# Practical Analysis of Semi-Rigid Frames

MUNZER BARAKAT and WAI-FAH CHEN


## ABSTRACT

This study attempts to introduce a simplified method of flexible frame analysis that builds on some aspects of the already established AISC/LRFD design approaches, namely the $B_1$ and $B_2$ amplification factor method. Two idealized connection models are proposed: The first is a modified initial stiffness representation; the second is a model determined by the beam-line method. The connection models are designed for implementation in a first-order analysis of the nonsway and sway configurations of frame, thus determining the moment values $M_{nt}$ and $M_{lt}$, respectively. The design moment $M_u$ is obtained in a procedure similar to that conducted for rigid frames using the amplification factor method. The effective length factor concept is utilized with some modifications to account for connection flexibility. A modified relative stiffness factor for elastically restrained members is suggested, which allows the use of existing alignment charts for determining the effective length factor of columns.


## 1. INTRODUCTION

The steel framework is one of the most commonly used structural systems in modern construction. The analysis of such structural systems is governed by the assumptions employed in modeling these elements, especially those concerning the behavior of beam-to-column connections. Conventional methods of steel frame analysis used highly idealized joint models: the rigid-joint model and the pinned-joint model. Since the actual behavior of frame joints always falls in between these two extremes, more attention has been directed in recent years toward a more accurate modeling of such joints. The extensive research work on flexible connections resulted in a considerable amount of knowledge that prompted changes in the design provisions.

Section A2.2 of the LRFD Specification[1] identifies Type PR (partially-restrained) construction as one of two basic types of construction. In this type, the structural joints are presumed to offer some restraint to the members they connect. For the case when the flexibility of connections is considered in the analysis and design of frames, the LRFD Specification permits the evaluation of the flexibility of connections by rational analytical or empirical means. For the other basic type of construction (Type FR or fully-restrained), the specification provides for a simplified second-order elastic analysis with $B_1$ and $B_2$ amplification factors. In contrast to this, the design specification provides for Type PR only broad principles for analysis and design. It is left to the individual engineer to implement these principles in a quantitative manner.

There are several computer-based methods available for the analysis of Type PR frames.[2,6,9,10,19,21,23] Most of these methods involve, because of second-order effects, either a complicated mathematical formulation or a shortage in versatility for practical design applications. Moreover, some of the proposed methods often employ cumbersome and time-consuming numerical techniques in order to ensure convergent solutions.

In this study, a simplified procedure for the design analysis of frames with semi-rigid connections is proposed. The procedure is based on the elastic analysis for design which is permitted by Sec. A5 of the AISC LRFD Specification. It follows the basic philosophy of the so-called $B_1$ and $B_2$ amplification factor method of analysis, in conjunction with the concept of effective length for columns. Bearing in mind that, in order for $B_1$ and $B_2$ philosophy to be applicable, a structural system must behave linearly, among other conditions. Thus, the proposed procedure attempts to satisfy this requirement by linearizing the problem, particularly the moment-rotation ($M-\theta_r$) relationship of the connection. It focuses on the notion that if a linear relationship could be assumed between the connection moment and the connection rotation in the form of secant stiffness, then the $B_1$ and $B_2$ factor method may also be applied to the analysis of Type PR construction as long as the connection flexibility is considered in the first-order analysis.

## 2. $B_1$ AND $B_2$ METHOD OF ANALYSIS

The AISC LRFD Specification recommends the following design limitations for sway and nonsway beam-columns:

For $\dfrac{P_u}{\phi_c P_n} \geq 0.2$

$$\frac{P_u}{\phi_c P_n} + \frac{8}{9}\left[\frac{M_{ux}}{\phi_b M_{nx}} + \frac{M_{uy}}{\phi_b M_{ny}}\right] \leq 1.0 \qquad (2.1a)$$

*Munzer Barakat is graduate assistant, School of Civil Engineering, Purdue University, West Lafayette, Indiana.*

*Wai-Fah Chen is professor and head of structural engineering, School of Civil Engineering, Purdue University, West Lafayette, Indiana.*

For $\dfrac{P_u}{\phi_c P_n} < 0.2$

$$\frac{P_u}{2\phi_c P_n} + \left[\frac{M_{ux}}{\phi_b M_{nx}} + \frac{M_{uy}}{\phi_b M_{ny}}\right] \leq 1.0 \qquad (2.1b)$$

The principal unknowns in Eq. 2.1 are $M_{ux}$ and $M_{uy}$ which are the required flexural strength in X and Y planes. In structures designed on the basis of elastic analysis, $M_u$ is determined from a second-order elastic analysis using factored loads. As an alternate, the LRFD provisions provide for two amplification factors ($B_1$ and $B_2$) to be used with the results of a first-order analysis to estimate the design moment $M_u$ as follows:

$$M_u = B_1 M_{nt} + B_2 M_{lt} \qquad (2.2)$$

where

$M_{nt}$ = maximum moment in the member assuming no lateral translation of the frame, calculated by using a first-order elastic analysis (see Fig. 1a)
$M_{lt}$ = maximum moment in the member as a result of lateral translation of the frame only, calculated by using a first-order elastic analysis (see Fig. 1b)
$B_1$ = $P - \delta$ moment amplification factor, given by

$$B_1 = \frac{C_m}{1 - \dfrac{P_u}{P_{ek}}} \geq 1 \qquad (2.3)$$

where

$C_m$ = coefficient whose value depends upon column curvature caused by applied moments
$P_{ek}$ = $\pi^2 EI/(KL)^2$, in which $K$ is the non-sway effective length factor in the plane of bending
$B_2$ = $P - \Delta$ moment amplification factor, given by

$$B_2 = \frac{1}{1 - \Sigma P_u \left(\dfrac{\Delta_{oh}}{\Sigma HL}\right)} \qquad (2.4)$$

or alternatively

$$B_2 = \frac{1}{1 - \dfrac{\Sigma P_u}{\Sigma P_{ek}}} \qquad (2.5)$$

$\Sigma P_u$ = axial load on all columns in a story
$\Delta_o$ = first-order translational deflection of the story under consideration
$\Sigma H$ = sum of all story horizontal forces producing $\Delta_o$
$L$ = story height
$P_{ek}$ = $\pi^2 EI/(KL)^2$, in which $K$ is the sway effective length factor in the plane of bending

As can be seen, the analysis procedure is reduced to merely determining the values of $M_{nt}$, $M_{lt}$, $B_1$ and $B_2$. For this procedure to be applicable to frames with semi-rigid connections, proper modifications need to be made to reflect their existence and to take into consideration the effects of connection flexibility on force distribution. The selection of rational and simple models of the connection moment-rotation ($M-\theta_r$) relationship is an essential part for this effort to be successful.

## 3. MODELING OF CONNECTION $M-\theta_r$ RELATIONSHIP

As part of linearizing the analysis of flexible frames and revising the $B_1$ and $B_2$ method to a form that will account for connection flexibility, linear $M-\theta_r$ relationships are proposed in the following sections. In light of the fact that loading of frames in real life occurs in essentially sequential manner (meaning that after most gravity loads are applied, horizontal forces are induced), the connection stiffness changes noticeably during the loading process. Hence, in the simplified modeling of connection behavior, two secant stiffnesses are proposed. The first is a reduced initial stiffness, termed here as the modified initial stiffness $R_{ko}$, and the second is a secant stiffness $R_{kb}$ determined by the beam-line method. These stiffnesses are expected to resemble the average connection behavior under gravity and horizontal loads as treated by the $B_1$ and $B_2$ method. The first stiffness is intended for implementation in the first-order analysis of the nonsway frame devised by this method (Fig. 1a), in which the principal outcome is $M_{nt}$. The second stiffness is to be used in the first-order analysis of the sway frame (Fig. 1b) which determines $M_{lt}$.

### 3.1 Modified Initial Connection Stiffness ($R_{ko}$)

Existing connection data[15] shows that the initial values of the connection tangent stiffness are relatively high compared to those at advanced loading stages. Researchers have extensively used the initial connection stiffness $R_{ki}$ in the analysis of flexible frames[3,5,12,14,18,24] because of the relative ease of determining such value either graphically or analytically.

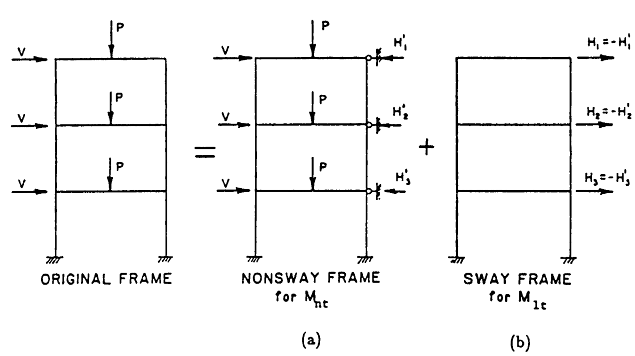



*Fig. 1. Two fictional frames for determining frame moments.*

In spite of the comfort that the initial stiffness provides for an elastic analysis, it is fair to say that $R_{ki}$ is too high a value for use throughout the analysis. Among others, Goto and Chen (1987) showed the inadequacy of using a unique initial stiffness value throughout the analysis of flexible frames. A moderately softer stiffness than the initial $R_{ki}$ is, therefore, desired.

This objective will be achieved in what follows by introducing the modified initial stiffness $R_{ko}$. This stiffness is defined as the secant modulus corresponding to the initial rotation $\theta_o$ of the connection (Fig. 2). The initial rotation $\theta_o$ is that corresponding to the intersection of the initial tangent modulus $R_{ki}$ and the ultimate moment $M_u$. The proposed stiffness will be evaluated in conjunction with the $M-\theta_r$ curve models proposed by Lui and Chen[20] and Kishi and Chen.[16] The later model provides for connection properties ($M-\theta_r$ relation, $R_{ki}$ and $M_u$) required to determine the secant modulus $R_{ko}$. The adoption of this model stems from the fact that the connection parameters are determined directly from its material and geometric properties. The procedure is systematic and can easily be implemented in a computerized method of analysis.

As a good representation of the semi-rigid type of connections, the top- and seat-angle with double web-angle connection will be primarily used here and in subsequent sections. For this type of connection, Kishi and Chen[16] present the following expressions in conjunction with Fig. 3 for determining the initial stiffness $R_{ki}$, ultimate moment $M_u$ and $M-\theta_r$ relation. The initial connection stiffness is determined by:

$$R_{ki} = \frac{3(EI_t)d_l^2}{g_l(g_l^2 + 0.78t_t^2)} + \frac{6(EI_a)d_3^2}{g_3(g_3^2 + 0.78t_a^2)} \tag{3.1}$$

where

$$g_l = g_t' - \frac{w}{2} - \frac{t_t}{2}$$

$$g_3 = g_c - \frac{w}{2} - \frac{t_a}{2}$$

$$d_l = d + \frac{t_s}{2} + \frac{t_t}{2}$$

$$d_3 = \frac{d}{2} + \frac{t_s}{2}$$

$EI_a = E\dfrac{l_p(t_a)^3}{12}$, bending stiffness of the leg of web angle adjacent to the column face

$EI_t = E\dfrac{l_t(t_t)^3}{12}$, bending stiffness of the leg of top angle adjacent to the column face

$w$ = width across flats of bolt

The ultimate moment $M_u$ is determined as the sum of the plastic capacities of connection components as follows

$$M_u = M_{os} + M_{pt} + V_{pt}d_2 + 2V_{pa}d_4 \tag{3.2}$$

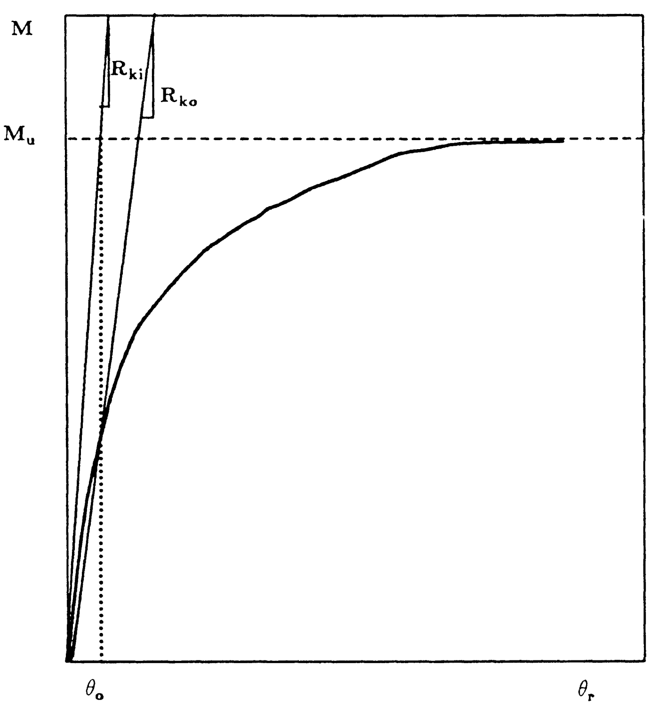



*Fig. 2. Initial and modified initial stiffness for $M_{nt}$ calculations.*

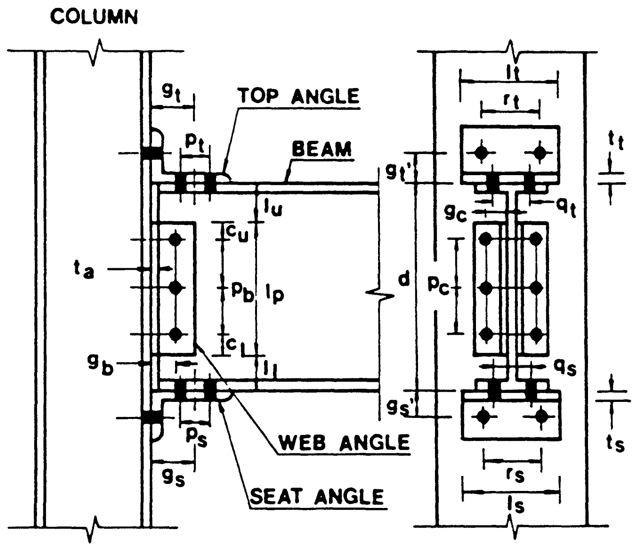



*Fig. 3. Typical top- and seat-angle with double web-angle connection.*

where

$$M_{os} = \frac{\sigma_y l_s (t_s)^2}{4}$$

$$M_{pt} = \frac{V_{pt} g_2}{2}$$

$V_{pt}$ is determined by solving the following equation

$$\left(\frac{V_{pt}}{V_{ot}}\right)^4 + \frac{g_2}{t_t}\left(\frac{V_{pt}}{V_{ot}}\right) - 1 = 0$$

$$V_{ot} = \frac{\sigma_y l_t t_t}{2}$$

$$V_{pa} = \left(\frac{V_{py} + V_{oa}}{2}\right) l_p$$

$$V_{oa} = \frac{\sigma_y t_a}{2}$$

$$g_2 = g_t' - k_t - \frac{w}{2} - \frac{t_t}{2}$$

$$g_y = \left(\frac{g_c - k_a}{l_p}\right) y$$

$$d_2 = d + \frac{t_s}{2} + k_t$$

$$d_4 = \frac{2V_{pu} + V_{oa}}{3(V_{pu} + V_{oa})} l_p + l_l + \frac{t_s}{2}$$

Using the initial connection stiffness $R_{ki}$ and the ultimate moment capacity $M_u$ of the connection, the moment-rotation $M-\theta_r$ relationship can be adequately represented by:[16]

$$M = \frac{R_{ki}\theta_r}{[1 + (\theta_r/\theta_o)^n]^{1/n}} \tag{3.3}$$

### 3.2 Connection Stiffness by Beam-Line Method

At advanced stages of loading, the connection sustains increasing rotations and consequently exhibits declining stiffness values. In sway (unbraced) frames, the connection is presumed to undergo most of such activity when the action of lateral forces adjoins that of gravity loads. In terms of the analysis procedure at hand ($B_1$ and $B_2$ method), this situation may be considered as the phase of determining the values of $M_{lt}$, which in effect the phase of analyzing a structure subjected to loads causing sideways. Thus, if any idealized connection stiffness is to be considered in the analysis of frame under sway loads (determination of $M_{lt}$), such stiffness should be less than that used for determining $M_{nt}$ moments caused by nonsway loads. The other major factors that may influence the degree to which the connection stiffness has decreased before lateral loads commence are the intensity of gravity loads and the stiffness of the beam. With these considerations in mind, the secant stiffness determined by the beam-line is chosen as the average connection stiffness that approximates the connection behavior in the analysis for $M_{lt}$. The determination of this stiffness is presented in the following.

For an elastic behavior, the relationship between the beam end moment $M_E$ and end rotation $\phi_b$ can be determined by using the slope-deflection equations and is expressed by:

$$M_E = M_F - \frac{2EI}{L}\phi_b \tag{3.4}$$

where $M_F$ is the fixed-end moment. For a uniformly loaded beam, Eq. 3.4 can be written as

$$M_E = \frac{wL^2}{12}\left(1 - \frac{\phi_b}{\phi_{bo}}\right) \tag{3.5}$$

where $\phi_{bo} = \frac{wL^3}{24EI}$, end rotation for a uniformly loaded, simply supported beam. For any value of $w$, Eq. 3.5 represents a straight line (beam-line) on an end moment vs. end rotation diagram (Fig. 4).

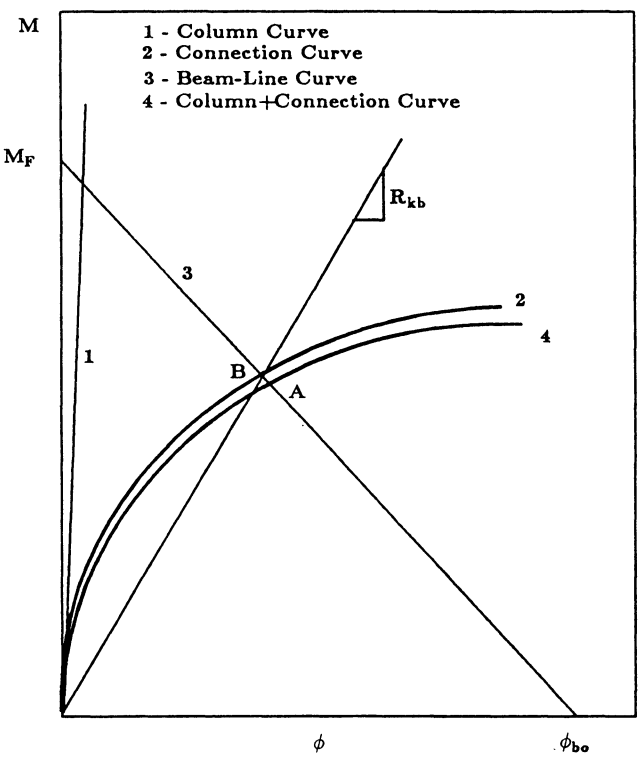



*Fig. 4. Moment-rotation curves for joint components and the determination of $R_{kb}$ for $M_{lt}$*

Figure 4 schematically shows the moment-rotation relationships for the three components of a beam-joint-column assembly.[22] Curve 1 represents the moment-rotation relationship at the column end, curve 2 is the connection $M-\theta_r$ relation and curve 3 is the beam line. Compatibility of rotational deformation at a joint combining these elements will be satisfied at the intersection of the beam line (curve 3) with curve 4 which combines the connection and column rotations (curve 1 + curve 2). This is shown as point A in the figure. Noting that the end rotation of the column is usually very small compared to that of the connection, curve 4 can therefore be approximated by curve 2 for the determination of point A which is, consequently, replaced by point B. The simplified linear model of the connection $M-\theta_r$ relation is chosen as the secant modulus that passes through the intersection point of the beam line and the moment-rotation curve of connection (point B). This modulus is denoted here as $R_{kb}$ and will be implemented in the first-order analysis of the sway frame of Fig. 1b for the determination of $M_{lt}$ moments.

Having prepared the two linear connection models ($R_{ko}$ and $R_{kb}$) intended for use in the analysis procedure, we now proceed to deal with the question of effective length of columns under flexible restraint conditions.

## 4. EFFECTIVE LENGTH OF COLUMNS

### 4.1 Relative Stiffness Factor ($G$)

In designing rigid frames, it is common practice to isolate each member from the frame and design it as an individual beam-column. The influence of adjacent members (end-restraint conditions) on the behavior of the particular member is usually accounted for by using the effective length of the member in question. The effective length $l$ of an end-restrained column is defined as the length of an equivalent pin-ended column that will give the same critical load as the end-restrained column.

One convenient way of determining the column effective length is to use the concept of effective length factor $K$ ($l = KL$, where $L$ is the actual length of column, $K = \sqrt{P_e/P_{cr}}$, where $P_e$ is the Euler buckling load and $P_{cr}$ is the critical load of column with actual end restraints). The determination of the effective length factor $K$ for a framed member is a complex procedure, because the stiffness of all adjacent members, as well as the rigidities of connections, must be included in the process. As an alternate, the AISC LRFD Specification[1] allows the use of alignment charts (Fig. 5) based on a procedure proposed by Julian and Lawrence[13] where $K$ is determined by evaluating the relative stiffness

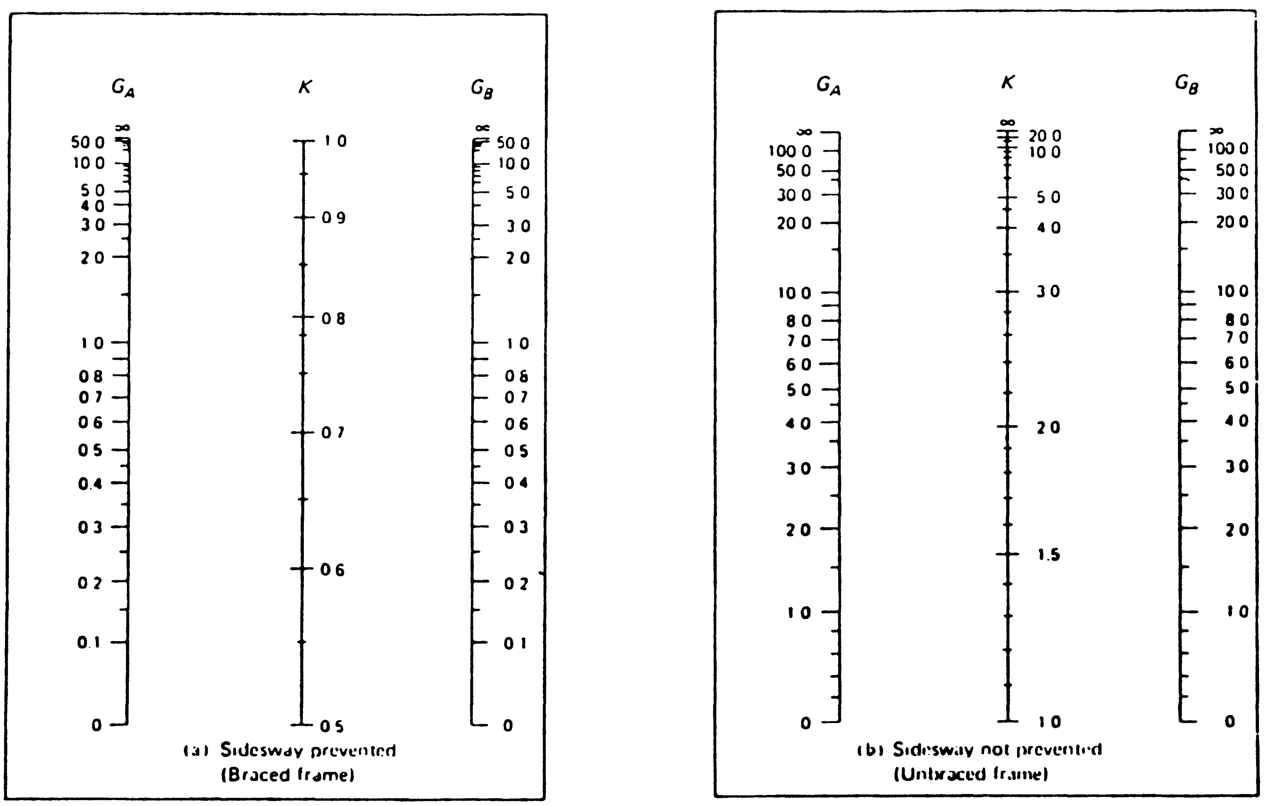



*Fig. 5. Alignment charts for the determination of the effective length factor K.*

factors $G_A$ (at column end A) and $G_B$ (at column end B). A straight line joining the two values $G_A$ and $G_B$ will intersect the middle line which gives the value of $K$. The relative stiffness factor $G$ at any end is expressed as:

$$G = \frac{\Sigma\left(\frac{EI}{L}\right)_c}{\Sigma\left(\frac{EI}{L}\right)_b} = \qquad (4.1)$$

$$\frac{\text{sum of column stiffnesses meeting at the joint}}{\text{sum of beam stiffnesses meeting at the joint}}$$

In practical terms of the AISC LRFD design format, the concept of effective column length is embedded in the amplification factors $B_1$ and $B_2$ (Eqs. 2.3 and 2.5) in which $P_{ek}$ is expressed by $\frac{\pi^2 EI}{(KL)^2}$. To be able to implement this concept in the design analysis of flexible frames, and to justify the adoption of the AISC format for such frames, proper modifications have to be made to account for the reduced amount of restraint at column ends due to the presence of semi-rigid connections. This can be achieved by expressing the apparent effects through a modified representation of the relative stiffness factor $G$ which, consequently, provides for an updated effective length factor $K$.

Since the design format of the $B_1$ and $B_2$ method involves the determination of effective length factors for both nonsway (determination of $B_1$) and sway frames (determination of $B_2$), the two cases are considered in the following derivation of a modified relative restraint factor.

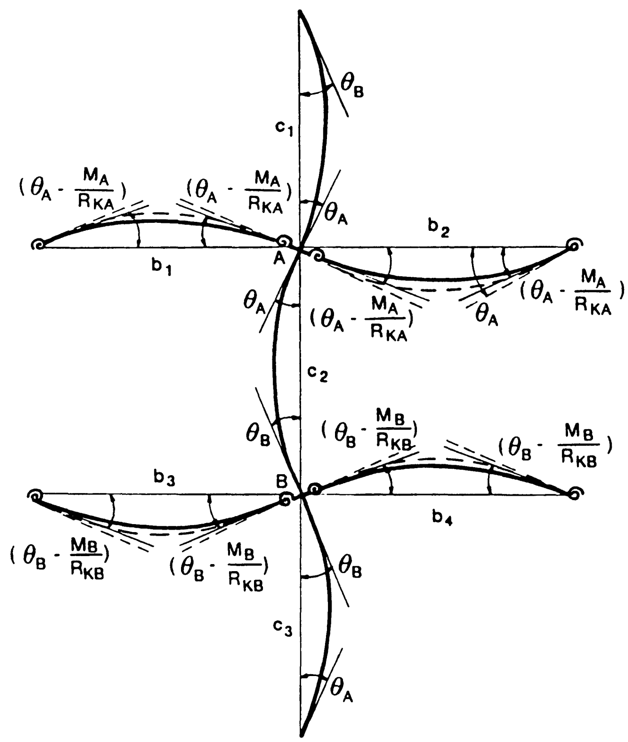



*Fig. 6. Subassemblage model for braced (nonsway) frame.*

## 4.2 Modified Relative Stiffness Factor for Nonsway Frames

The derivation of a relationship for the relative stiffness factor $G$ that accounts for frame flexibility is similar to that provided for rigid frames.[7] The model used for this purpose is shown in Fig. 6 which illustrates an assumed deflected shape at the bifurcation state of a subassemblage of a braced frame. Semi-rigid connections are modeled as elastic springs attached to the ends of beam members. The column under consideration is Column $C_2$. The assumptions used for the model are:

1. All members are prismatic and behave elastically.
2. Beam-to-column connections behave linearly with identical stiffness parameters in each floor.
3. The axial force in beam members is negligible.
4. All columns in the frame buckle simultaneously.
5. At a joint, the restraining moment provided by the beam is distributed among the columns in proportion to their stiffnesses.
6. At buckling, the rotations at the near and far end of beams are equal in magnitude and opposite in direction (beams are bent in single curvature). Using the standard form of the slope-deflection equations and the modified form for relative joint translation or elastic and restraints[7] as applicable, the equilibrium equations for the subassemblage can be written in the form:[4]

$$\begin{bmatrix} s_{ii} + \frac{2}{G'_A} & s_{ij} \\ s_{ij} & s_{ii} + \frac{2}{G'_B} \end{bmatrix} \begin{pmatrix} \theta_A \\ \theta_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (4.2)$$

where

$$G' = \frac{\Sigma\left(\frac{EI}{L}\right)_c}{\Sigma(\alpha_{nt})_b \left(\frac{EI}{L}\right)_b} \qquad (4.3)$$

$$(\alpha_{nt})_{bi} = \left[\frac{1}{1 + 2\frac{\left(\frac{EI}{L}\right)_{bi}}{R_{kA}}}\right] \qquad (4.4)$$

At bifurcation, the determinant of the coefficient matrix vanishes which leads to the following governing equation.

$$\frac{G'_A G'_B}{4}\left(\frac{\pi}{K}\right)^2 + \left(\frac{G'_A + G'_B}{2}\right)\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + \frac{2\tan(\pi/2K)}{\pi/K} - 1 = 0 \tag{4.5}$$

Equation 4.5 is identical to that developed previously by Julian and Lawrence[13] for rigid frames, except that $G$ at each member end is now replaced by $G'$. The modified relative stiffness factor $G'$ accounts for the presence of elastic beam-to-column connections. A relationship $G'$ and $G$ can now be established in the form

$$G' = \bar{\alpha}_{nt} G \tag{4.6}$$

where

$$\bar{\alpha}_{nt} = \frac{\Sigma\left(\frac{EI}{L}\right)_b}{\Sigma(\alpha_{nt})_b\left(\frac{EI}{L}\right)_b} \tag{4.7}$$

$\bar{\alpha}_{nt}$ is the scaling factor by which the relative stiffness factor $G'$ for members in flexible frames with no lateral translation is obtained from $G$.

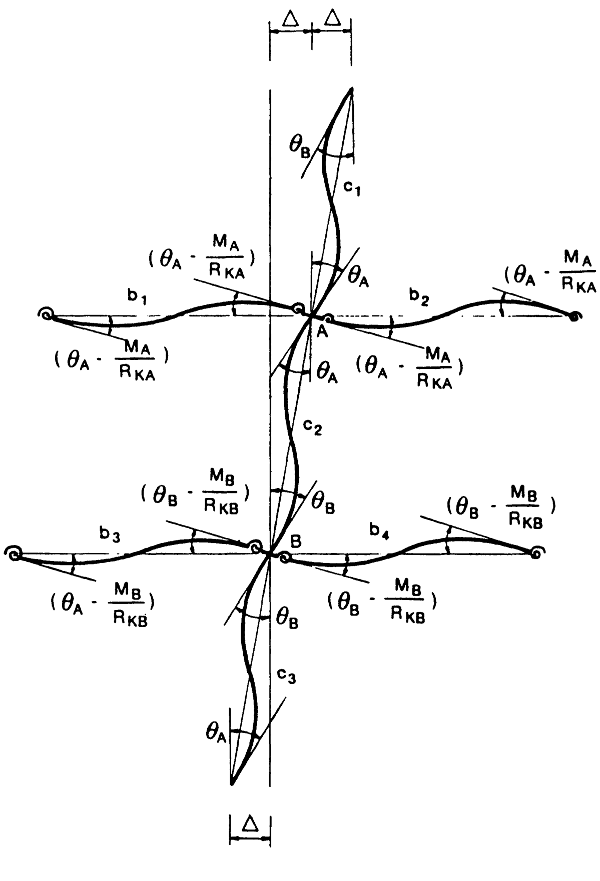



*Fig. 7. Subassemblage model for unbraced (sway) frame.*

For rigid frames, the relationship between $G_A$, $G_B$ and $K$ can be expressed by an alignment chart as used in the AISC LRFD Specification (Fig. 5). Realizing that Eq. 4.6 represents a linear relationship, the same alignment chart can be used for determining the effective length factor of an elastically restrained column by entering the values of $G'_A$ and $G'_B$.

### 4.3 Modified Relative Stiffness Factor for Sway Frames

Following a similar procedure, we proceed now to derive the relationship governing the end restraint factors for sway frames with semi-rigid connections. The model used for this purpose is shown in Fig. 7 which illustrates an assumed deflected shape at the bifurcation state of subassemblage of a flexible frame where lateral translation is not prohibited. Again, the column under consideration is Column $C_2$. The assumptions used for this model are the same as for the nonsway case except for point 6 which is modified to the following: At buckling, the rotations at the near and far ends of the beam are equal and in the same direction (i.e., the beams are bent in double curvature).

Similar to the case of nonsway frames described in Sec. 4.2, the state of bifurcation leads to the following governing equation:

$$\frac{G'_A G'_B(\pi/K)^2 - 36}{6(G'_A + G'_B)} - \frac{(\pi/K)}{\tan(\pi/K)} = \tag{4.8}$$

where

$$G' = \frac{\Sigma\left(\frac{EI}{L}\right)_c}{\Sigma(\alpha_{lt})_b\left(\frac{EI}{L}\right)_b} \tag{4.9}$$

which is identical to the equation derived for unbraced rigid frames,[7] except that $G$ has been replaced by $G'$ which reflects the effects of elastic joint flexibility. The relationship between $G'$ and $G$ can be expressed by

$$G' = \bar{\alpha}_{lt} G \tag{4.10}$$

where

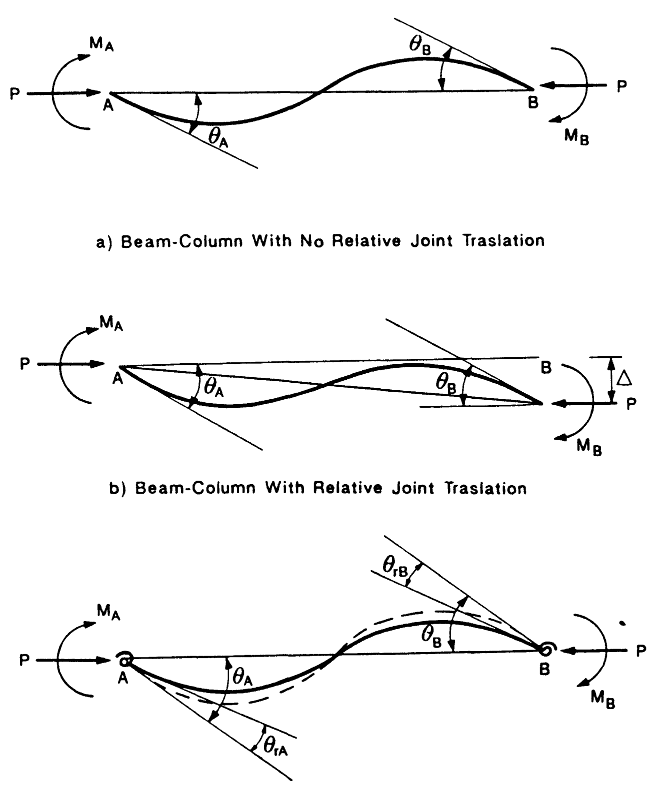
$$\bar{\alpha}_{lt} = \frac{\Sigma\left(\frac{EI}{L}\right)_b}{\Sigma(\alpha_{lt})_b\left(\frac{EI}{L}\right)_b} \tag{4.11}$$

The symbol $\bar{\alpha}_{lt}$ is the scaling factor by which the relative stiffness factor $G$ is scaled to give the equivalent relative stiffness factor for flexible frames $G'$. Equation 4.10 is a linear relationship which makes it possible to use the AISC LRFD alignment chart for unbraced rigid frames (Fig. 5) to determine the effective length factor for column members in flexible sway frames by entering the values of $G'$ as $G$.

It is worth mentioning that the expressions for the modified relative stiffness factor $G'$ provided in Sec. 4.2 and 4.3 reduce to $G$ when very stiff (rigid) connections are used. For rigid joints ($R_k \rightarrow \infty$), the scaling factor $\bar{\alpha}$ becomes equal to unity, which leads to $G' = G$. For simple joints ($R_k \rightarrow 0$), $\bar{\alpha}$ becomes equal to zero and the relative restraint factor $G' = G = 0$.

## 5. MODIFIED STIFFNESS OF BEAM $(EI')_b$

Sections 4.2 and 4.3 showed the applicability of the existing alignment charts for determining the effective length factor of columns in a linearized flexible frame analysis. This was achieved by deriving the governing equations (Eqs. 4.5 and 4.8) relating $G'_A$, $G'_B$ and $K$. This section employs a similar but much simpler approach that focuses on the beam member only to show that the effects of elastic joint flexibility can be taken into account by using a reduced beam stiffness $(EI')_b$. Since the modulus of elasticity $E$ is a constant parameter for steel members, the actual change will only involve the second moment of inertia of the beam element. This will be accomplished by examining a general case of beam member restrained by different connections at its ends. Further, special cases of a member with same end connections will be considered.




a) Beam-Column With No Relative Joint Traslation

b) Beam-Column With Relative Joint Traslation

c) Beam-Column With Elastically-Restrained Ends

*Fig. 8. Beam-column subjected to end moments.*

### 5.1 Modified Slope-Deflection Equations

Figure 8c shows a beam-column member with elastic end restraints. The member is loaded with end forces. The slope-deflection equations for this member have the form, in the usual notations

$$M_A = \frac{EI}{L}\left[ s_{ii}(\theta_A - \theta_{rA}) + s_{ij}(\theta_B - \theta_{rB}) \right] \tag{5.1}$$

$$M_B = \frac{EI}{L}\left[ s_{ij}(\theta_A - \theta_{rA}) + s_{ii}(\theta_B - \theta_{rB}) \right] \tag{5.2}$$

Substituting $\theta_{rA} = \dfrac{M_A}{R_{kA}}$ and $\theta_{rB} = \dfrac{M_B}{R_{kB}}$, we get:

$$M_A = \frac{EI}{L}\left[ s'_1\theta_A + s'_2\theta_B \right] \tag{5.3}$$

$$M_B = \frac{EI}{L}\left[ s'_2\theta_A + s'_3\theta_B \right] \tag{5.4}$$

where $s'_1$, $s'_2$ and $s'_3$ are the modified stability functions due to the presence of flexible connections, expressed as:

$$s'_1 = \frac{R'_{kA}(R'_{kB}s_{ii} + s^2_{ii} - s^2_{ij})}{R'_{kA}R'_{kB} + s_{ii}(R'_{kA} + R'_{kB}) + s^2_{ii} - s^2_{ij}} \tag{5.5}$$

$$s'_2 = \frac{R'_{kA}R'_{kB}s_{ij}}{R'_{kA}R'_{kB} + s_{ii}(R'_{kA} + R'_{kB}) + s^2_{ii} - s^2_{ij}} \tag{5.6}$$

$$s'_3 = \frac{R'_{kB}(R'_{kA}s_{ii} + s^2_{ii} - s^2_{ij})}{R'_{kA}R'_{kB} + s_{ii}(R'_{kA} + R'_{kB}) + s^2_{ii} - s^2_{ij}} \tag{5.7}$$

where

$$R'_{kA} = \frac{L}{EI}R_{kA} \tag{5.8}$$

$$R'_{kB} = \frac{L}{EI}R_{kB} \tag{5.9}$$

Equations 5.3 and 5.4 are the modified slope-deflection equations that account for linear flexible connections at member ends. Similar expressions have been presented in Ref. 18, where incremental formulation was considered for nonlinear analysis.

Two idealized cases of beam members will now be considered. These are the cases expressing the two deflection modes associated with the bifurcation of braced frames (single curvature bending) and unbraced frames (double curvature bending).

### 5.2 Beam Member Bent in Single Curvature

For a beam member with identical end connections ($R_{kA}$ =

$R_{kB} = R_k$) and single curvature bending, the end rotations are equal in magnitude and opposite in direction ($\theta_B = -\theta_A$). Considering that for beam members $s_{ii} = 4$ and $s_{ij} = 2$, Eqs. 5.5, 5.6, and 5.7 can be reduced to

$$s_1' = \frac{4R_k'^2 + 12R_k'}{R_k'^2 + 8R_k' + 12} \tag{5.10}$$

$$s_2' = \frac{2R_k'^2}{R_k'^2 + 8R_k' + 12} \tag{5.11}$$

$$s_3' = s_1' \tag{5.12}$$

The end moments (Eqs. 5.3 and 5.4) are determined by

$$M_A = \frac{2E}{L} I \left[ \frac{1}{1 + \dfrac{2EI}{R_k L}} \right] \theta_A \tag{5.13}$$

$$M_B = -M_A \tag{5.14}$$

Comparing Eq. 5.13 with the equation derived for rigidly connected beam member, in which the moment at each end is expressed by

$$M = \frac{2EI}{L}\theta \tag{5.15}$$

we can write Eq. 5.13 in a similar form as

$$M_A = \frac{2EI_{nt}'}{L}\theta_A \tag{5.16}$$

where

$$I_{nt}' = \left[ \frac{1}{1 + \dfrac{2EI}{R_k L}} \right] I \tag{5.17}$$

$I_{nt}'$ is the modified moment of inertia of beam element with linear flexible connections at both ends, for which loading conditions produce single curvature bending. Note that the expression in brackets is actually the coefficient $\alpha_{nt}$ in Sec. 4.2. Consequently, the modified relative stiffness factor can be directly written in the following form

$$G' = \frac{\Sigma \left( \dfrac{EI}{L} \right)_c}{\Sigma \left( \dfrac{EI_{nt}'}{L} \right)_b} \tag{5.18}$$

or

$$G' = \bar{\alpha}_{nt} G \tag{5.19}$$

where

$$\bar{\alpha}_{nt} = \frac{\Sigma I_b}{\Sigma (I_{nt}')_b} \tag{5.20}$$

$\bar{\alpha}_{nt}$ is the nonsway scaling factor which allows the use of existing alignment charts for determining the effective length of elastically restrained columns.

### 5.3 Beam Member Bent in Double Curvature

The modified stability functions $s_1'$, $s_2'$, and $s_3'$ remain the same as expressed in Equations 5.10, 5.11, and 5.12. The only difference for this case is that $\theta_A = \theta_B = \theta$. Substituting these values into Eqs. 5.3 and 5.4 gives

$$M_A = \frac{6E}{L} I \left[ \frac{1}{1 + \dfrac{2EI}{R_k L}} \right] \theta_A \tag{5.21}$$

$$M_B = M_A \tag{5.22}$$

Comparing this equation with that derived for rigidly connected beam member, in which the moment at each end is expressed by

$$M = \frac{6EI}{L}\theta \tag{5.23}$$

Equation 5.21 can be written in a similar form as

$$M_A = \frac{6EI_{lt}'}{L}\theta_A \tag{5.24}$$

where

$$I_{lt}' = \left[ \frac{1}{1 + \dfrac{6EI}{R_k L}} \right] I \tag{5.25}$$

$I_{lt}'$ is the modified moment of inertia of a beam element with elastic end restraints, bent in double curvature. The expression in brackets is the same as coefficient $\alpha_{lt}$ introduced in Sec. 4.3. The modified relative stiffness factor has the same form as Eq. 5.18 or Eq. 5.19, except that the scaling factor $\bar{\alpha}_{nt}$ is now $\bar{\alpha}_{lt}$ and the equation becomes

$$G' = \bar{\alpha}_{lt} G \tag{5.26}$$

where

$$\bar{\alpha}_{lt} = \frac{\Sigma I_b}{\Sigma (I_{lt}')_b} \tag{5.27}$$

Since $I_b'$ is a reduced moment of inertia as opposed to $I_b$ (see Eq. 5.25), it can be seen that the modified relative stiffness factor $G'$ is numerically larger than $G$ which in effect means reduced restraint at the column end, and which ultimately results in higher values of the effective length factor $K$.

## 6. OUTLINE OF THE PROPOSED ANALYSIS PROCEDURE

Having developed the appropriate connection models and

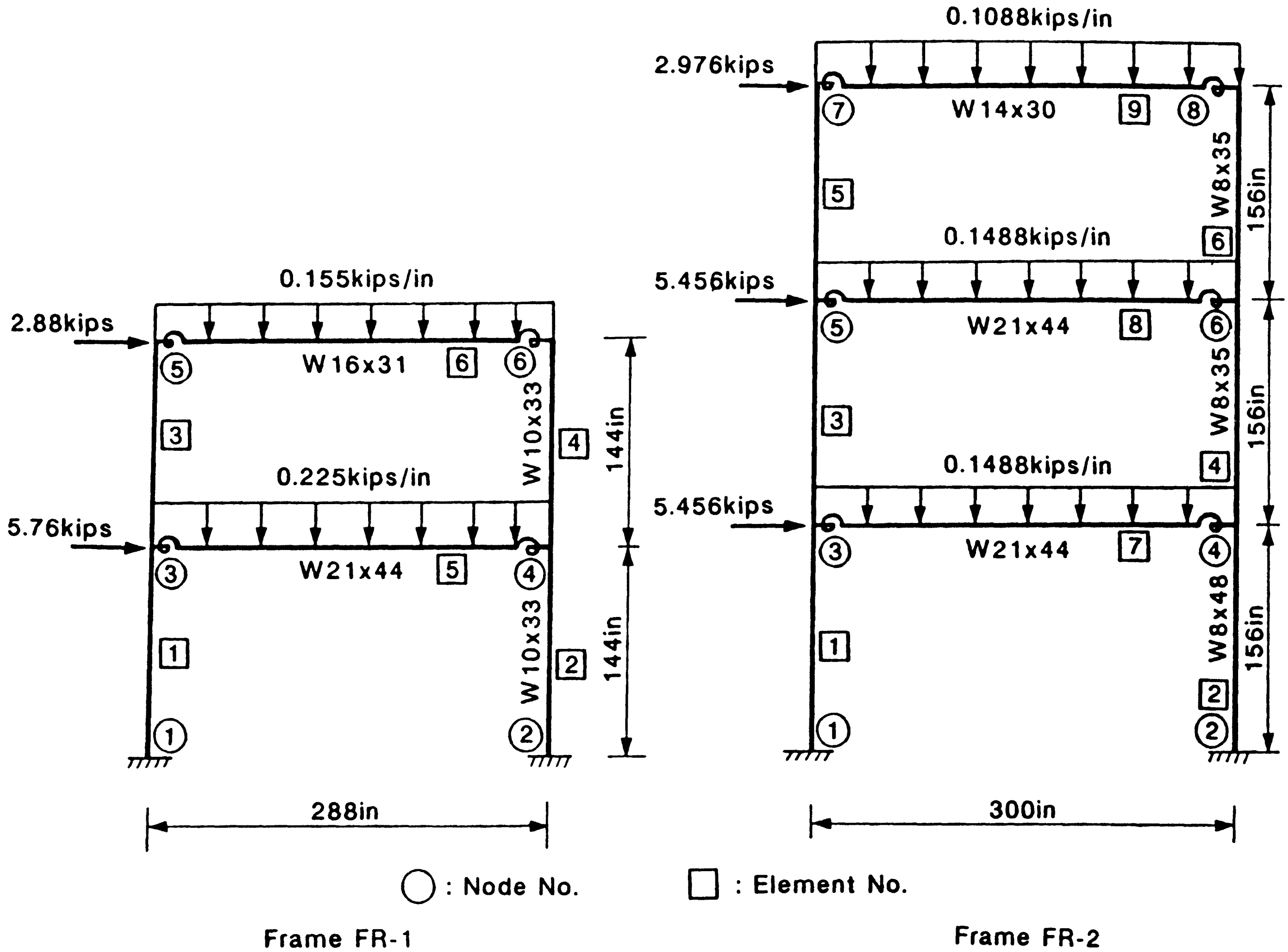



*Fig. 9. Partially-restrained frames FR-1 and FR-2.*

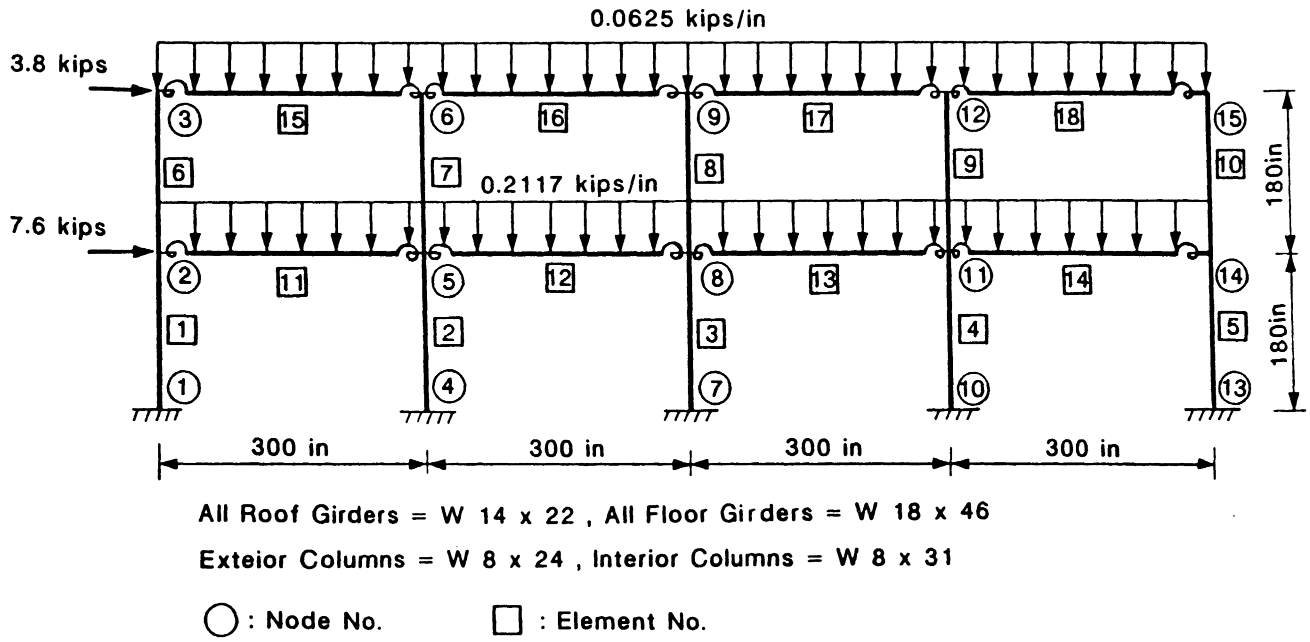



*Fig. 10. Partially-restrained frame FR-3.*

procedural modifications, the analysis procedure can now be outlined in the following steps.

1. Determine connection stiffness values: $R_{ko}$ as described in Sec. 3.1, and $R_{kb}$ as described in Sec. 3.2.
2. For the frame with loading arrangements and boundary conditions outlined in Fig. 1a, use a first-order elastic analysis which incorporates the linear connection stiffness $R_{ko}$. This step determines the column moments $M_{nt}$.
3. For the frame with loading arrangements and boundary conditions outlined in Fig. 1b, use a first-order elastic analysis which incorporates the linear connection stiffness $R_{kb}$. This step determines the column moments $M_{lt}$.
4. Determine the modified stiffnesses of the beam elements ($I'_{nt}$ and $I'_{lt}$) due to the presence of flexible connections at their ends.
5. Using the modified stiffness for beam members, determine the modified relative stiffness factors $G'$ at each column end for both nonsway and sway cases.
6. Determine the effective length factor $K$ for column members by entering the values $G'$ as $G$ into the alignment charts.
7. Determine the values of $P_{ek}$ for each compression member.
8. Evaluate the amplification factors $B_1$ and $B_2$ according to Eqs. 2.3 and 2.5, respectively.
9. The design column moment is determined by Eq. 2.2.

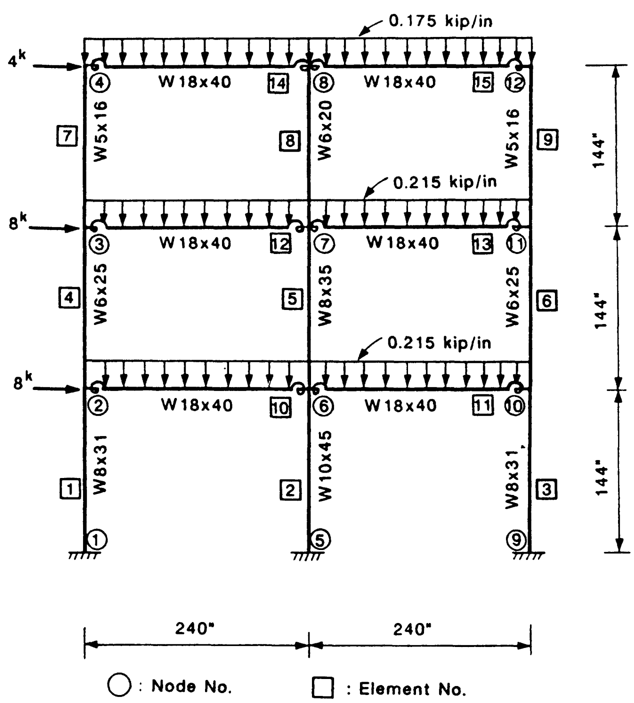



*Fig. 11. Partially-restrained frame FR-4.*

## 7. NUMERICAL STUDY

As proposed in this study, a modified initial stiffness $R_{ko}$ and a secant stiffness determined by the beam-line method $R_{kb}$ are used to analyze the nonsway and sway frames according to Figs. 1a and 1b, respectively. This section attempts to illustrate how these stiffness values comply with the concept of their usage. For this purpose, and for illustrating the proposed method of analysis, a number of frame-connection combinations are used. The frames are labeled FR-1, FR-2, FR-3 and FR-4 and are shown in Figs. 9, 10, and 11. The connections (selected from Ref. 15 and labeled III-11, III-14, III-16 and III-17) are of type top- and seat-angle with double web angles and are shown simultaneously in Fig. 12. Exact second-order (with actual $M-\theta_r$ connection curve) and linear first-order analyses were conducted using the computer program FLFRM.[11]

According to the proposed procedure, $R_{ko}$ is used to represent the average connection stiffness in the process of determining $M_{nt}$. To examine this concept, a comparative analysis is conducted using all four frames and implementing each of the four semi-rigid connections. Exact second-order elastic analysis is conducted using sequential loading where gravity loads are applied as the first loading sequence, the horizontal loads are then added as the second loading sequence. The results of the two approaches of analysis are plotted in Figs. 13 and 14. It can be seen that the modified initial stiffness predicts the actual connection behavior and, consequently,

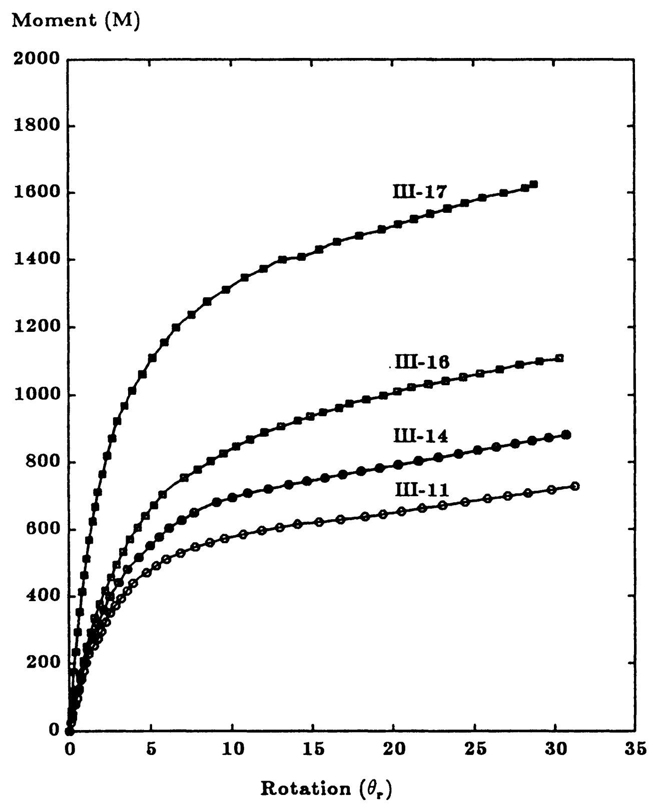



*Fig. 12. Experimental connection curves (Kishi and Chen, 1986).*

the overall frame behavior under gravity loads very closely.

The performance of $R_{kb}$ as the average connection stiffness in the calculations for $M_{lt}$ is now examined. First, an exact second-order analysis using actual connection curves is performed for load sequence 2 which determines column moments $M_{exact}$. Secondly, a first-order analysis is performed implementing connection stiffness $R_{kb}$ which gives column moments $M_{lt}$. The results are presented in Figs. 15 and 16. It can be seen that the use of connection stiffness $R_{kb}$ allows to predict, to a sufficient degree of accuracy, the behavior of connection expressed by moment distribution in frame columns.

More results of frame analysis are presented in a tabulated

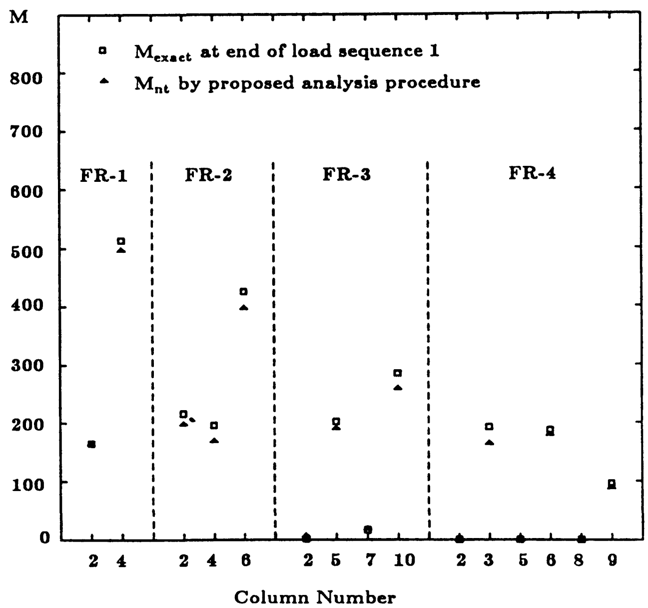



*Fig. 13. Comparison of column moments due to nonsway loads in frames with connection III-14 of medium rigidity.*

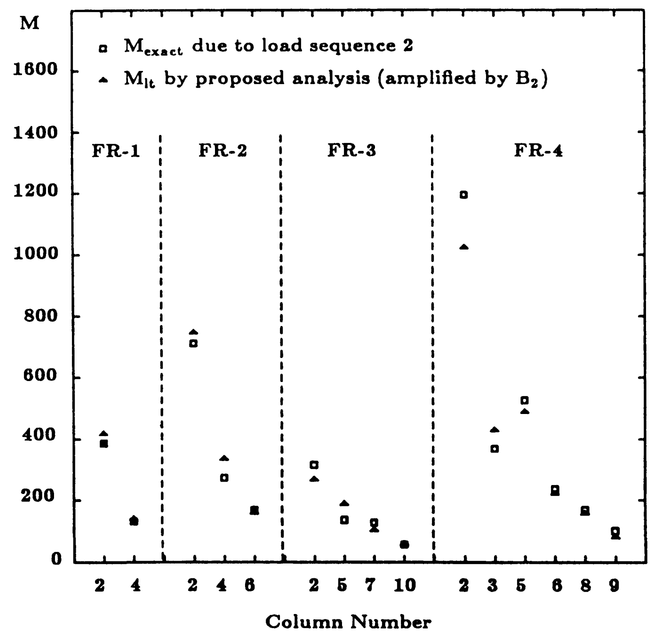



*Fig. 15. Comparison of column moments due to sway loads in frames with connection III-14 of medium rigidity.*

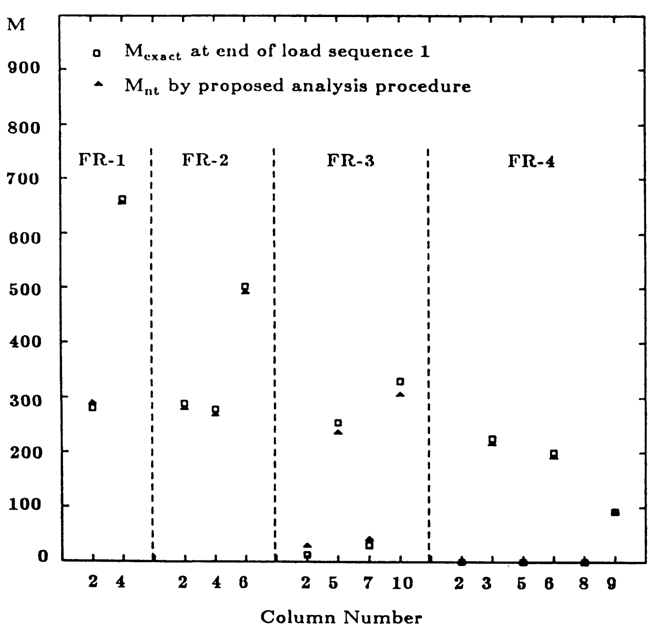



*Fig. 14. Comparison of column moments due to nonsway loads in frames with the more rigid connection III-17.*

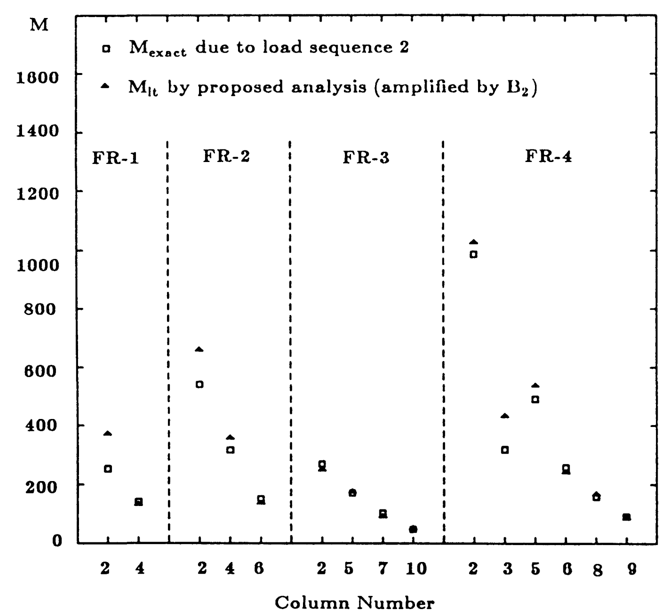



*Fig. 16. Comparison of column moments due to sway loads in frames with the more rigid connection III-17.*

form. Table 1 shows moment values of column members $M_{exact}$ for all four frames determined by exact second-order analysis using connection III-16, as well as moment values $M_u$ determined by the proposed analysis procedure. It can be seen that the procedure offers very good predictions for the design column moments in these frames. Tables 2 and 3 contain the normalized (by exact solution, i.e., $M_{exact}$) column moments in frames FR-1 and FR-2, respectively. It is evident from these tables that the predictions of the design column moments by the proposed method of analysis are very good and conservative for most members. The unconservative moment values (all of which happened to occur in top floor columns) are considered to be within the allowable tolerance in engineering practice (less than 5 percent). In addition, these values represent, in most cases, better estimates than those obtained by the conventional frame analysis.

The decision of selecting a different, and at the same time, softer stiffness (i.e., $R_{ko}$) as opposed to the initial connection stiffness ($R_{ki}$) was made after conducting a considerable amount of parametric calculations which showed the inadequacy of $R_{ki}$ for the simplified method of analysis. A sample of data supporting this suggestion is presented in Table 4. It can be seen that the design moment $M_u$ determined by using $R_{ko}$ for the calculation of $M_{nt}$ are closer to the exact solution than those obtained by using $R_{ki}$. As compared to $R_{ko}$, the higher stiffness $R_{ki}$ causes larger $M_{nt}$ moments to be allocated at beam ends which are, consequently, transferred to column ends. It is evident that the modified initial stiffness $R_{ko}$ is a more adequate and reasonable choice for the simplified analysis as opposed to the initial stiffness $R_{ki}$.

The wisdom of using two different connection stiffnesses in the analysis instead of a single average stiffness is demonstrated in Tables 5 and 6. The tables show the results of frame analysis in the form of normalized (by exact solution) moment values obtained by implementing different selections of idealized connection stiffnesses. First, a single average stiffness $R_{ki}$ was used for determining both $M_{nt}$ and $M_{lt}$. Then, the initial stiffness $R_{ki}$ was used in the

**Table 1.**
**Moment Values Determined by the Proposed ($M_u$) and Exact ($M_{exact}$) Analyses (Connection III-16 Is Used)**

| Frame | Code | $M_u$ | $M_{exact}$ | $M_u/M_{exact}$ |
|---|---|---|---|---|
| FR-1 (Fig.9) | Col. 2 | −636 | −543 | 1.170 |
| | Col. 4 | −704 | −691 | 1.020 |
| FR-2 (Fig. 9) | Col. 2 | −998 | −891 | 1.109 |
| | Col. 4 | −604 | −522 | 1.156 |
| | Col. 6 | −596 | −611 | 0.976 |
| FR-3 (Fig. 10) | Col. 2 | −308 | −302 | 1.03 |
| | Col. 5 | −420 | −369 | 1.137 |
| | Col. 7 | −128 | −133 | 0.968 |
| | Col. 10 | −341 | −337 | 1.009 |
| FR-4 (Fig. 11) | Col. 2 | −1072 | −1102 | 0.973 |
| | Col. 3 | −637 | −559 | 1.139 |
| | Col. 5 | −551 | −512 | 1.076 |
| | Col. 6 | −427 | −430 | 0.993 |
| | Col. 8 | −174 | −166 | 1.047 |
| | Col. 9 | −182 | −186 | 0.978 |

**Table 2.**
**Design Moments $M_u$ in FR-1 Normalized by the Exact Solution $M_{exact}$ (see Fig. 9)**

| Connection Used (Fig. 12) | Normalized Design Moment | |
|---|---|---|
| | Col. 2 | Col. 4 |
| Connection III-11 | 1.065 | 1.055 |
| Connection III-14 | 1.101 | 1.002 |
| Connection III-16 | 1.170 | 1.020 |
| Connection III-17 | 1.239 | 0.993 |

**Table 3.**
**Design Moments $M_u$ in FR-2 Normalized by the Exact Solution $M_{exact}$ (see Fig. 9)**

| Connection Used (Fig. 12) | Normalized Design Moment | | |
|---|---|---|---|
| | Col. 2 | Col. 4 | Col. 6 |
| Connection III-11 | 1.023 | 1.287 | 0.977 |
| Connection III-14 | 1.059 | 1.241 | 0.968 |
| Connection III-16 | 1.110 | 1.156 | 0.976 |
| Connection III-17 | 1.159 | 1.089 | 0.971 |

**Table 4.**
**Analysis Results for Moments $M_{nt}$ and $M_{lt}$ Using Combinations of $R_{ki}$ and $R_{ko}$ with $R_{kb}$ for Linear Models of Connection III-16**

| Frame Code | Column No. | Analysis with $R_{ki}$ and $R_{kb}$ | | | Analysis with $R_{ko}$ and $R_{kb}$ | | | Exact Analysis |
|---|---|---|---|---|---|---|---|---|
| | | $M_{nt}$ | $M_{lt}$ | $M_u$ | $M_{nt}$ | $M_{lt}$ | $M_u$ | |
| FR-1 | 2 | 297 | 406 | 720 | 208 | 406 | 636 | 543 |
| (Fig. 9) | 4 | 662 | 141 | 806 | 558 | 141 | 704 | 691 |
| FR-2 | 2 | 284 | 726 | 1043 | 222 | 726 | 998 | 891 |
| (Fig. 9) | 4 | 273 | 368 | 662 | 204 | 368 | 604 | 522 |
| | 6 | 496 | 156 | 656 | 435 | 156 | 596 | 611 |
| FR-3 | 2 | 31 | 263 | 321 | 13 | 263 | 308 | 302 |
| | 5 | 237 | 185 | 441 | 212 | 185 | 420 | 369 |
| (Fig. 10) | 7 | 44 | 100 | 147 | 24 | 100 | 128 | 133 |
| | 10 | 307 | 57 | 366 | 281 | 57 | 341 | 337 |
| FR-4 | 2 | 00 | 997 | 1058 | 00 | 997 | 1072 | 1102 |
| | 3 | 214 | 419 | 659 | 186 | 419 | 637 | 559 |
| (Fig. 11) | 5 | 00 | 486 | 533 | 00 | 486 | 551 | 512 |
| | 6 | 190 | 222 | 433 | 175 | 222 | 426 | 429 |
| | 8 | 00 | 157 | 172 | 00 | 157 | 174 | 166 |
| | 9 | 89 | 82 | 179 | 91 | 82 | 182 | 186 |

analysis for determining $M_{nt}$, with $R_{kb}$ for determining $M_{lt}$. Finally, the proposed selection of connection stiffnesses: $R_{ko}$ for determining $M_{nt}$ and $R_{kb}$ for determining $M_{lt}$. The moments determined by the traditional rigid frame assumption with $B_1$ and $B_2$ method of analysis are also included. It is evident that $R_{ko}$ and $R_{kb}$ represent a better choice as idealized connection stiffness in a simplified method of analysis.

## 8. SUMMARY AND CONCLUSIONS

This study is an attempt to put together a simple method of frame analysis that accounts for connection flexibility in unbraced frames. Procedural simplicity and design practicality were kept in mind to produce a simple, yet sufficiently accurate, method of design analysis. One of its important advantages is believed to be its reliance on the concepts of the well-established $B_1$ and $B_2$ method of analysis recommended by the AISC LRFD Specification.

Two connection models are proposed in the form of linear connection stiffnesses: the modified initial stiffness $R_{ko}$ and connection stiffness $R_{kb}$ developed by the beam-line concept. It has been shown that the proposed connection models enable us to adequately depict moment distribution in column members. The effective length concept associated with the $B_1$ and $B_2$ method of analysis is utilized with the appropriate modifications to account for elastic end restraints. This is done through a modified (reduced) stiffness of beam members meeting at the joint which, in turn, results in a modified relative stiffness factor $G'$.

Extensive numerical studies are then conducted to substantiate the selection of the proposed connection models and related assumptions, and to evaluate the overall analysis procedure. While the proposed method of analysis is approximate in nature, it has been shown, however, that it can predict the design moments in flexible frames with a very good margin of accuracy for such a complicated type of problems. Based on the parametric and numerical studies presented, the following conclusions can be made.

1. For design analysis, the behavior of semi-rigid connections can be adequately represented by linear models

**Table 5.**
**Moments in Frame FR-1 Determined by the Proposed Method Using the LRFD $B_1$ and $B_2$ Factors (Normalized by Exact Solution)**

| Connection Code (Fig. 12) | Column No. (Fig. 9) | Rigid Frame | Flexible Frame Connection Stiffness Taken As | | |
|---|---|---|---|---|---|
| | | | $R_{ki}$ | $R_{ki}$ & $R_{kb}$ | $R_{ko}$ & $R_{kb}$ |
| III-11 | 2 | 1.308 | 1.103 | 1.142 | 1.065 |
| | 4 | 1.291 | 1.167 | 1.173 | 1.055 |
| III-14 | 2 | 1.348 | 1.138 | 1.197 | 1.101 |
| | 4 | 1.352 | 1.121 | 1.121 | 1.002 |
| III-16 | 2 | 1.366 | 1.236 | 1.324 | 1.170 |
| | 4 | 1.260 | 1.143 | 1.168 | 1.020 |
| III-17 | 2 | 1.357 | 1.285 | 1.331 | 1.239 |
| | 4 | 1.091 | 1.039 | 1.058 | 0.993 |

**Table 6.**
**Moments in Frame FR-2 Determined by the Proposed Method Using the LRFD $B_1$ and $B_2$ Factors (Normalized by Exact Solution)**

| Connection Code (Fig. 12) | Column No. (Fig. 9) | Rigid Frame | Flexible Frame Connection Stiffness Taken As | | |
|---|---|---|---|---|---|
| | | | $R_{ki}$ | $R_{ki}$ & $R_{kb}$ | $R_{ko}$ & $R_{kb}$ |
| III-11 | 2 | 1.0016 | 1.1096 | 1.0559 | 1.0226 |
| | 4 | 1.6101 | 1.3586 | 1.3619 | 1.2868 |
| | 6 | 1.1965 | 1.0487 | 1.0551 | 0.9770 |
| III-14 | 2 | 1.0480 | 1.0883 | 1.1003 | 1.0587 |
| | 4 | 1.5110 | 1.3049 | 1.3129 | 1.2416 |
| | 6 | 1.1536 | 1.0352 | 1.0492 | 0.9675 |
| III-16 | 2 | 1.0916 | 1.0849 | 1.1710 | 1.1096 |
| | 4 | 1.3523 | 1.2449 | 1.2670 | 1.1561 |
| | 6 | 1.1006 | 1.0411 | 1.0743 | 0.9763 |
| III-17 | 2 | 1.1640 | 1.1956 | 1.1992 | 1.1598 |
| | 4 | 1.1881 | 1.1371 | 1.1484 | 1.0894 |
| | 6 | 1.0285 | 0.9899 | 1.1530 | 0.9706 |

within the framework of the proposed linearized analysis procedure.

2. The concepts of the amplification factor method ($B_1$ and $B_2$ method) can be utilized, with appropriate modifications, for the design analysis of flexible frames.
3. In the analysis of flexible frames using $B_1$ and $B_2$ procedures, two different connection stiffnesses should be used for determining $M_{nt}$ and $M_{lt}$ with the smaller stiffness value used for determining $M_{lt}$.
4. The loss in the amount of end restraint provided by the beam elements due to connection flexibility can be accounted for by using a modified (reduced) stiffness of beam members meeting at the joint.
5. The proposed method is at least as accurate as the AISC LRFD $B_1$ and $B_2$ method with the traditional assumptions of simple and rigid framing. By disregarding the steps associated with the presence of semi-rigid connections, the method reduces to rigid/pin cases.

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