

Table 2.
Theoretical K Factors for Columns of Frame B

$\beta = P_{left\ column} / P_{right\ column}$	$K_{left\ column}$	$K_{right\ column}$
1	2.00	2.00
2	1.73	2.45
4	1.59	3.17
6	1.53	3.76
8	1.51	4.26
10	1.49	4.72

Table 3.
Theoretical K Factors for Columns of Frame C

$\gamma = L_{left\ column} / L_{right\ column}$	$K_{left\ column}$	$K_{right\ column}$
1.0	2.00	2.00
1.5	1.51	2.27
2.0	1.16	2.31
2.5	0.93	2.32

nomenon is depicted in Table 4 in which theoretical values for the effective length factor of the "strong" column are shown as the applied load on the "weak" column increases.

The theoretical K factors for the leaner column are not shown in the table but they can be determined as follows: Recognizing that the leaner column is being "braced" by the rigid column, one can develop a simple model for the leaner column. The model is shown in Fig. 3. The portion of the frame which provides lateral stability to the structure is represented by a translational spring with a spring stiffness of S. An eigenvalue analysis of this system yields the solutions.

$$(\beta P)_{cr} = \text{smaller of} \begin{cases} \dfrac{\pi^2 EI}{L^2} \\ SL \end{cases} \tag{5}$$

Substituting Eq. 5 for P_i' into Eq. 4 gives

$$K = \text{larger of} \begin{cases} 1 \\ \sqrt{\dfrac{\pi^2 EI}{SL^3}} \end{cases} \tag{6}$$

For frames of usual proportions, the quantity $\pi^2 EI / SL^3$ normally does not exceed unity and so $K = 1$ often governs. In the context of design, a designer can use $K = 1$ for the leaner columns provided that accurate values of K are used for the rigid columns. In evaluating K factors for the rigid columns, the P–Δ effect generated in the leaner column must be considered to reflect the destabilizing influence of the leaner column has on the overall stability of the frame.

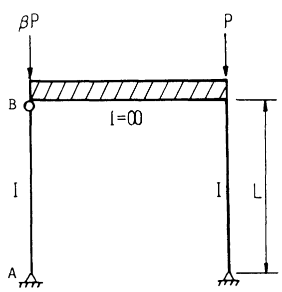


Fig. 2. Leaner-column frame.

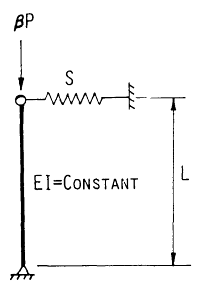


Fig. 3. Model for the leaner column.

Table 4.
Theoretical K Factors for the Rigid Column of the Leaner Column Frame

$\beta = P_{left\ column} / P_{right\ column}$	$K_{right column}$
0	2.000
2	3.249
4	4.139
6	4.871
8	5.502
10	6.077

For all the cases presented in the preceding discussions, a direct use of the alignment chart gives $K = 2$ for all the rigid columns. Significant errors are observed for a number of cases because the alignment charts were not developed to account for the interaction effect that occurs among columns having different values of $L\sqrt{P / EI}$. In what follows, a simple yet accurate procedure to determine the elastic K factors for framed columns will be developed. The procedure makes use of the correlation between stability and magnification effects on frames. The validity of the proposed procedure will be demonstrated by numerical examples.

DERIVATION OF THE PROPOSED *K* FACTOR

When members of a frame are subjected to compressive forces, two types of instability effects will arise. Member instability (or P–δ) effect arises as the axial force in the member acts through the lateral displacement of the member relative to its chord. Frame instability (or P–Δ) effect arises as the axial force acts through the relative end displacements of the member. Both types of instabilities affect the effective length factor of the member. Member instability reduces the flexural rigidity of the member whereas frame instability increases the drift and hence the overturning moment of the frame. This increase in moment is often accounted for in design by a moment magnification factor. It should be noted that a correlation exists between this moment magnification effect and the K factor. This correlation will be explored in further detail in a later section of this paper. In the meantime, a simple formula for calculating K factors for framed columns will be derived. The proposed K factor formula accounts for both the member and frame instability effects explicitly and it gives accurate results for frames which exhibit the strong column-weak column phenomenon.

For the sake of clarity, member instability and frame instability effects will be treated separately in the formulation.

Member Instability Effect

In a theoretical context member instability (P–Δ) effect is accounted for by the use of stability functions.[12] For the member shown in Fig. 4, the slope-deflection equations relating the member-end moments (M_A, M_B) and member-end rotations (θ_A, θ_B) are given by

$$M_A = \frac{EI}{L}(s_{ii}\theta_A + s_{ij}\theta_B) \tag{7a}$$

$$M_B = \frac{EI}{L}(s_{ij}\theta_A + s_{ii}\theta_B) \tag{7b}$$

where EI is the flexural rigidity and L is the length of the member. s_{ii} and s_{ij} are stability functions which are expressed in terms of the axial force P in the member. Expressions for these functions are given in Ref. 12 and will not be shown here. Simplified forms for these functions will be used in this paper.

If the axial force P in the member is negligible, Eqs. 7a and 7b reduce to

$$M_A = \frac{EI}{L}(4\theta_A + 2\theta_B) \tag{8a}$$

$$M_B = \frac{EI}{L}(2\theta_A + 4\theta_B) \tag{8b}$$

For the case in which the member bends in reverse curvature so that $\theta_A = \theta_B = \theta$, Eqs. 7a and 7b become

$$M_A = M_B = \frac{EI}{L}(s_{ii} + s_{ij})\,\theta \tag{9}$$

and Eqs. 8a and 8b become

$$M_A = M_B = \frac{6EI}{L}\theta \tag{10}$$

Using Taylor series expansion for ($s_{ii} + s_{ij}$), we obtain

$$s_{ii} + s_{ij} = 6 - \frac{PL^2}{10EI} + \ldots \tag{11}$$

Substituting Eq. 11 into Eq. 9, we have

$$M_A = M_B \approx \frac{6EI}{L}\left(1 - \frac{PL^2}{60EI}\right)\theta \tag{12}$$

The approximation sign is used in the above equation because only two terms are retained in the Taylor series expansion.

Upon comparison of Eq. 12 with Eq. 10, it can be concluded that when a member bends in reverse curvature, the member instability effect reduces the flexural rigidity of the member by an amount of $1 - PL^2 / 60EI$.

Similarly, for the case in which the member bends in single curvature so that $\theta_A = -\theta_B = \theta$, Eqs. 7a and 7b become

$$M_A = -M_B = \frac{EI}{L}(s_{ii} - s_{ij})\,\theta \tag{13}$$

and Eqs. 8a and 8b become

$$M_A = -M_B = \frac{2EI}{L}\theta \tag{14}$$

Again, using Taylor series expansion for $s_{ii} - s_{ij}$, we obtain

$$s_{ii} - s_{ij} = 2 - \frac{PL^2}{6EI} + \ldots \tag{15}$$

Retaining the first two terms in the series and substituting the result into Eq. 13, we have

$$M_A = -M_B \approx \frac{2EI}{L}\left(1 - \frac{PL^2}{12EI}\right)\theta \tag{16}$$

Upon comparison of Eq. 16 with Eq. 14, it can be seen that the member instability effect for a member bends in single curvature reduces the flexural rigidity of the member by an amount of $(1 - PL^2 / 12EI)$.

Finally, for the case in which one of the member-end moment (say, M_A) is zero, Eq. 7b becomes

$$M_B = \frac{EI}{L}\left(s_{ii} - \frac{s_{ij}^2}{s_{ii}}\right)\theta_B \tag{17}$$

and Eq. 8b becomes

$$M_B = \frac{3EI}{L}\theta_B \tag{18}$$

A Taylor series expansion for the terms in parenthesis in Eq. 17 gives

$$s_{ii} - \frac{s_{ij}^2}{s_{ii}} = 3 - \frac{PL^2}{5EI} + \ldots \tag{19}$$

which, upon substitution into Eq. 17 gives

$$M_B \approx \frac{3EI}{L}\left(1 - \frac{PL^2}{15EI}\right)\theta_B \tag{20}$$

A comparison between Eq. 20 and Eq. 18 reveals that the member instability effect reduces the flexural rigidity of this member by a factor of $1 - PL^2 / 15EI$.

In the foregoing discussions, it was seen that when $M_A / M_B = 1$, member instability effect would reduce the flexural stiffness of the member by a factor of $1 - PL^2 / 60EI$. When $M_A / M_B = 1$, this stiffness reduction factor was $1 - PL^2 / 15EI$, and when $M_A / M_B = 0$ the factor was $1 - PL^2 / 15EI$. Assuming that the stiffness reduction factor varies parabolically from a moment ratio of -1 to 1, a general stiffness reduction factor suitable for any moment ratio which can be

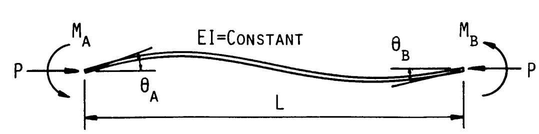


Fig. 4. A beam-column element.

used to account for the member instability effect can be written as

$$r = 1 - \frac{P}{5\eta L} \tag{21}$$

where

$$\eta = \frac{(3 + 4.8m + 4.2m^2)EI}{L^3} \tag{22}$$

In the above equations, r is the member instability (P–δ) stiffness reduction factor, η is the member stiffness index, P is the compressive axial force in the member, EI is the flexural rigidity of the member, and m is the ratio of the smaller to larger end moments of the member; m is taken as positive if the member bends in reverse curvature and it is taken as negative if the member bends in single curvature. Theoretically, the end moments used for calculating this moment ratio should be the moments developed in the member when the frame buckles. Since exact values for these moments are difficult to obtain, a simplified procedure will be used to obtain approximate values for these moments. In this procedure, a small disturbing force equal to a fraction of the story gravity loads is applied laterally to the frame. The moments developed in the member due to this disturbing force are used to calculate the moment ratio in Eq. 22. This procedure is demonstrated in an illustrative example in a later section of this paper.

Equation 21 indicates that the effect of member instability (i.e., the P–δ effect) can be expressed as a function of the moment ratio of the member. The use of moment ratio to account for the P–δ effect is not uncommon in design practice. For instance, the P–δ moment magnification factor B_1 used in the current AISC-LRFD Specification[1] is also expressed as a function of moment ratio of the member under consideration. The inclusion of member instability effect in the formulation of a K factor equation is indispensable if the interaction effect between member and frame instability is to be accounted for. The use of moment ratio implicitly takes account of the interaction effect of the various members of the frame. If the alignment charts were used, this interaction effect was accounted for by the G factors. A drawback for the G factors is that they only account for the interaction effect of members in the immediate neighborhood of the member under investigation. The proposed approach does not suffer from this shortcoming because the member-end moments to be used in Eq. 22 are determined from a global frame analysis.

Frame Instability Effect

In the context of design, frame instability is conveniently accounted for by the use of the story stiffness concept. If we denote s_{Im} as the first-order member lateral stiffness and S as the story stiffness accounting for the P–Δ effect. The two stiffness are related by the equation

$$S = \Sigma r s_{Im} - \Sigma\left(\frac{P}{L}\right) \tag{23}$$

where r is the member instability reduction factor defined in Eq. 21, $\Sigma(P / L)$ is the sum of the axial load to length ratio for all members of the story.

Since the first-order member lateral stiffness s_{Im} is proportional to the member stiffness index η defined in Eq. 22, Eq. 23 can be written in the form

$$S = \Sigma r\left(\frac{\eta}{\Sigma\eta}S_I\right) - \Sigma\left(\frac{P}{L}\right) \tag{24}$$

or

$$S = \left[\Sigma\left(r\frac{\eta}{\Sigma\eta}\right) - \Sigma\left(\frac{P}{L}\right)\left(\frac{1}{S_I}\right)\right]S_I \tag{25}$$

where S_I is the first-order story stiffness.

Using Eq. 21 and substituting $\Sigma H / \Delta_I$ (where ΣH is story lateral loads producing Δ_I, and Δ_I is the first-order inter-story deflection) for S_I into Eq. 25, we obtain

$$S = \left[\left(1 - \frac{\Sigma(P / L)}{5\Sigma\eta}\right) - \Sigma\left(\frac{P}{L}\right)\left(\frac{\Delta_r}{\Sigma H}\right)\right]S_I \tag{26}$$

The terms in brackets is the stiffness reduction factor for the story. Inverse of this factor is the moment magnification factor, A_F

$$A_F = \frac{1}{\left(1 - \frac{\Sigma(P / L)}{5\Sigma\eta}\right) - \Sigma\left(\frac{P}{L}\right)\left(\frac{\Delta_I}{\Sigma H}\right)} \tag{27}$$

The similarity in form between Eq. 27 and Eq. H1-5 of the AISC-LRFD Specification[1] is apparent. In fact, if the member instability effect is ignored, the term $\Sigma(P / L) / 5\Sigma\eta$ vanishes and Eq. 27 will be reduced to Eq. H1-5.

Proposed K factor Formula

Equation 27 is applicable to all members of the story. Suppose we are interested in calculating the K factor for the i-th member, we can equate Eq. 27 with the member moment magnification factor[12]

$$(A_F)_i = \frac{1}{1 - \frac{P_i}{(P_{ek})_i}} \tag{28}$$

where $(P_{ek})_i = \pi^2 EI_i / (K_i L_i)^2$.

Equating Eqs. 27 and 28, and solving for K_i, we obtain

$$K_i = \sqrt{\left(\frac{\pi^2 EI_i}{P_i L_i^2}\right)\left[\left(\Sigma\frac{P}{L}\right)\left(\frac{1}{5\Sigma\eta} + \frac{\Delta_I}{\Sigma H}\right)\right]} \tag{29}$$

Equation 29 is the proposed K factor formula. In the equation, EI_i and L_i are the flexural rigidity and length of the member, respectively. P_i is the axial force in the member, $\Sigma(P / L)$ is the sum of the axial force to length ratio of *all* members in a story, ΣH is the story lateral loads producing Δ_I, Δ_I is the first-order inter-story deflection, and η is the member stiffness index defined in Eq. 22. It is important to note that the term ΣH used in Eq. 29 is *not* the actual applied lateral load. Rather, it is a small disturbing force (taken as a fraction of the story gravity loads) to be applied to each story of the frame. This disturbing force is applied in a direction such that the displaced configuration of the frame will resemble its buckled shape. The member-end moments calculated using a first-order analysis under the action of this disturbing force will be used in Eq. 22 to evaluate the member stiffness index.

The derivation of Eq. 29 takes into account both the P–δ and P–Δ effects that are present in the frame at buckling. As a result, the equation is expected to give accurate results for design. In applying Eq. 29, the designer must perform a first-order frame analysis under a small disturbing force ΣH to determine Δ_I and the member-end moments. The member stiffness index η (Eq. 22) is then calculated for each member. Once η and Δ_I are calculated, Eq. 29 can be used to calculate K. The procedure will be demonstrated in an illustrative example in a following section.

Before proceeding any further, it is of interest to compare Eq. 29 with Eq. 4. In Eq. 4, the term P_i' is the axial force in the column at buckling (i.e., the critical load). Both the P–δ and P–Δ effects are implicit in P_i'. In Eq. 29, P_i is the axial force in the column without accounting for the two instability effects. These effects are accounted for explicitly by the terms in brackets. A relationship between P_i' and P_i can be obtained by equating the two equations giving

$$\frac{P_i}{P_i'} = \left(\Sigma\frac{P}{L}\right)\left(\frac{1}{5\Sigma\eta} + \frac{\Delta_I}{\Sigma H}\right) \tag{30}$$

An advantage of using Eq. 29 over Eq. 4 is that all terms in Eq. 29 can be obtained readily by inspection or from a simple first-order analysis. As will be demonstrated later, despite the simplicity in form, Eq. 29 gives sufficiently accurate results for design purposes.

FURTHER STUDIES OF THE PROPOSED *K* FACTOR EQUATION

As mentioned earlier in this paper, a correlation exists between the K factor and the moment magnification effect. This correlation is rather transparent in Eq. 29. Recalling that the terms in brackets represent the instability effects associated with frame buckling, it is not difficult to infer that as these effects intensify, the K factor increases for the member. From Eq. 28, it can be seen readily that A_F increases with K. Thus, an accurate assessment for K is rather important in a valid limit state design of frames subjected to heavy gravity loads.

In what follows, it will be shown that the proposed K factor equation can be reduced to other K factor formulas proposed in the past by other researchers.

Consider the case in which the P–δ effect is negligible, we can ignore the member stiffness reduction effect and disregard the term $\Sigma(P / L) / 5\Sigma\eta$ in Eq. 29. Setting $\Sigma(P / L) / 5\Sigma\eta = 0$ and substituting Eq. H1-5 of the AISC-LRFD Specification,[1] i.e.,

$$A_F = \frac{1}{1 - \dfrac{\Sigma P\Delta_I}{\Sigma HL}} \quad or \quad \frac{\Sigma P\Delta_I}{\Sigma HL} = 1 - \frac{1}{A_F} \tag{31}$$

into Eq. 29, we obtain

$$K_i = \sqrt{\frac{\pi^2 EI_i}{P_i L_i^2}\left[1 - \frac{1}{A_F}\right]} \tag{32}$$

Equation 32 was proposed by Cheong-Siat-Moy.[13] In Ref. 13, A_F is defined as the ratio of the second-order deflection to the first-order deflection of a given story. Thus, the use of Eq. 32 necessitates a second-order frame analysis. On the contrary, the use of the proposed K factor equation (Eq. 29) only requires the designer to perform a first-order analysis.

Now, suppose we use Eq. H1-6 of the AISC-LRFD Specification[2] as the P–Δ moment magnification factor, i.e.,

$$A_F = \frac{1}{1 - \dfrac{\Sigma P}{\Sigma P_{ek}}} \tag{33}$$

Substituting the above equation for A_F into Eq. 32, we obtain

$$K_i = \sqrt{\frac{\pi^2 EI_i}{P_i L_i^2}\left(\frac{\Sigma P}{\Sigma P_{ek}}\right)} \tag{34}$$

Equation 34 was proposed by LeMessurier[14] for the evaluation of effective length factors for framed columns. A more elaborate formula for K was also proposed by LeMessurier.[4] However, the application of the LeMessurier's formulas requires the use of the alignment chart for solutions. The use of Eq. 29, on the other hand, is completely independent of the alignment chart solutions.

AN ILLUSTRATIVE EXAMPLE

To demonstrate the procedure for applying Eq. 29 to determine K factors for columns in sway frames, the frame shown in Fig. 5a will be used. To initiate sway in a buckling analysis, a small lateral load (a disturbing force) ΣH equals to 0.1 percent of the story gravity loads (i.e., $\Sigma H = 0.1$ percent $\times$ $5P = 0.005P$) is applied to the frame. This is shown in Fig. 5b. The value of 0.1 percent was selected purely for conceptual purpose. In practice, any value can be chosen since the quantities $\Delta_I / \Sigma H$ and M_A / M_B required for applying Eq. 29

Table 5									
Column	***I***	***L***	***P***	M_A	M_B	$m = M_A / M_B$	η	***P / L***	***K***
left	288	240	2*P*	0.419*P*	0.530*P*	0.791	5.89	0.00833*P*	1.35
right	480	480	2*P*	0.245*P*	0.256*P*	0.957	1.49	0.00625*P*	0.71

are not affected by the value of lateral load used. This is because in a first-order analysis, all quantities vary linearly with the applied load and so the ratio of the quantities will remain unchanged.

It is important to note that the term ΣH represents a small disturbing force. It is *not* the actual lateral load that the frame may be subjected to. In fact, for frames which are subjected to a system of lateral loads, these lateral loads should be removed in the analysis for the effective length factor *K*. The reason for this is that in a buckling analysis, only the effect of axial forces but not the lateral forces should be considered. The purpose of applying a small disturbing force to the frame is merely to establish an adjacent equilibrium configuration for the frame. This adjacent equilibrium configuration will be the preferred configuration for the frame when the original configuration ceases to be stable once the axial loads in the columns reach their critical values. In theory, this adjacent equilibrium configuration is the buckled shape of the frame. The exact buckled configuration of the frame can be obtained from an eigenvalue analysis. In practice, this buckled configuration can be approximated by subjecting the frame to a small disturbing force as shown in Fig. 5b. The direction of this disturbing force is applied from left to right for this problem because the structural geometry and loading are such that the frame will most likely buckle in that direction. For frames which exhibit no preferred direction for buckling (e.g., frames which are symmetric in terms of both structural geometry and loading), the direction of this disturbing force is unimportant.

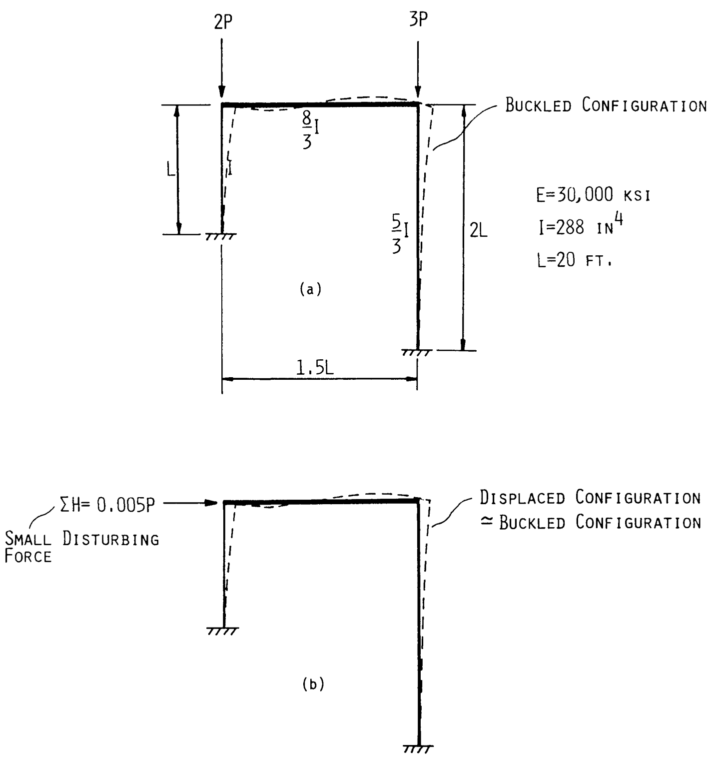


Fig. 5. An unequal leg frame.

Applying a disturbing force ΣH of 0.005*P* to the frame, a first-order analysis yields $\Delta_I = 7.13 \times 10^{-4}P$. So $\Delta_I / \Sigma H = 0.143$. The remaining calculations are depicted in tabulated form (see Table 5). (Units are in kips and inches).

For comparison, the theoretical *K* values obtained from an eigenvalue analysis[15] are 1.347 and 0.710 for the left and right columns, respectively. Thus, excellent correlation is observed.

If one uses the alignment chart, the *K* factor are obtained as 1.09 for the left column and 1.07 for the right column. The errors are rather significant. Using Eq. 34, the *K* factors are calculated to be 1.44 and 0.76 for the left and right columns, respectively.

NUMERICAL EXAMPLES

A valid *K* factor formula suitable for design application must satisfy the following criteria:

1. Simple to use
2. Transparent in form
3. Versatile in application
4. Accurate for design purpose

The proposed formula is simple to apply since it only requires the user to perform a first-order analysis; the use of special charts are not required. It is transparent in form because the two instability effects (P–δ and P–Δ) that have a predominant influence on *K* are explicitly accounted for in the equation. In what follows, it will be demonstrated that the proposed equation is also applicable to a variety of conditions and it gives sufficiently accurate results for design application.

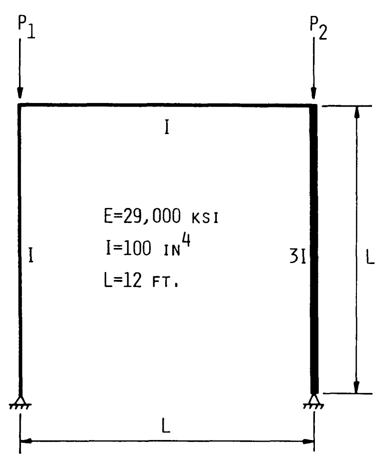
Example 1

The objective of this example is to demonstrate that the proposed *K* factor equation is applicable for frames with unequal distribution of column stiffness and gravity loads. The demonstration frame is shown in Fig. 6. The frame is a simple portal frame and consists of one beam and two col-

Table 6					
		K Factors			
Load Case	Column	Eq. 29	Theoretical	Alignment Chart	Eq. 34
A	left	1.42	1.46	2.3	1.35
B	left	2.01	1.99	2.3	1.91
	right	3.48	3.44	2.9	3.32
C	right	2.46	2.38	2.9	2.34

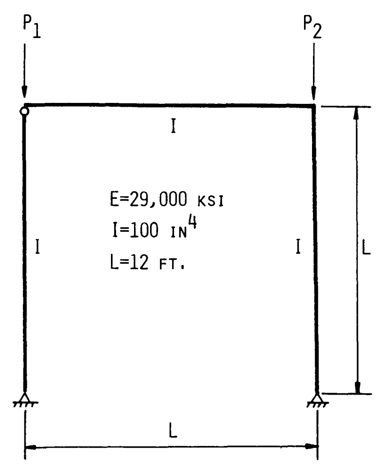
Table 7					
		K Factors			
Load Case	Column	Eq. 29	Theoretical	Alignment Chart	Eq. 34
A	right	3.70	3.69	2.60	3.68
B	right	2.62	2.64	2.60	2.60

umns. The flexural rigidity of the right column is three times that of the beam and the left column. Three load cases are investigated. In Load Case A, a gravity load of $2P$ is applied to the left column only. In Load Case B, the gravity load of $2P$ is evenly distributed on the columns. In Load Case C, all gravity loads are applied on the right column. As in the illustration example shown earlier, a small disturbing force of 0.1 percent times the total gravity loads acting on the frame (i.e., 0.1 percent × $2P$ = $0.002P$) was applied laterally to the frame to establish an adjacent equilibrium configuration for the frame from which the moment ratios were calculated using a first order analysis. The K factors for the loaded columns evaluated using Eq. 29 are compared with those evaluated using an eigenvalue analysis as well as those evalu-



Load Case	P_1	P_2
A	2P	0
B	P	P
C	0	2P

Fig. 6. Frame for Example 1.



Load Case	P_1	P_2
A	P	P
B	0	2P

Fig. 7. Frame for Example 2.

Table 8							
First-story							
Column	***I***	***L***	***P***	***m***	η	***P/L***	***K***
left	920	138.96	352.5	0.735	87.5	2.537	1.39
middle	626	138.96	604.5	0.935	75.5	4.350	0.87
right	472	138.96	252.0	0.878	53.3	1.813	1.18
Second-story							
Column	***I***	***L***	***P***	***m***	η	***P/L***	***K***
left	470	120	105.0	0.382	42.9	0.875	1.58
middle	470	120	183.4	0.947	89.2	1.528	1.21
right	200	120	78.4	0.800	32.0	0.653	1.20

Table 9					
		K Factors			
Story	**Column**	**Eq. 29**	**Theoretical**	**Alignment Chart**	**Eq. 34**
first	left	1.39	1.36	1.19	1.40
	middle	0.87	0.86	1.06	0.88
	right	1.18	1.16	1.07	1.18
second	left	1.58	1.73	1.25	1.40
	middle	1.21	1.31	1.11	1.06
	right	1.20	1.30	1.12	1.05

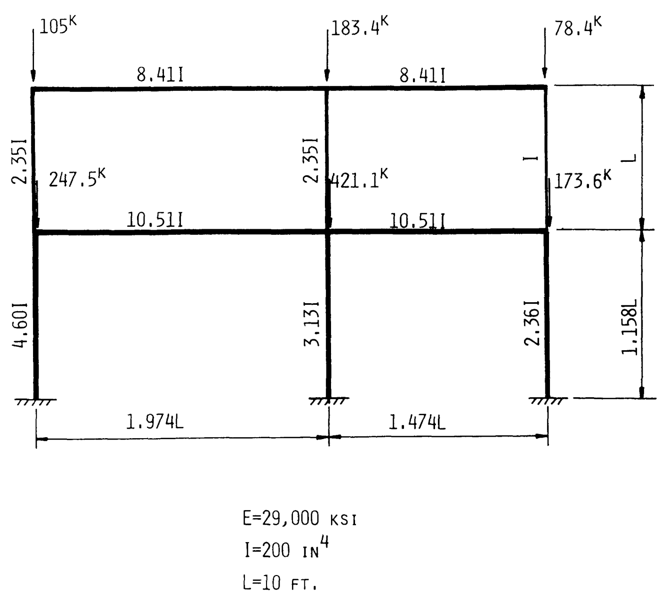


Fig. 8. Frame for Example 3.

ated using the alignment chart and Eq. 34 in Table 6. (Note that K is not defined for the unloaded columns).

From Table 6, it can be seen that Eq. 29 gives sufficiently accurate results for K and that significant errors are incurred by using the alignment charts. If the alignment charts are to be used, the solutions must be refined by using Eq. 34.

Example 2

In this example the ability of the proposed K factor equation to evaluate effective length factors for columns in frame with leaner columns will be demonstrated. Such a frame is shown in Fig. 7. Two load cases are used. In Load Case A, a gravity load of P is applied on each column and in Load Case B, the entire gravity load of $2P$ is applied on the right column. To establish an adjacent equilibrium position for the frame, a small disturbing force of 0.1 percent $\times$ $2P = 0.002P$ was applied laterally to the frame and the moment ratios were calculated using a first-order analysis. The K factors for the right column evaluated using Eq. 29, the alignment chart, and Eq. 34 are compared with the theoretical K factors evaluated using an eigenvalue analysis in Table 7.

Again, the accuracy of Eq. 29 and the inability of the alignment chart to give correct values of K are demonstrated.

Example 3

As a final example, Eq. 29 is used to calculate the K factors for the six columns of the two-story two-bay frame shown in Fig. 8. The small disturbing forces required to establish an adjacent equilibrium position for the frame are calculated from the equation 0.1 percent × Applied Story Gravity Load. This gives a value of 0.367 kips for the top story and 0.842 kips for the bottom story. Subjecting the frame to these disturbing forces, a first-order analysis gives $\Delta_I / \Sigma H$ = 0.00483 for the first story and 0.00792 for the second story. The remaining calculations are depicted in tabulated form shown in Table 8. (Units are in kips and inches.)

For purpose of comparison, values of the K factors obtained using different approaches are shown in Table 9.

SUMMARY AND CONCLUSIONS

A simple and effective formula for evaluating elastic effective length factors for framed columns in sway frames was derived. The formula takes into consideration the member instability and frame instability effects explicitly. As a result, in addition to providing the users with a clear physical picture of the two destabilizing influences on column stability, the formula gives reasonably accurate results for design application. The explicit consideration of the two instability effects also eliminates the need for a second-order analysis. The application of the proposed formula only requires the user to perform a first-order analysis. No special charts or iterations are required for solutions. The formula provides sufficiently accurate estimates for K factors of columns in frames with unequal distribution of column stiffness, unequal distribution of gravity loads and for frames with leaner columns. The validity of the proposed K factor equation when applied to these cases was demonstrated by numerical examples.

The applicability of Eq. 29 for determining K factors of columns stressed into the inelastic range is currently being investigated. The approach makes use of the tangent modulus concept and uses the tangent modulus E_t in place of the elastic modulus E in determining the various parameters in Eq. 29. Detailed discussion for determining this inelastic K factor will be addressed in a subsequent paper.

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