

DISCUSSION

On Beam-Column Moment Amplification Factor

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Discussion by **Pierre Dumonteil**

In a sway-prevented frame, a column is elastically restrained by adjacent beams and girders. For the first-order bending moments in that column to be amplified, or changed in any manner, by second-order effects (the so-called $P - \delta$ action), the column end rotations must also change. This is shown by the equations governing second-order beam-column behavior. Therefore, the restraining end moments will also change. While Eqs. 7 are correct, the moments M_a and M_b to be entered in the formulae are not the first-order moments but the second-order ones. Of course, the problem is how to obtain these second-order moments.

The following analysis proposes to show how this problem could be approached. Consider the non-swaying structure of Fig. 1(a) and, more particularly, column AB. This column is restrained at both ends by its adjoining members. A first-order frame analysis gives the end moments \bar{M}_a and \bar{M}_b (the superscript bar will be reserved to indicate moments obtained by first-order calculations).

To evaluate the elastic restraints and the actions of the structure on the column, introduce pins, that is, member releases at the column ends A and B. The structure then becomes that of Fig. 1(b) and could be analyzed by a first-order method to obtain the rotations ϕ_a and ϕ_b at the nodes A and B, as well as the rotational stiffnesses k_a and k_b .

The end restraints may be modeled by two beam springs AA' and BB', as shown on Fig. 2. These beams are pinned at the ends A' and B' and rigidly connected to the column in A and B. They may be given the same moment of inertia I as that of the column, so that their lengths are $r_a L$ and $r_b L$:

$$r_a L = \frac{3EI}{k_a} \quad (1.1a)$$

$$r_b L = \frac{3EI}{k_b} \quad (1.1b)$$

To restore the structure to its original state of Fig. 1(a), remove the pins and enforce continuity; this determines the first-order bending moments in the column. The relations are:

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$$0_a - \frac{r_a L \bar{M}_a}{EI} = \frac{L}{3EI} (\bar{M}_a - \frac{1}{2} \bar{M}_b) \quad (1.2a)$$

$$0_b - \frac{r_b L \bar{M}_b}{EI} = \frac{L}{3EI} (\bar{M}_b - \frac{1}{2} \bar{M}_a) \quad (1.2b)$$

If the restraints r_a and r_b are known, the rotations ϕ_a and ϕ_b may be obtained from the results of the first-order analysis on the original structure:

$$\phi_a = \frac{L}{eEI} [(r_a + 1) \bar{M}_a - \frac{1}{2} \bar{M}_b] \quad (1.3a)$$

$$\phi_b = \frac{L}{3EI} [-\frac{1}{2} \bar{M}_a + (r_b + 1) \bar{M}_b] \quad (1.3b)$$

It was shown elsewhere¹ that estimating rather accurately the restraint factors r_a and r_b in a truss or a braced frame is not a very difficult task.

To introduce second-order effects in the column, Eqs. 1.2 are modified:

$$\phi_a - \frac{r_a L M_a}{EI} = \frac{L}{3EI} [f(u) M_a - g(u) M_b] \quad (1.4a)$$

$$\phi_b - \frac{r_b L M_b}{EI} = \frac{L}{3EI} [-g(u) M_a + f(u) M_b] \quad (1.4b)$$

In these equations, we have:

$$u = \sqrt{\frac{PL^2}{EI}} \quad (1.5)$$

$$f(u) = 3 \frac{\sin u - u \cos u}{u^2 \sin u} \quad (1.6)$$

$$g(u) = 3 \frac{u - \sin u}{u^2 \sin u} \quad (1.7)$$

Substituting the values of the rotations ϕ_a and ϕ_b given by Eqs. 1.3 into Eqs. 1.4, we obtain the fundamental equations expressing the second-order moments M_a and M_b in function of the first-order moments \bar{M}_a and \bar{M}_b :

$$(f + r_a) M_a - g M_b = (r_a + 1) \bar{M}_a - \frac{1}{2} \bar{M}_b \quad (1.8a)$$

$$-g M_a + (f + r_b) M_b = -\frac{1}{2} \bar{M}_a + (r_b + 1) \bar{M}_b \quad (1.8b)$$

Thus, the explicit calculation of the rotations ϕ_a and ϕ_b is not necessary. Leaving aside the singularities that may exist in Eqs. 1.8 for a moment, they may be solved to give:

$$M_a = c_{11}\bar{M}_a + c_{12}\bar{M}_b \quad (1.9a)$$

$$M_b = c_{21}\bar{M}_a + c_{22}\bar{M}_b \quad (1.9b)$$

with:

$$c_{11} = \frac{1}{2D}[2(r_a+1)(f+r_b)-g] \quad (1.10a)$$

$$c_{12} = \frac{1}{2D}[2(r_b+1)g-f-r_a] \quad (1.10b)$$

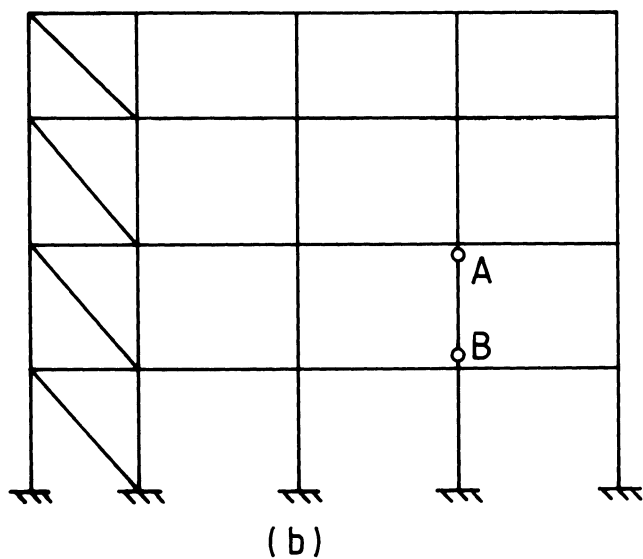
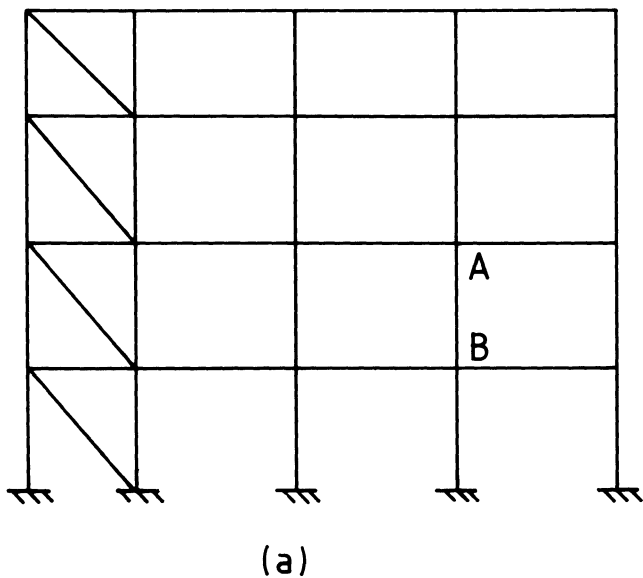


Fig. 1. Structure (a) with column AB rigidly connected and (b) with pins at column ends A and B.

$$c_{21} = \frac{1}{2D}[2(r_a+1)g-f-r_b] \quad (1.10c)$$

$$c_{22} = \frac{1}{2D}[2(r_b+1)(f+r_a)-g] \quad (1.10d)$$

$$D = (f+r_a)(f+r_b) - g^2 \quad (1.11)$$

If the determinant $D(u)$ is equal to zero, the system is unstable and the column buckles. The smallest positive root u_e of the equation $D(u) = 0$ determines the buckling load P_e and the effective length factor is $K = \pi/u_e$. The system is elastically stable for u smaller than the critical value u_e . For $u = \pi$, and regardless of the restraint values, both f and g become infinite. However, the coefficients c_{11}, \dots, c_{22} remain finite and are in fact continuous, the singularity disappearing upon solving Eqs. 1.8.

It is to the second-order moments M_a and M_b that Eqs. 6 to 9 given by the authors (with $\gamma L = u$) must be applied to obtain a correct result, and not to the first-order moments \bar{M}_a and \bar{M}_b .

Sample calculations were made with $r_a = r_b = 0.50$. The critical load is obtained for $u_e = 4.9113$, or $K = 0.6397 = 0.64$. The buckling load P_e is therefore 2.44 times that of the pinned column.

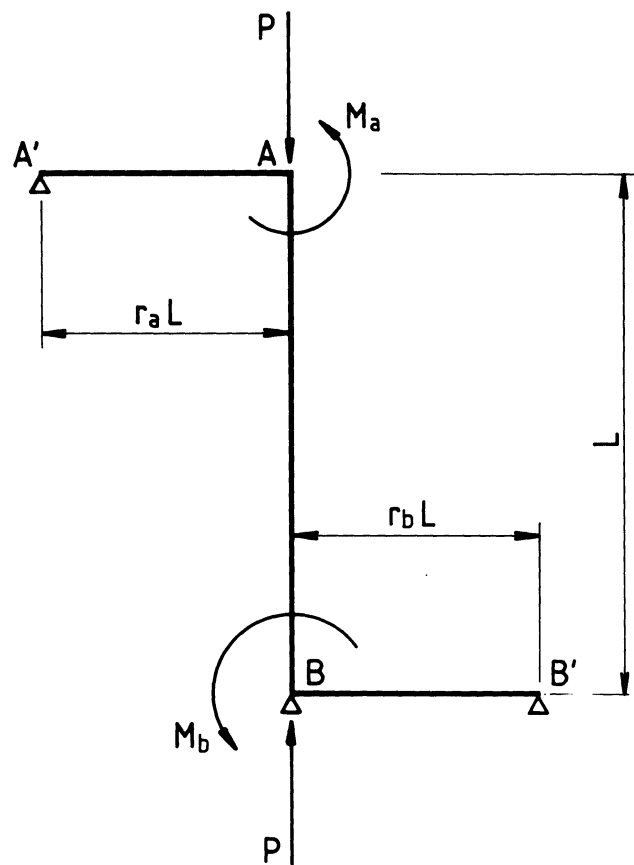


Fig. 2. Model of elastically restrained beam-column.

u	c_{11}	c_{12}
0.0	1.0000	1.0000
0.2	0.9986	0.0011
0.4	0.9943	0.0044
0.6	0.9871	0.0099
0.8	0.9768	0.0178
1.0	0.9633	0.0282
1.2	0.9463	0.0414
1.4	0.9254	0.0577
1.6	0.9003	0.0775
1.8	0.8704	0.1012
2.0	0.8348	0.1296
2.2	0.7927	0.1636
2.4	0.7428	0.2045
2.6	0.6833	0.2537
2.8	0.6117	0.3138
3.0	0.5248	0.3878
3.2	0.4175	0.4808
3.4	0.2821	0.6000
3.6	0.1065	0.7575
3.8	-0.1304	0.9741
4.0	-0.4677	1.2886
4.2	-0.9889	1.7838
4.4	-1.9071	2.6727
4.6	-3.9833	4.7153
4.8	-13.4448	14.1380
4.9	-143.5639	144.2354

Because of symmetry, $c_{11} = c_{22}$ and $c_{12} = c_{21}$. The coefficients c_{11} and c_{12} are listed in Table 1, where it is seen that c_{11} decreases from 1.0 for $u = 0$ to a very large negative value for u nearly equal to u_e . On the other hand, c_{12} increases from 0 for $u = 0$ to a very large positive value for u nearing u_e .

Two cases were considered. The first case is that of a constant moment bending the column in single curvature, that is, $\bar{M}_a = \bar{M} = -\bar{M}_b$, for which:

$$M_a = (c_{11} - c_{12})\bar{M} = -M_b \quad (1.12)$$

Eqs. 6 and 7a show that the maximum second-order bending moment is then:

$$M_{max} = |M_a \sec \frac{u}{2}| \quad (1.13)$$

For $u = \pi$, $c_{11} = c_{12}$, so that $M_a = M_b = 0$: the column behaves as if it were pinned at both ends, which was to be expected. All of the existing or proposed amplification formulae are quite unconservative when compared to the results shown in Table 2. A reasonable representation would be, for this particular case:

$$B_1 = (1+0.6P/P_e) / (1-P/P_e) \quad (1.14)$$

The other case is of course the other extreme, with equal

u	M_a/M	M_{max}/M
0.0	1.000	1.000
0.2	0.997	1.003
0.4	0.990	1.010
0.6	0.977	1.023
0.8	0.959	1.041
1.0	0.935	1.065
1.2	0.905	1.096
1.4	0.868	1.135
1.6	0.823	1.181
1.8	0.769	1.237
2.0	0.705	1.305
2.2	0.629	1.387
2.4	0.538	1.486
2.6	0.430	1.606
2.8	0.298	1.753
3.0	0.137	1.936
3.2	-0.063	2.168
3.4	-0.318	2.467
3.6	-0.651	2.866
3.8	-1.104	3.416
4.0	-1.756	4.220
4.2	-2.773	5.492
4.4	-4.580	7.782
4.6	-8.699	13.056
4.8	-27.583	37.406
4.9	-287.799	373.653

end moments bending the column in double curvature. With $\bar{M}_a = \bar{M}_b = \bar{M}$, then:

$$M_a = M_b = (c_{11} + c_{12})\bar{M} \quad (1.15)$$

For $u < \pi$, $M_{max} = |M_a|$, but when $u > \pi$:

$$M_{max} = |M_a \operatorname{cosec} \frac{u}{2}| \quad (1.16)$$

Except when the axial force P is very close to the buckling load P_e , the second-order moment is not amplified but in fact reduced, and it is acceptable to take $B_1 = 1.0$. This was to be expected: the only eigenfunction which could be amplified is the second buckling mode, whose shape is closest to that of a beam in double curvature. However, this second buckling mode is too far removed to be significantly amplified, since it occurs for $u = 7.4528$.

Similar results were obtained with other restraint values, always for a column loaded by end moments only: there is no appreciable moment amplification, rather a small reduction, in the antisymmetric case ($\bar{M}_a = \bar{M}_b = \bar{M}$), but all amplification formulae appear too unconservative in the constant bending moment case ($\bar{M}_b = \bar{M} = -\bar{M}_a$).

Table 3.
Second-Order Moments M_a and M_{max} with
Antisymmetric First-Order Bending Moment
($M_a = M = M_b$) for $r_a = r_b = 0.50$

u	M_a/M	M_{max}/M
0.0	1.000	1.000
0.2	1.000	1.000
0.4	0.999	0.999
0.6	0.997	0.997
0.8	0.995	0.995
1.0	0.992	0.992
1.2	0.988	0.988
1.4	0.983	0.983
1.6	0.978	0.978
1.8	0.972	0.972
2.0	0.964	0.964
2.2	0.956	0.956
2.4	0.947	0.947
2.6	0.937	0.937
2.8	0.925	0.925
3.0	0.913	0.913
3.2	0.898	0.899
3.4	0.882	0.890
3.6	0.864	0.887
3.8	0.844	0.892
4.0	0.821	0.903
4.2	0.795	0.921
4.4	0.766	0.947
4.6	0.732	0.982
4.8	0.693	1.026
4.9	0.671	1.053

In reality, most practical cases occur between these two extremes, and actual structures behave in a strongly non-linear manner. To account for this non-elastic behavior, one could apply the Dutheil-Yura method,^{1,2} in a manner not unlike that which has been successfully used with trusses.

REFERENCES

1. Dumonteil, Pierre, "In-Plane Buckling of Trusses," *Canadian Journal of Civil Engineering*, 16:(1989)504-516.
2. Yura, J. A., "The Effective Length of Columns in Unbraced Frames," *Engineering Journal*, 8:(2nd Quarter, 1971).

Addendum/Closure by Lian Duan, Iqbal S. Sohal and Wai-Fah Chen

The writers would like to thank Mr. Pierre Dumonteil for his interest in and comments on the paper.

Although the elastic moment amplification factor, Eqs. 7, has been derived from an isolated beam-column (Fig. 1) subjected to applied end-moments M_a and M_b , combined with axial compression P ,^{5,10} this expression is also valid and can

be used for beam-columns in a frame as long as the end-moments of the members are properly determined. We agree therefore with Mr. Pierre Dumonteil's viewpoint that to correctly determine the maximum elastic second-order moment in a framed beam-column, the second-order elastic end-moments should be entered in Eqs. 7. However, when using the B_1 factor put forward by the LRFD Specification for estimating approximately the maximum second-order moment in a framed member, the end-moments to be amplified are not the second-order moments but the first-order ones. This is in contrast to the general belief that if second-order end-moments are available in a $P - \Delta$ analysis, they should be entered and amplified by the B_1 factor to obtain a better maximum $P - \delta$ moment in the member. This point will be illustrated in the forthcoming.

In the LRFD interaction Eqs. 1, M_u is based on a second-order elastic analysis. Structural engineers are encouraged to use a direct elastic second-order analysis.¹² When only a first-order analysis is performed, the LRFD Specification provides an approximate second-order analysis based on the B_1 and B_2 amplification factors for the determination of M_u in lieu of an exact second-order analysis. It is clearly and rightfully stated in the specification that if the B_1 factor is used to estimate the $P - \delta$ moment in a framed member, the M_{nt} should be the non-sway first-order end-moments, not the second-order ones.

It is well-known that the B_1 factor is an approximate formula for Eqs. 7 as shown in the present paper. But the applications of Eqs. 7 and B_1 factor are rather different. To this end, we will analyze two simple braced frames as shown in Figs. 6 and 7. The beam is loaded by a uniformly distributed load w . After the full value of w is reached, the column is then loaded by a monotonically increasing concentric load P . Figures 8 and 9 show comparison of the exact second-order analysis and the approximate B_1 factor for the determination of the maximum moment in Column AB. In these figures, the solid lines are exact solutions obtained by Eqs. 7 with the second-order end-moments, the dashed lines are obtained by the LRFD B_1 formula with the alignment chart $K = 0.839$ (hinged-ended frame) and 0.59 (fixed-ended frame) and with the first-order end-moments, and the dot-dashed lines are the LRFD B_1 formula with the same $K = 0.839$ (hinged-ended frame) and 0.59 (fixed-ended frame) but with the second-order end-moments. For a further comparison, the solutions of Eqs. 7 with first-order end-moments are also plotted (star-solid lines). The following highlights can be observed:

1. The use of Eqs. 7 with the first-order end-moments provides very conservative results (star-solid curves).
2. The use of the B_1 factor with the second-order end-moments does not necessarily provide a better estimation of the maximum $P - \delta$ moment in the columns (dot-dashed curves).

3. For low and intermediate axial compressions, the use of the B_1 factor with the first-order end-moments provides very good but conservative results for the maximum $P - \delta$ moment in the column. For higher axial compressions, the B_1 factor gives less conservative results.
4. Although the restraining end-moments change due to the $P - \delta$ effect, the approximate B_1 method combined with the first-order analysis does provide a simple, practical and good estimation of the second-order $P - \delta$ moment in a framed member, because axial compression in most framed columns in actual frames is not so high. For frames subjected to higher axial compression, a direct second-order analysis may be necessary for an accurate determination of the second-order $P - \Delta$ end-moments for members in a frame.

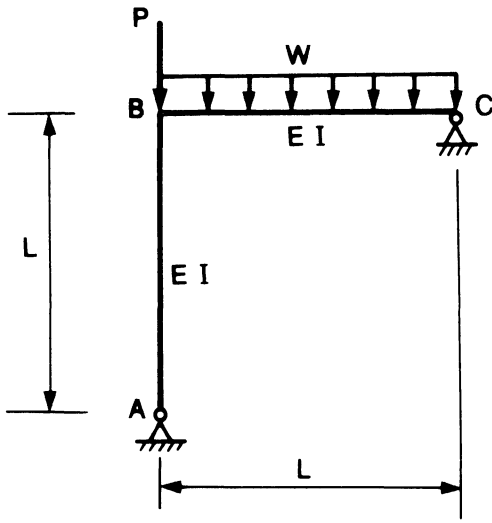


Fig. 6. Hinged-ended two-member frame.

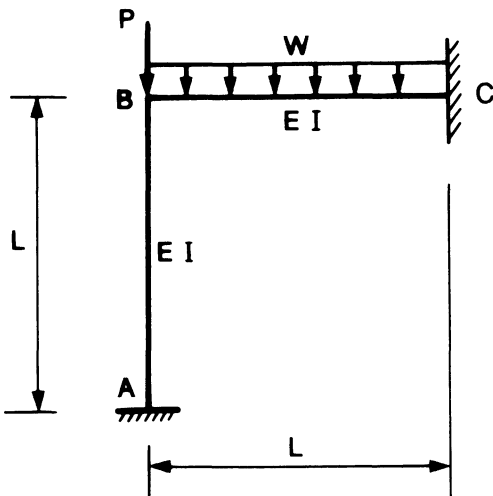


Fig. 7. Fixed-ended two-member frame.

Once this is done, the maximum $P - \delta$ moment in each member can be determined exactly by entering the second-order end-moments in Eqs. 7, but not in the B_1 factor. Of course, the exact maximum second-order moments in framed members can always be obtained by solving the governing differential equations of the structural system.

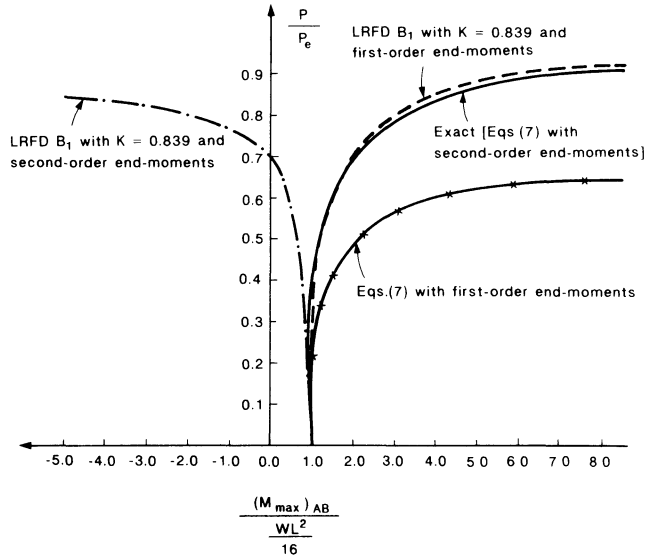


Fig. 8. Maximum moment in Column AB of hinged-ended two-member frame.

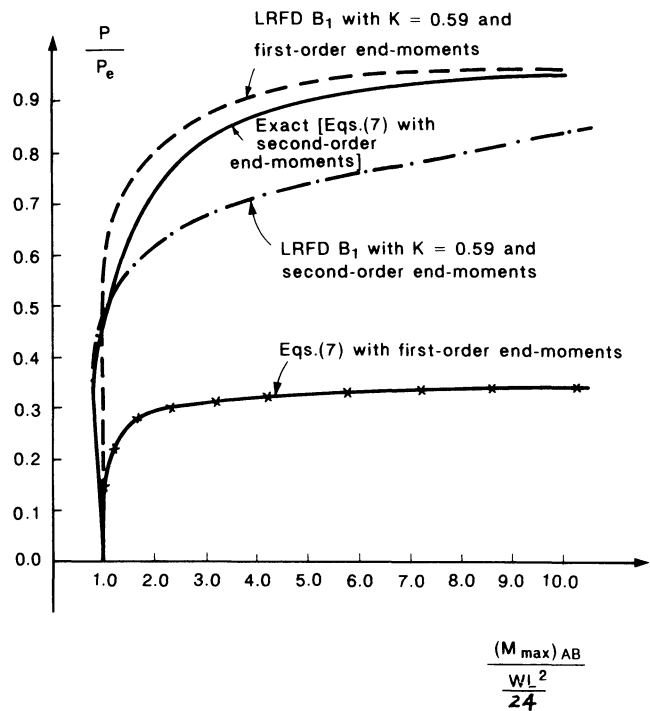


Fig. 9. Maximum moment in Column AB of fixed-ended two-member frame.