A Simplified Look at Partially Restrained Beams

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INTRODUCTION

Over the past few years, a great deal of effort has been expended within the research arm of the structural engineering profession addressing the topic of semi-rigid connections and their impact on structural response. A review of the recent literature shows that this interest is being maintained and even expanded. At the 1990 National Steel Construction Conference sponsored by the American Institute of Steel Construction, at least three papers dealt with semi-rigid connections[1] while at the 1989 conference, the T.R. Higgins Lecture addressed semi-rigid connections.[6] The AISC LRFD Specification,[2] in an effort to increase the design awareness of this type of construction, altered the previously defined types of construction by combining the former type 2 and type 3 into a single category PR, partially restrained. Although the ninth edition of the ASD Specification[3] retained the previously defined three types, there appears to be a heightened recognition that some attention must be given to moment-rotation characteristics of connections, even when it is anticipated that they are really behaving as pins. The intention of this paper is to take a step back and look at some simple analysis techniques and simple structures, thereby generating an improved level of understanding of the overall impact and importance that this type of connection has on building structures.

THE MODEL

The first step in looking at partially restrained beams is to form a mathematical model of the uniformly loaded beam which includes the characteristics of the connections. The full range of connection behavior, from the truly pinned to the fully rigid connection, may be modeled as a rotational spring with a specified stiffness, n, so that the moment in the spring will be given by:

$$M_{con} = n\theta_{con} \tag{1}$$

If these connections are attached to the ends of a simple beam, with a uniformly distributed load as shown in Fig. 1, a classical indeterminate analysis may be performed to relate the moment in the spring to the load and to the spring and beam stiffnesses. Using the method of consistent deformations, the springs are first removed from the beam leaving a simple beam as shown in Fig. 2a. Then the moments that would be applied by the springs are applied independently to the beam ends as shown in Figs. 2b and 2c. The rotations at end A for these three cases are given by:

$$\theta_a = WL^2/24EI \tag{2}$$

$$\theta_{aa} = -M_aL/3EI \tag{3}$$

and

$$\theta_{ab} = -M_bL/6EI \tag{4}$$

Superposition of these rotations yields the final rotation on the beam at A or

$$\theta_{final} = \theta_a + \theta_{aa} + \theta_{ab} \tag{5}$$

Since the final beam rotation and the final spring rotation must be the same, substitution of Eqs. 1 through 4 into Eq. 5 yields

$$M_{con}/n = WL^2/24EI - M_aL/3EI - M_bL/6EI \tag{6}$$

Taking into account the symmetry of the structure and recognizing that the moment in the spring is the moment on the

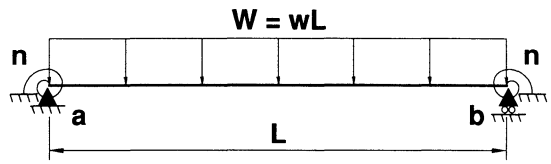

Fig. 1. Beam and connection model.

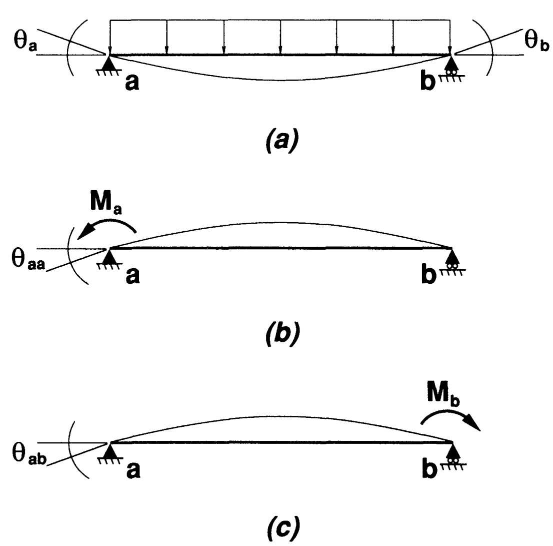

Fig. 2. The cut-back structure.

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beam, $M_{con} = M_a = M_b$, Eq. 6 may be rearranged to solve for this moment so that

$$M_{con} = \frac{WL^2/24EI}{1/n + L/2EI} \tag{7}$$

In order to simplify this expression, the ratio of beam stiffness to spring stiffness is defined as

$$u = (EI/L)/n \tag{8}$$

or

$$n = EI/uL \tag{9}$$

Substitution of this new representation of the spring stiffness into Eq. 7 and simplifying yields

$$M_{con} = \frac{WL^2/24}{(uL + L/2)} = \frac{WL/12}{(2u + 1)} \tag{10}$$

The beam bending moment diagram is shown in Fig. 3. Superposition of the simple beam moment diagram on the beam with end moments yields a positive center line moment of

$$M_{pos} = \frac{WL}{8} - \frac{WL/12}{(2u + 1)} \tag{11}$$

or

$$M_{pos} = \frac{6u + 1}{4u + 2}\frac{WL}{12} \tag{12}$$

Both the connection and the center line moment are written as a coefficient times the fixed end moment. If these coefficients are plotted as a function of the spring stiffness ratio, the full response of the beam can be represented as shown in Fig. 4.

DEFLECTIONS

The center line deflection may now be determined using the method of conjugate beam. The beam and the corresponding conjugate beam are shown in Fig. 5. The area of the *M/EI* diagram above the beam represents the influence of the load on the simple beam while that below the beam represents the influence of the negative end moments. The end rotation may be determined by taking moments of these areas about end B of the conjugate beam such that

$$R_a = M_sL/3EI - M_{con}L/2EI \tag{13}$$

The deflection at the center line, *D,* may now be determined by taking moments about the conjugate beam center line which yields, after simplification,

$$D = \frac{5M_sL^2}{48EI} - \frac{M_{con}L^2}{8EI} \tag{14}$$

The first term in Eq. 14 represents the center line deflection of a uniformly loaded simple beam, D_{simp}, while the second term represents the reduction in center line deflection as a result of the end moments, D_{-M}. The ratio of these terms will show the overall reduction in deflection due to the end restraint. If Eq. 10 is substituted for the moment in the connection, the deflection ratio becomes

$$\frac{D_{-M}}{D_{simp}} = \frac{4}{5(2u + 1)} \tag{15}$$

The deflection ratio, given as a function of the spring stiffness ratio, is plotted in Fig. 6. It can be seen that for a fixed end condition, $u = 0$, the deflection will be reduced by 80% of the simple beam deflection. For spring stiffness ratios greater than zero, the reduction in deflection will be correspondingly less.

THE BEAM LINE

Another useful and common approach to the solution of the interaction of partially restrained connections and the beam to which they are attached is available using the beam line.[4] Figure 7 shows the relationship between moment and rotation on the end of a uniformly loaded prismatic beam. Note that the rotation is zero for a fixed-end beam with the resulting fixed-end moment and the moment is zero for a simply

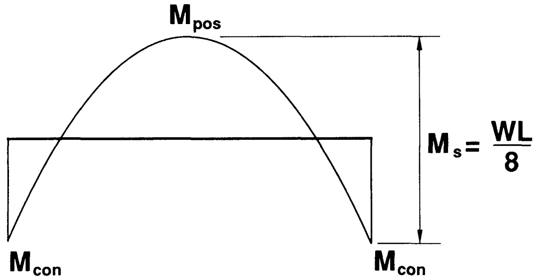

Fig. 3. Bending moment diagram.

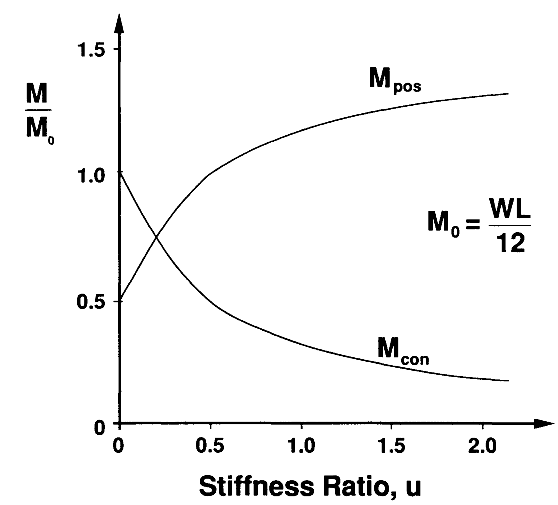

Fig. 4. Bending moment coefficients vs. connection stiffness ratio.

supported beam with the resulting simple beam rotation. A straight line connects these two extreme conditions. Since the connection is represented by Eq. 1, it too may be plotted on the graph of Fig. 7 as a straight line with a slope of n. The intersection of these two lines represents the final equilibrium condition for the beam with the given partially restrained connections. Thus, for a connection with a known stiffness ratio, u, the solution will again be given by Eq. 12.

ELASTIC DESIGN

Figure 8 combines the two views of the beam and connection interaction. The normal approach to design would have a connection capable of developing up to 20% of the fixed end moment considered as a pinned connection and one capable of developing at least 90% of the fixed end moment considered fixed.[4] These two regions are shaded on both portions of Fig. 8. They represent the area below a value of $u = 0.0555$ and above the value of $u = 2.0$. Beam connection combinations falling within the unshaded area should be treated so as to include connection behavior. Neither the LRFD nor the ASD specification directly recommend these assumptions but rather suggest that any combination which is not fully pinned or fully rigid be treated in a way that reflects actual behavior.

In order to fully understand the impact that the use of flexible connections may have on beam design, it is important to consider further the results presented in Fig. 8b. The maximum moment on the beam is indicated by the maximum coefficient. This will occur on the end of the beam for values of $u = 0$ to $u = 0.167$. For values of $u > 0.167$, the maximum moment will occur at the beam center line. The most economical design from the standpoint of the beam would occur at the point where the end moment and the center line moment would be the same, a connection with a value of $u = 0.167$. Unfortunately, any slight deviation from this value will result in a beam design moment larger than anticipated. Thus, the beam would no longer be adequate to carry the design loads. Considering a beam designed for the fixed end condition, $u = 0$, it can be seen that a range of stiffness ratios up to $u = 0.5$ will still permit the beam to adequately carry the design moment, thus allowing for some inaccuracies in

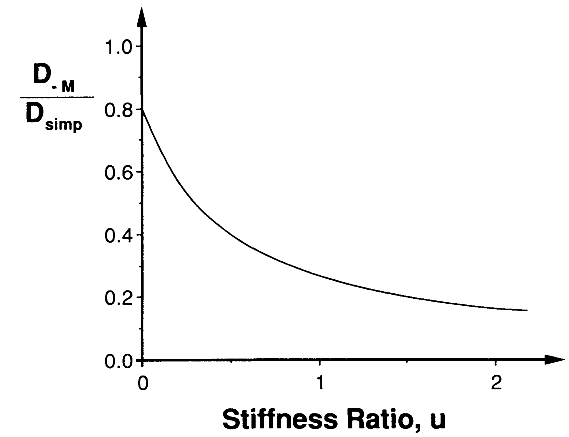

Fig. 6. Deflection reduction vs. connection stiffness ratio.

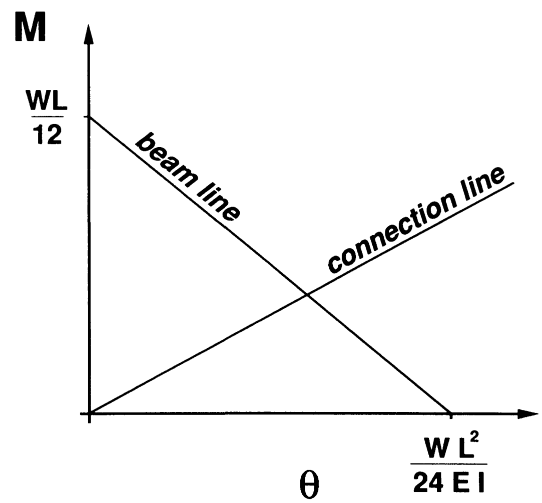

Fig. 7. Moment-rotation diagram—the beam line.

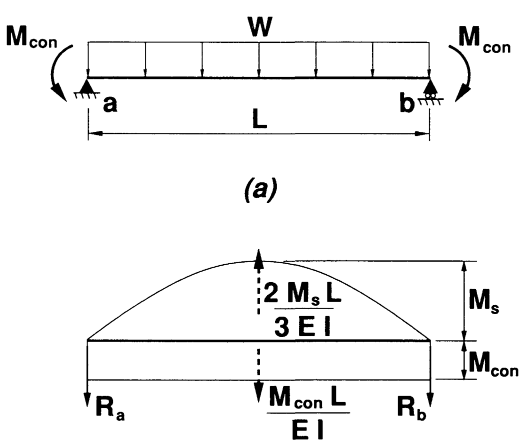

Fig. 5. Conjugate beam.

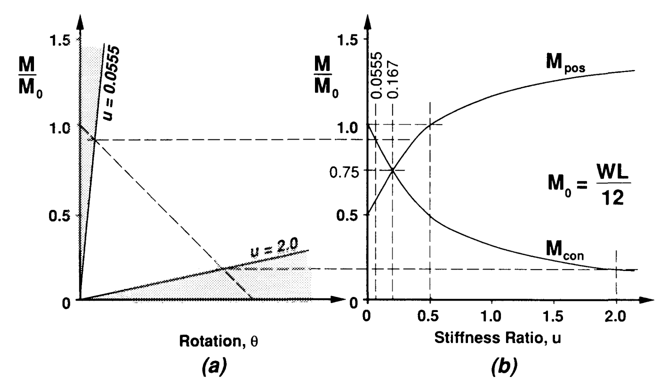

Fig. 8. Combined views of moment-rotation-stiffness diagrams.

the determination of connection stiffness. If the beam is designed as a simple beam, u equal to infinity, any connection, regardless of its stiffness ratio will still result in an acceptable beam. For any connection with a stiffness ratio between these two extremes, there is always the potential that an inaccuracy in determining the connection stiffness could result in the beam feeling a moment larger than that for which it was designed.

Recent papers would seem to suggest that extreme care is not required in modeling connection stiffness[6] or that the actual shape of the moment rotation curve is not really critical.[5] However, currently available connection models may actually predict a stiffness that varies from the actual stiffness by a factor of plus or minus 2.[5] Thus, from the above it would appear that connection stiffness, as measured by the stiffness ratio, may be quite important for a broad range of possible situations. In addition, if sufficient care is not exercised, the resulting design may be significantly inadequate.

NON-RIGID SUPPORTS

The previously developed equations were based on the assumption that the connection was attached to a non-yielding support. Since in most real structures the beams are attached to columns or other flexible elements, it will be informative to investigate the situation presented in Fig. 9. As with the single beam already considered, the beam of Fig. 9 is symmetrical and loaded with a uniform load. The spring stiffness and stiffness ratio are defined as in Eq. 1 and Eq. 8. The support members are defined with the stiffness EI_B/L_B as shown in Fig. 9a. In this situation, the connection rotation is no longer equal to the final beam rotation but is instead equal to the final beam rotation less the support rotation as shown in Fig. 9b. Thus, with the inclusion of the support rotation, Eq. 6 becomes

$$M/n = WL^2/24EI - ML/3EI - ML/6EI - ML_B/4EI_B \tag{16}$$

Simplifying Eq. 16 and solving for the moment yields

$$M = \frac{WL^2/24EI}{1/n + L_B/4EI_B + L/2EI} \tag{17}$$

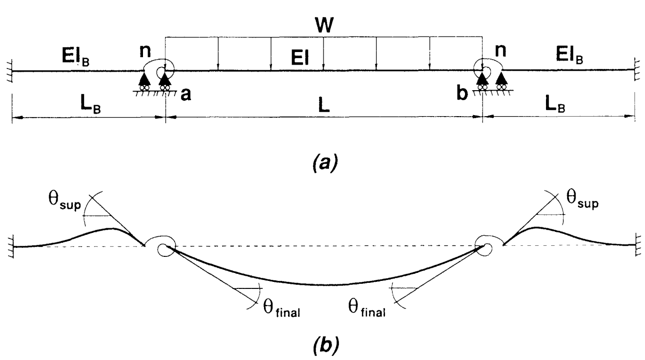

Fig. 9. Semi-rigid connection with flexible supports.

Inspection of Eq. 17 reveals that the first two terms in the denominator represent the spring and support respectively. If the support beam is infinitely rigid, the second term may be eliminated and Eq. 17 becomes Eq. 7. If, at the other extreme, the spring is made infinitely rigid, Eq. 17 will yield the results for a three span beam. If these two terms are combined and defined as an effective spring representing both the connection and the support, such that

$$1/n_{eff} = 1/n + L_B/4EI_B \tag{18}$$

the moment on the end of the beam may be given by Eq. 7 with n being replaced by n_{eff}. It then becomes clear that the range of responses available for the beam is the same as shown in Fig. 8. In addition, regardless of the structure which may provide the support, an effective spring can be defined which will dictate the beam response.

PLASTIC ANALYSIS

A beam with semi-rigid connections may also be investigated through a plastic analysis. The primary requirement is that the connection be capable of maintaining the plastic moment while undergoing significant rotation. If the plastic moment capacity of the beam is defined as M_p and the plastic moment capacity of the connection is defined as M_{pc}, the plastic mechanism and corresponding moment diagram are as shown in Fig. 10. Equilibrium requires that the simple beam moment,

$$M_s = M_p + M_{pc} \tag{19}$$

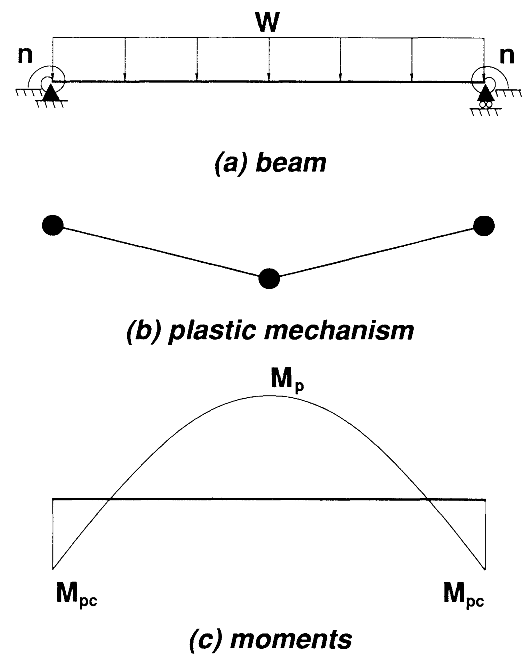

Fig. 10. Plastic analysis of beam with semi-rigid connections.

If the connection capacity is taken as a certain portion of the beam capacity such that

$$M_{pc} = aM_p \tag{20}$$

then for $a = 0$ the connection is pinned and for $a = 1.0$ the connection has the same capacity as the beam, independent of rotation. Substitution of Eq. 20 into Eq. 19 and rearrangement yields

$$M_p = M_s/(1 + a) \tag{21}$$

Eq. 21 represents the plastic moment capacity required for the beam to carry the applied load. A plot of Eq. 21 is provided in Fig. 11. Since the most economical beam design would result when the connection is capable of resisting the full plastic moment capacity of the beam, $a = 1.0$, the design by plastic analysis would require only that the connection be capable of attaining that moment. Its actual moment rotation characteristics, how it got there, would not be important.

REAL CONNECTIONS

The moment-rotation characteristics for real connections normally exhibit a nonlinear behavior. Two comprehensive collections of connection data have been reported[7,8] which provide the designer with a starting point for considering true connection behavior. Figure 12 shows representative curves for connections which might be considered pinned, fixed and semi-rigid. It is obvious that the linear model used in the earlier sections of this paper does not accurately describe the full range of behavior of these true connections. However, as shown in Fig. 13, if the intersection of the beam and connection lines was known, an effective linear connection could be determined with a stiffness, $1/n_{eff}$, which would provide the same solution as the true connection curve. This again shows that, regardless of the complexity of the connection model, the beam will consistently respond as shown in Fig. 8b.

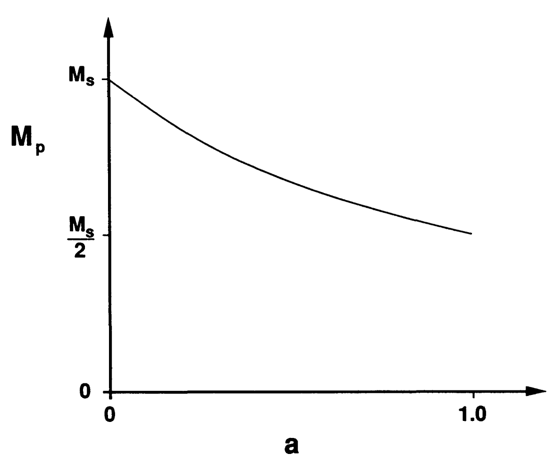

Fig. 11. Required plastic moment capacity of beam.

CONCLUSIONS

Behavior of prismatic beams with semi-rigid or partially restrained connections has been presented through the use of classical methods of analysis. It has been shown that, regardless of the connection model or the support to which the connection is attached, the beam will behave in a consistent manner which may be defined as a function of the effective linear stiffness. In addition, it can be seen that in all cases between the purely pinned and the truly fixed connection, there is the possibility for both a reduction in the required beam size if true connection behavior is considered as well as errors which may cause the actual beam to be overloaded. Partially restrained connections may be valuable additions to the designer's toolkit if they are properly understood and used.

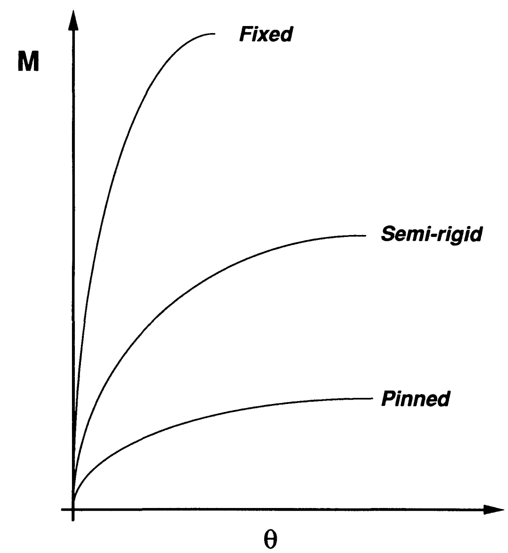

Fig. 12. Connection moment-rotation curves.

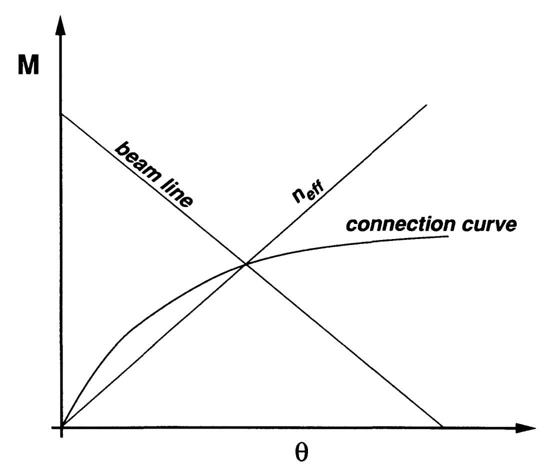

Fig. 13. Beam line with true connection and effective stiffness.

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