Simple Equations for Effective Length Factors

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In theory at least, the design of a column or of a beam-column starts with the evaluation of the elastic restraints at both ends of the column, from which the effective length factor K is then derived. To get a K-factor, the designer is much more likely to use the two charts provided in the Column Design section of the AISC Manuals,[1,2] rather than to solve the transcendental equations on which the charts are based.

However, having to read K-factors from an alignment chart in the middle of an electronic computation, in a spreadsheet for instance, prevents full automation and can be a source of errors. The fact that spreadsheets cannot accept so-called circular references makes their use awkward for the automatic solution of transcendental equations. A side benefit of an excellent article by Barakat and Chen[3] was the demonstration of how powerful an engineering tool the electronic spreadsheet can be: it automates many routine calculations, and it is well suited for tedious column and beam-column calculations. Barakat and Chen did not elaborate on how they obtained the K-factors used in their examples; from the context, it seems that the factors were manually entered into the spreadsheet. Obviously, it would be convenient to have simple equations take the place of the charts in the AISC Manuals. The American Concrete Institute[4] does publish equations, but their lack of accuracy may be why they seem not to be used in steel design. Better equations have been available in the *French Design Rules for Steel Structures*[5] since 1966, and have been included in the European Recommendations[6] of 1978, with only a change in notation. These equations are accurate, yet simple enough to be easily programmed within the confines of a spreadsheet cell. For this reason, they may be useful to North American engineers.

1. EXACT AND APPROXIMATE EQUATIONS

Consider a column AB elastically restrained at both ends. The rotational restraint at one end, A for instance, is represented by a restraint factor G_A, expressing the relative stiffness of all the columns connected at A to that of all the beams framing into A:

$$G_A = \frac{\Sigma(I_c / L_c)}{\Sigma(I_b / L_b)} \tag{1}$$

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1.1. Braced Frames

Braced frames are frames in which the sidesway is effectively prevented, and, therefore, the K-factor is never greater than 1.0. The "sidesway inhibited" alignment chart is the graphic solution of the following mathematical equation:

$$\frac{G_A G_B}{4}(\pi / K)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\pi / K}{\tan \pi / K}\right) + 2\frac{\tan \pi / 2K}{\pi / K} = 1 \tag{2}$$

This equation is mathematically exact, in that certain physical assumptions are exactly translated in mathematical terms. Whether these assumptions can be reasonably extended to a specific structure is a matter for the designer to decide.

For the transcendental Eq. 2, which can only be solved by numerical methods, the French Rules propose the following approximate solution:

$$K = \frac{3G_A G_B + 1.4(G_A + G_B) + 0.64}{3G_A G_B + 2.0(G_A + G_B) + 1.28} \tag{3}$$

Slightly simpler equations apply to special cases. If the column is hinged at B, G_B is infinitely large, and $1 / G_B = 0$:

$$K = \frac{3G_A + 0.64}{3G_A + 1.28} \tag{4}$$

If, instead, the column is fully fixed at B, $G_B = 0$:

$$K = \frac{0.7G_A + 0.32}{G_A + 0.64} \tag{5}$$

Finally, in the not infrequent case where $G_A = G_B = G$:

$$K = \frac{G + 0.4}{G + 0.8} \tag{6}$$

1.2. Sway Frames

If a rigid frame depends solely on frame action to resist lateral forces, its sidesway is not prevented. In this case, the K-factor is never smaller than 1.0. The mathematical equation for the "sway uninhibited" case is:

$$\frac{G_A G_B(\pi / K)^2 - 36}{6(G_A + G_B)} = \frac{\pi / K}{\tan \pi / K} \tag{7}$$

Table 1.
Comparison of Eqs. 2 and 3—Braced Frames

G_A	0.10	0.25	0.10	0.25	0.50	0.10	0.25	0.50
G_B	0.40	0.25	0.90	0.75	0.50	1.90	1.75	1.50
K exact	0.603	0.611	0.648	0.680	0.686	0.683	0.716	0.751
K approx	0.61	0.62	0.65	0.67	0.69	0.72	0.76	0.78
Error, %	0.9	1.3	0.4	0.8	0.9	0.4	0.7	0.6
G_A	1.00	0.50	1.00	2.50	0.50	1.00	2.50	5.00
G_B	1.00	4.50	4.00	2.50	9.50	9.00	7.50	5.00
K exact	0.774	0.792	0.840	0.877	0.806	0.858	0.913	0.930
K approx	0.78	0.80	0.84	0.88	0.81	0.86	0.91	0.93
Error, %	0.5	0.7	0.4	0.2	0.8	0.5	0.2	0.1

Table 2.
Comparison of Eqs. 7 and 8—Unbraced Frames

G_A	0.10	0.25	0.10	0.25	0.50	0.10	0.25	0.50
G_B	0.40	0.25	0.90	0.75	0.50	1.90	1.75	1.50
K exact	0.603	0.611	0.648	0.680	0.686	0.683	0.716	0.751
K approx	0.61	0.62	0.65	0.67	0.69	0.72	0.76	0.78
Error, %	0.9	1.3	0.4	0.8	0.9	0.4	0.7	0.6
G_A	1.00	0.50	1.00	2.50	0.50	1.00	2.50	5.00
G_B	1.00	4.50	4.00	2.50	9.50	9.00	7.50	5.00
K exact	0.774	0.792	0.840	0.877	0.806	0.858	0.913	0.930
K approx	0.78	0.80	0.84	0.88	0.81	0.86	0.91	0.93
Error, %	0.5	0.7	0.4	0.2	0.8	0.5	0.2	0.1

Although simpler than Eq. 2, this equation cannot be solved in closed form either. The French Rules recommend the following approximate solution:

$$K = \sqrt{\frac{1.6G_AG_B + 4.0(G_A + G_B) + 7.5}{G_A + G_B + 7.5}} \qquad (8)$$

For a hinge at B, the formula simplifies to:

$$K = \sqrt{1.6G_A + 4.0} \qquad (9)$$

For complete fixity at B, the approximation is:

$$K = \sqrt{\frac{4.0G_A + 7.5}{G_A + 7.5}} \qquad (10)$$

When $G_A = G_B = G$:

$$K = \sqrt{0.8G + 1.0} \qquad (11)$$

1.3. Accuracy of Equations

The accuracy that we can readily measure is of course the mathematical accuracy, that is, the comparison of the results given by an approximate formula to those obtained by solving the corresponding "exact" equation. The accuracy of the alignment charts depends essentially on the size of the charts, and on the reader's sharpness of vision. For the small charts in the Column Section of the AISC Manuals, this accuracy may be about five percent. In view of the many simplifying assumptions needed to arrive at Eqs. 2 and 7, this accuracy is certainly sufficient.

The formula proposed by the ACI for braced frames gives $K = 0.7$ for a beam fully fixed at both ends, instead of 0.5. If $G_A = G_B = 3.0$, it yields $K = 1.0$, instead of the expected 0.89. The equations for unbraced frames are somewhat better: for $G_A = G_B = 2.0$ for instance, they yield $K = 1.56$, instead of 1.61.

The French Rules indicate that Eq. 3 has an accuracy of –0.5 percent to +1.5 percent, while Eq. 8 is accurate within two percent. Tables 1 and 2 report the accuracies found at a few sample points. Again because of the nature of the surrounding assumptions, Eqs. 3 and 8 may be considered mathematically exact.

2. BACKGROUND

We have not been able to trace the origin of these equations, although similar closed-form approximations are said to have been published by Donnell.

In the European Recommendations, Eqs. 3 and 8 are given in function of two factors β_A and β_B (rather than K_A and K_B as in the French Rules). The definition of β differs from that of G, since, at each column end:

$$\beta = \frac{\Sigma(I_b / L_b)}{\Sigma(I_c / L_c) + \Sigma(I_b / L_b)} \qquad (12)$$

The mathematical relation between G and β is simple:

$$\beta = 1/(1+G) \tag{13}$$

Europeans tend to prefer β to G because a hinge means $\beta = 0$ and fixity $\beta = 1$. Obviously, the K-factor will be the same if the same elements are introduced in G and β.

Another approach is also described in the French Rules. The two beams AA′ and BB′ of Fig. 1 model the rotational restraints of column AB. These beams have the same moment of inertia I as AB, and are hinged at their far ends A′ and B′. Their respective lengths are $\rho_A L$ and $\rho_B L$, with ρ_A and ρ_B such that the rotational flexibilities of AA′ and BB′ at A and B are equal to the flexibilities f_A and f_B of the actual restraints. Applying a unit moment to AA′ at A for instance, must give a rotation θ_A equal to f_A:

$$\theta_A = \frac{\rho_A L}{3EI} = f_A \tag{14}$$

or

$$\rho_A = \frac{3EI}{L} f_A \tag{15}$$

Full fixity, $\rho = 0$, means a very short beam spring; a very long beam, ρ infinite, represents a perfect hinge.

2.1. Braced Frames and Trusses

The K-factor for braced frames and trusses is:

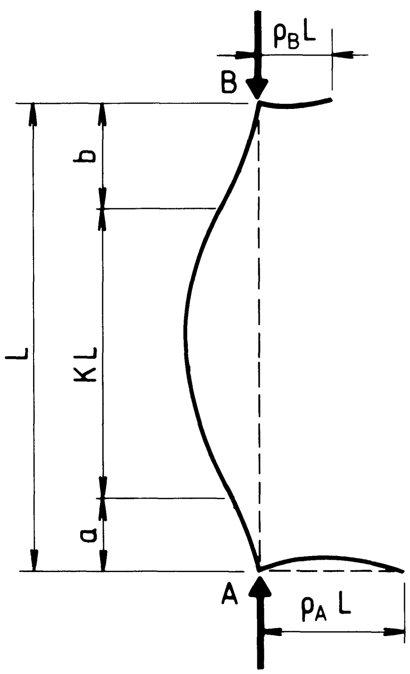

Fig. 1. Column with beam springs in the non-swaying mode.

$$K = \frac{\rho_A\rho_B + 0.7(\rho_A + \rho_B) + 0.48}{\rho_A\rho_B + \rho_A + \rho_B + 0.96} \tag{16}$$

The buckling mode is a sine curve, the half wave of which is KL; one point of inflexion is at a distance a from A:

$$\frac{a}{L} = \frac{0.3\rho_B + 0.12}{\rho_A\rho_B + 0.6\rho_A + \rho_B + 0.48} \tag{17}$$

The other point of inflexion is at a distance b, obviously equal to $L - KL - a$. The buckled shape is therefore easy to determine.

Consider a symmetrical frame, braced against sidesway. In the buckled shape of Fig. 2, the bending moments in the upper and lower beams are constant because of symmetry. To maintain that symmetry, applying unit moments at A and A′ causes a rotation θ_A:

$$\theta_A = \frac{L_b}{2EI_b} \tag{18}$$

so that

$$\rho_A = \frac{3}{2}\frac{I_c}{L_c}\frac{L_b}{I_b} = 1.5G_A \tag{19}$$

For braced frames, the equivalent flexural springs are obtained with $\rho_{A,B} = 1.5G_{A,B}$. Equation 3 was derived by substituting these values in Eq. 16.

2.2. Sway Frames

For a frame free to sway, the effective length factor K is:

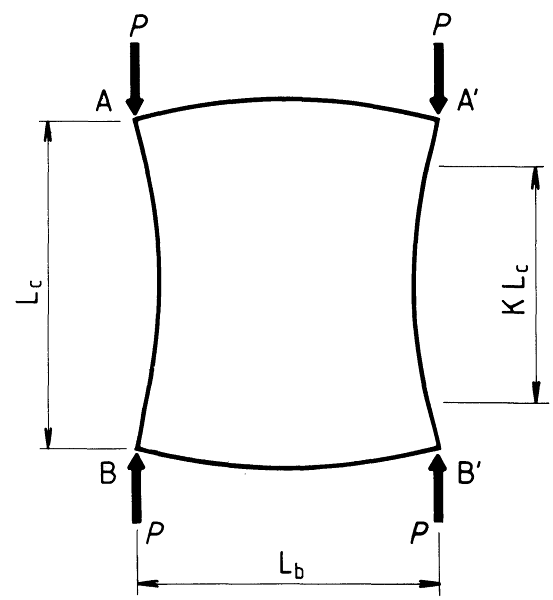

Fig. 2. Symmetrical buckling of symmetrical frame (non-swaying mode).

$$K = \sqrt{\frac{3.2\rho_A\rho_B + 4\rho_A + 4\rho_B + 3.75}{\rho_A + \rho_B + 3.75}} \tag{20}$$

If $\rho_B > \rho_A$, dimension a measured from point A sets the position of the point of inflexion in Fig. 3:

$$\frac{a}{L} = \tfrac{1}{2}\sqrt{\frac{4\rho_B - 2\rho_A + 3.75}{\rho_A + \rho_B + 3.75}} \tag{21}$$

If $\rho_A > \rho_B$, the point of inflexion is located by dimension b measured from point B:

$$\frac{b}{L} = \tfrac{1}{2}\sqrt{\frac{4\rho_A - 2\rho_B + 3.75}{\rho_A + \rho_B + 3.75}} \tag{22}$$

Note that the buckling mode has only one point of inflexion within the length L, the other one being obviously at a distance $KL > L$.

Referring to Fig. 4, which shows a symmetrical unbraced frame in the sidesway mode, it is seen that, because of symmetry, the beams present a point of inflexion at mid-span. The restraint on the columns is that provided by each half beam hinged at the axis of symmetry. Consequently, we find:

$$\rho_A = \frac{3EI_c(L_b/2)}{L_c \;\; 3EI_b} = 0.5G_A \tag{23}$$

Equation 8 was derived by substituting $\rho_{A,B} = 0.5G_{A,B}$ in Eq. 20. While one would normally use the G-ratios, there are cases where the flexural spring model may be better, in truss calculations[8] for instance.

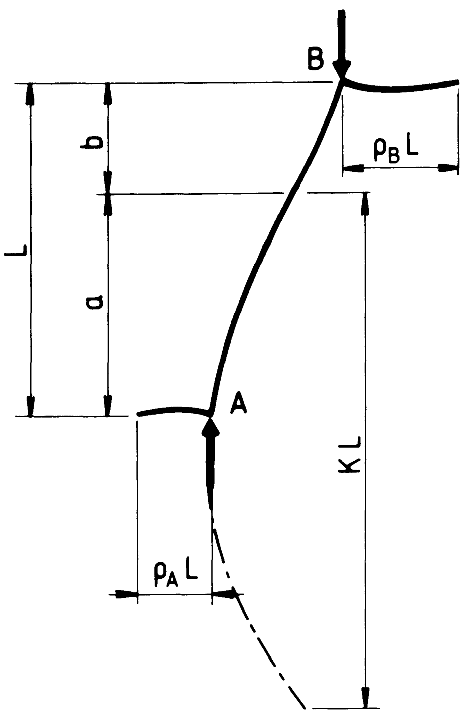

Fig. 3. Column with beam springs in the sidesway mode.

3. VALIDITY OF ASSUMPTIONS

The derivation of either the "exact" or the approximate equations requires several assumptions that are never exactly fullfilled, whether the frame is braced or not. Examine the frames of Figs. 2 and 4: it is evident that the assumed symmetries seldom exist. With the flexural spring model, one could move the points of inflexion along the beams to see how sensitive the K-factor is to their positions, but how much to move them can only be estimated. Fortunately, the K-factor is not too sensitive to variations in G_A and G_B, and its sensitivity is further dampened by the inelastic effect described by Yura.[10] Nonetheless, estimating a K-factor is sometimes difficult, and it would certainly be desirable to do away with K-factors and effective lengths altogether. There is a definite trend in modern codes to do precisely that.

In the AISC LRFD Specification, the designer has two options: either make a P–Δ calculation, or determine the required flexural strength M_u by means of Eq. H1-2. In the later case, one must establish not one, but two K-factors. The first one is calculated assuming that there is no lateral translation of the frame; always smaller than 1.0, it serves to calculate factor B_1. The other one produces B_2 which reflects the effects of sidesway; it is always larger than 1.0. The latest Canadian code[7] goes one step further: it eliminates K-factors altogether for unbraced frames and calls for a P–Δ analysis instead. Presumably, specifying $K = 1.0$ takes care of the second-order effects (or P–δ effects) within the beam-column itself. Professor McGuire[9] expresses a fairly common point

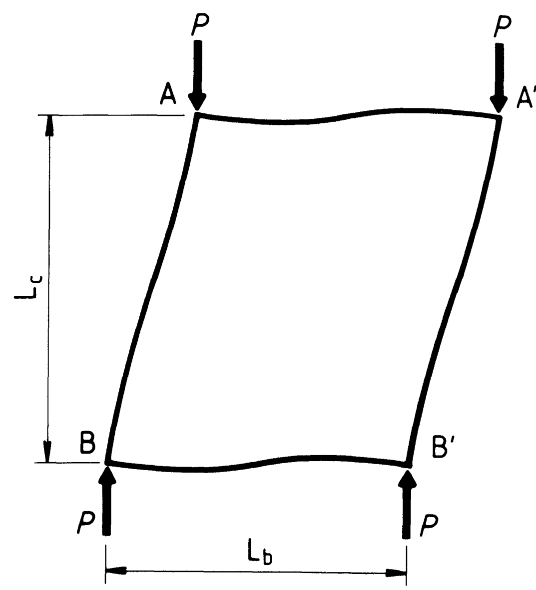

Fig. 4. Antisymmetrical buckling of symmetrical frame (sidesway mode).

of view when he states: "Right now, second-order elastic analysis programs that eliminate the need to calculate B_1 and B_2 factors...are available. I wish more engineers would use them."

However, Professor McGuire goes on to say: "But there are other places where effective lengths are still the best, or only, practical expedient for routine design, though future research may change this." Such seems to be the case whenever inelastic effects are introduced in the analysis, as in Yura's method.[10] For triangulated trusses[8] for instance, the ultimate strength can be safely predicted in both elastic and inelastic ranges, but only by making extensive use of the effective length concept. In fact, there is nothing wrong with effective lengths and K-factors whenever they are a convenient and accurate tool: at the same time it eliminates K-factors for unbraced frames, the Canadian code now explicitly allows K-factors substantially smaller than 1.0 for trusses in specific conditions. However undesirable they may seem, effective lengths and K-factors will be with us for some time yet.

CONCLUSION

The equations giving the K-factor in the French Rules are accurate enough for design purposes. Their simple closed form make them well suited for computer use, in particular in spreadsheets.

In some instances, the model with flexural beam springs considered in the French Rules may provide a better physical understanding, and lead to a better evaluation of the K-factor.

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