Automated Second-Order Elastic Analysis For Steel Space Frames

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INTRODUCTION

The magnification of bending moments and deflections due to interaction between axial compression and transverse loads is popularly known as second-order elastic effects (Fig. 1). The American Institute of Steel Construction (AISC) specifications for building construction[3,4] require inclusion of second-order elastic effects in the design of steel structures. In the past, these effects have been approximated by multiplying moments obtained through a linear analysis by a magnification factor. It is simpler and generally more accurate to compute second-order elastic effects within a displacement method computer analysis program. Several different approaches to automated computation are possible. Two methods of including second-order elastic effects in space frame analysis are described herein along with a FORTRAN computer program which executes the stiffness modification method while checking steel wide flange beam-columns for compliance with the AISC Load and Resistance Factor Design (LRFD) Specification.[3]

COMBINED STRESS FORMULAS

The AISC 1989 Specification[4] formulas for checking combined stress are the well known Eqs. H1-1 and H1-2. (These are Eqs. 1.6-1a and 1.6-1b from the 8th Edition.) The amplification factors $C_m/(1 - f_a/F'_e)$ in Eq. H1-1 are based on the assumption that bending stresses f_b are computed from moments obtained through a linear analysis. The amplification factor approximates second-order elastic effects on member moments and may be omitted if moments already include those effects.

The AISC LRFD[4] formula for checking adequacy of a beam-column with bending and substantial axial compression at a given location along the member is given below.

For $P_u/\phi P_n \geq 0.2$:

$$\frac{P_u}{\phi P_n} + \frac{8}{9}\left(\frac{M_{ux}}{\phi_b M_{nx}} + \frac{M_{uy}}{\phi_b M_{ny}}\right) \leq 1.0 \qquad \text{(1) (LRFD H-1a)}$$

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where:

P_u = Member axial force due to factored loads

ϕP_n = The axial strength of the member if no moment were present, considering end conditions and bracing

M_{ux} = Computed x-axis bending moment, including second-order elastic effect, due to factored loads

M_{uy} = Computed y-axis bending moment, including second-order elastic effect, due to factored loads

$\phi_b M_{nx}$ = The bending strength about the x-axis, if no axial force or y-axis bending were present, considering bracing conditions

$\phi_b M_{ny}$ = The bending strength about the y-axis (weak axis) if no axial force or x-axis bending were present

The LRFD alludes to the possibility of automated inclusion of second-order elastic effects in computed member moments in the following statement: "M_u may be determined from a second-order elastic analysis using factored loads." (AISC LRFD H1.2a)

COMPUTER ANALYSIS METHOD

Linear Analysis

Virtually all frame analysis programs use the "displacement"

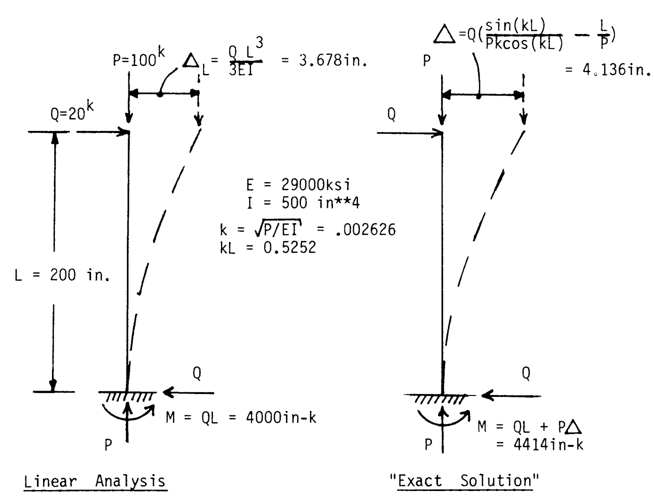


Fig. 1. Second-order elastic effect.

method. Linear analysis of space frames by this method is described by Vanderbilt, 1974.[2] Whereas the non-linear second-order elastic analyses described herein are extensions of the linear analysis, the basic concepts of such linear analysis are summarized below.

In the displacement method, the translations and rotations of each joint are first computed. Once these displacements (collective term for translations and rotations) have been computed, the support reactions and member actions (shear, moments, axial force) are computed therefrom.

Computation of the displacements requires the solution of a set of linear simultaneous equations. The number of equations and the number of terms in each equation are equal to the number of degrees of freedom (DOF) for the structure. Each joint in a space frame structure has six degrees of freedom (i.e., the movement of a space frame joint in response to applied loads can be completely described by six components). These degrees of freedom are the translations along the global X, Y, and Z coordinate axes and the rotations about these axes. Thus, the total number of space frame DOF, N, equals six times the number of joints or nodes in the structural model. The degrees of freedom are numbered sequentially with DOF 1–6 at node 1, DOF 7–12 at node 2, etc. At a given node, DOF numbers increase as follows: X translation, Y translation, Z translation, X rotation, Y rotation, Z rotation.

The set of simultaneous equilibrium equations which must be solved is given below:

$$\begin{array}{l} A_{1,1}X_1 + A_{1,2}X_2 + A_{1,3}X_3 + \ldots + A_{1,N}X_n = B_1 \\ A_{2,1}X_1 + A_{2,2}X_2 + A_{2,3}X_3 + \ldots + A_{2,N}X_n = B_2 \\ A_{3,1}X_1 + A_{3,2}X_2 + A_{3,3}X_3 + \ldots + A_{3,N}X_n = B_3 \\ \vdots \\ A_{N,1}X_1 + A_{N,2}X_2 + A_{N,3}X_3 + \ldots + A_{N,N}X_n = B_N \end{array} \tag{2}$$

where:

X_i = Displacement at DOF i

B_i = Applied force or moment at DOF i

$A_{i,j}$ = Force or moment at DOF i corresponding to a unit displacement at DOF j, with displacements at all other DOF equal to zero

$A_{i,j}X_j$ = Force or moment at DOF i corresponding to the actual displacement at DOF j, with displacements at all other DOF equal to zero

N = Number of DOF

In matrix form this set of equations is written "$AX = B$," where A is referred to as the global stiffness matrix, X is the matrix of unknown displacements, and B is the applied load matrix.

Terms, $A_{i,j}$, of the global stiffness matrix are computed by summing the stiffness contributions of the members. Since each member connects to two nodes and each node has six DOF, there are twelve DOF and 12 times 12, or 144 $A_{i,j}$ terms influenced by a given member. These member contributions are determined as follows. First, a member stiffness matrix is created in a local coordinate system. This system and the twelve member DOF are shown in Fig. 2. The local origin is at the member J node (i.e., the first node input to identify the member). The local x-axis passes through the K node. End nodes J and K, together with a noncolinear L node input for each member, define the plane and direction of the local y-axis. The local z-axis is perpendicular to the xy plane and follows the right hand rule. The member elastic stiffness matrix is given in Fig. 3. It is con-

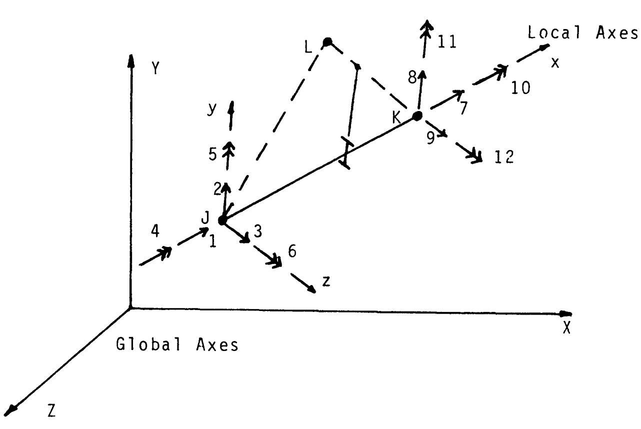


Fig. 2. Member coordinate system and DOF.

i \ j	1	2	3	4	5	6	7	8	9	10	11	12
1	Q						-Q					
2		R				S		-R				S
3			T		-U				-T		-U	
4				V						-V		
5			-U		2W				U		W	
6		S				2Y		-S				Y
7	-Q						Q					
8		-R				-S		R				-S
9			-T		U				T		U	
10				-V						V		
11			-U		W				U		2W	
12		S				Y		-S				2Y

Terms not shown are zero.

$Q = AE/L$
$R = 12EI_z/L^3$
$S = 6EI_z/L^2$
$T = 12EI_y/L^3$
$U = 6EI_y/L^2$
$V = GI_x/L$
$W = 2EI_y/L$
$Y = 2EI_z/L$

A = Member cross-section area
L = Member length
E = Modulus of Elasticity
G = Modulus of Rigidity (Shear Modulus)
I_x = Polar moment of inertia (about local x axis)
I_y = Weak axis moment of inertia (about local y axis)
I_z = Strong axis moment of inertia (about local z axis)

Fig. 3. Member elastic stiffness matrix (Terms $a_{i,j}$)

verted to the global coordinate system through the following multiplications:

$$SMR = RTRANS \times SM \times RT \tag{3}$$

where:

SMR = Member stiffness matrix rotated to global coordinates
RT = Rotation matrix shown in Fig. 4
RTRANS = Transpose of the rotation matrix, and
SM = Member stiffness matrix in local coordinates

Each of the 144 $ar_{i,j}$ terms of the SMR matrix is then added to the appropriate global stiffness coefficient $A_{i,j}$. For example, if the J node of a member is node 38, the X translation DOF for this node is number $[(38 \times 6) - 5]$, or 223. Thus, the $A_{223,223}$ term of the global stiffness matrix is incremented by $ar_{1,1}$ of the SMR matrix. After all member contributions have been included, the global stiffness matrix is modified for support conditions by adding the support spring constants to the $A_{i,i}$ terms of supported degrees of freedom i.

Applied load matrix B is the sum of the specified applied joint loads and the equivalent joint loads resulting from specified member loads (Fig. 5b). These equivalent joint loads are the negatives of the member fixed end actions rotated to global coordinates. Overall structure analysis for member loads plus fixed end actions (Fig. 5c) is not required since there are no joint displacements.

After the A and B matrices are completed, the equations $AX = B$ are solved for the unknown displacements X. Member end actions can then be computed as:

$$MEA = SM \times RT \times XMR + FEA \tag{4}$$

i \ j	1	2	3	4	5	6	7	8	9	10	11	12
1	CXx	CYx	CZx									
2	CXy	CYy	CZy									
3	CXz	CYz	CZz									
4				CXx	CYx	CZx						
5				CXy	CYy	CZy						
6				CXz	CYz	CZz						
7							CXx	CYx	CZx			
8							CXy	CYy	CZy			
9							CXz	CYz	CZz			
10										CXx	CYx	CZx
11										CXy	CYy	CZy
12										CXz	CYz	CZz

Terms not shown are zero.

CXx = Cosine of the angle from the Global X axis to the local x axis. etc.

Fig. 4. Rotation matrix, RT.

where:

MEA = Matrix of the twelve member end actions
XMR = Member end displacements in global coordinates
FEA = Member fixed end actions
RT = Rotation matrix previously given, and
SM = Member stiffness matrix

The terms of the above described member and global stiffness matrices depend only on the geometry of the structure and the properties of the members, and not upon the forces and displacements which result. The equations $AX = B$ are therefore linear, and the principle of superposition applies. Final moments, forces, and displacements of different basic load cases (i.e., different B matrices) may be factored and/or combined to determine the moments, forces and displacements for any load combination without resolving $AX = B$. Furthermore, many of the calculation required in the solution of these equations for the basic load cases only have to be performed for the first basic load case. This is because the stiffness matrix A is the same for each load case and does not have to be retriangularized or reinverted in each $AX = B$ solution.

P Delta Analysis

The described linear analysis gives member actions and displacements which will equilibriate the applied loads in their initial locations. In reality, many of the loads, especially gravity loads, move with the structure as it deforms to resist those loads. The real member actions and displacements must equilibriate the applied loads in their final locations. For structures with very small displacements, this second-order elastic effect may be negligible. For many steel structures, especially unbraced rigid frame high-rise buildings, it can be substantial.

In order to include second-order elastic effects in the computer analysis results, modifications to the linear analysis procedure are required. Member geometric stiffness, which refers to the change in its linear stiffness (Fig. 3) attributable to axial force, must be included. As illustrated by the example in Fig. 6, a lesser moment is required to rotate the beam end one radian if an axial compression P is present in the beam-column; if P equals the buckling load, an

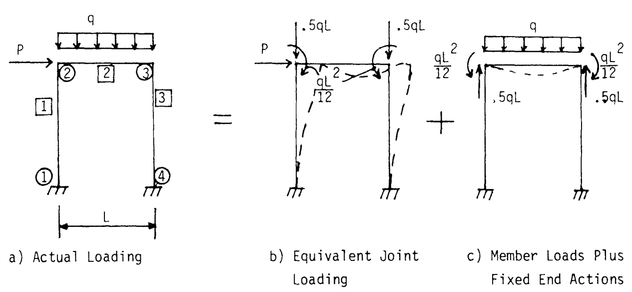


Fig. 5. Equivalent joint loading.

infinitesimally small moment will cause this rotation. The 12th row, 12th column term of the geometric stiffness matrix for this example equals $a'_{12,12}$ minus $a_{12,12}$, and obviously is a function of axial load P.

The member geometric stiffness matrix for a space frame member is given in Fig. 7. By adding this to the linear stiffness matrix of Fig. 3 for each member before rotating to global coordinates and insertion into the global stiffness matrix, second-order elastic effects are included. The theory behind the geometric stiffness matrix, and its form for a plane frame member, are given by Clough and Penzien, 1975.[1] Individual terms of this matrix are derived from the following equation:

$$ag_{i,j} = P\int_0^L N_i'(x)N_j'(x)dx \qquad (5)$$

where:

$ag_{i,j}$ = Term in the geometric stiffness matrix
P = Member axial force
L = Member length
x = Distance from beginning of member
$N_i(x)$ = Transverse deflection at x due to a unit deflection at degree of freedom i, and
$N_i'(x)$ = Slope at x due to a unit deflection at degree of freedom i

As can be seen, the terms of the member geometric stiffness matrix depend on the initially unknown member axial force. Thus, for this stiffness modification approach, an iterative procedure must be used in which the equilibrium equations $AX = B$ are repeatedly solved. For each cycle, the global stiffness matrix A is recreated based on the latest estimate of member axial forces. For this method, convergence is achieved when all calculated axial forces are within an acceptable tolerance of the assumed values. Since second-order elastic effects do not often change member axial forces significantly, convergence is generally achieved in one or two cycles beyond the linear analysis.

The stiffness modification procedure just described requires at least two, and possibly more, complete solutions of the $AX = B$ equations. An alternate, more approximate, approach is to model the geometric effects as additional pseudo loads instead of as stiffness changes. The displacements and member end actions resulting from these pseudo loads are calculated using the linear stiffness matrix and are added to the displacements and member end actions from the linear analysis. As with any other additional load case, the stiffness matrix does not have to be retriangularized to process these pseudo loads, thus saving computation time. The pseudo load vector is calculated as follows. After the linear analysis, the member axial forces from this analysis are used to compute member geometric stiffness matrices. Each member's geometric stiffness matrix is rotated to global coordinates (similar to Eq. 3) and multiplied by the member end displacement matrix in global coordinates. This gives an increment in the member's end actions in global coordinates due to its geometric stiffness. The negatives of all these incremental end actions are combined to form the global pseudo load vector. Additional cycles of pseudo load calculation may be made based on the latest member axial forces and end displacements.

Another issue that arises in the formulation of either of the described second-order elastic analyses is the effect of member axial force on the fixed end actions. Timoshenko and Gere[5] describe this effect for several cases. For example, the fixed end moment M for a uniformly loaded beam with a finite axial compression P is given by:

$$M = \frac{qL^2}{12}\frac{3(\tan(u)-u)}{u^2\tan(u)} \qquad (6)$$

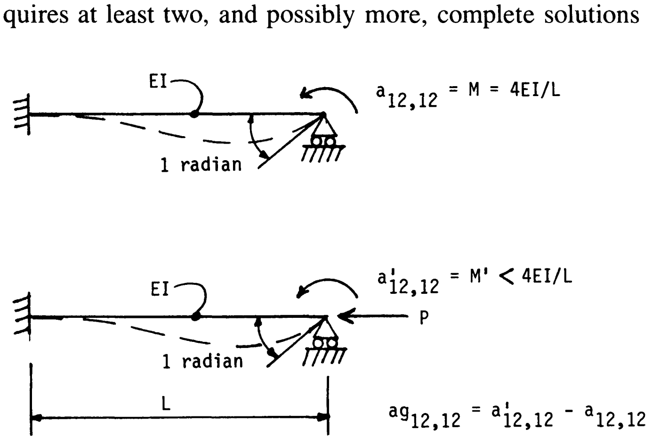


Fig. 6. Effect of axial compression on beam rotational stiffness.

i \ j	1	2	3	4	5	6	7	8	9	10	11	12
1												
2		a				b		-a				b
3			a		-b				-a		-b	
4												
5			-b		c				b		-d	
6		b				c		-b				-d
7												
8		-a				-b		a				-b
9			-a		b				a		b	
10												
11			-b		-d				b		c	
12		b				-d		-b				c

Terms not shown are zero.

a = 6P/5L
b = P/10
c = 2PL/15
d = PL/30

P = Member axial force (Tension positive)
L = Member length

(Note: This geometric stiffness matrix based on cubic polynomial approximations to the true deflected curve.)

Fig. 7. Member geometric stiffness matrix.

where:

q = Member uniform load
L = Member length
u = $5L\sqrt{P/EI}$, and
EI = Member flexural stiffness

For most structures, few members have both high axial compression and significant transverse loading between the joints. In these cases the effects of axial forces on fixed end actions are negligible.

In order to include axial force effects on fixed end actions, it is first necessary to have the appropriate formulas (such as Eq. 6) for the types of member loads to be handled by the computer program. Extension of the work by Timoshenko and Gere[5] led to the following methods of computing fixed end actions for a concentrated load and for a triangular load on the entire length of beam.

Concentrated Load: solve the following four linear simultaneous equations for unknowns M_L, V_L, C, and D.

$$M_L(-\cos(ka)/P)+V_L(\sin(ka)/Pk)-C(\cos(ka))-D(\sin(ka))=0$$
$$M_L(k\sin(ka)/P)+V_L(\cos(ka)/\mathrm{P})+C(k\sin(ka))-D(k\cos(ka))=Q/P$$
$$M_L(1./P)-V_L(L/P)+C(\cos(kL))+D(\sin(kL))=Q(a-L)/P$$
$$M_L(0.)+V_L(1./P)+C(k\sin(kL))-D(k\cos(kL))=Q/P \qquad (7)$$

where:

M_L = Fixed end moment at left end (counterclockwise positive, M_L will be positive for downward Q)
V_L = Fixed end shear at left end (up positive)
C,D = Boundary condition constants for right portion of beam ($a < x < L$)
P = Beam axial compression
k = $\sqrt{P/EI}$
EI = Beam flexural stiffness
L = Beam length
Q = Applied concentrated load, and
a = Distance from left end that load Q is applied

Using statics the fixed end actions at the right end can be determined.

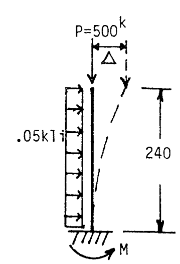
Triangular Load: Zero at left end, maximum value q at right end; on entire length of beam.

$$M_L=\frac{-1.}{\mathrm{DENOM}}(.5qL(\sin(kL)-kL)+kqL^2(1.-\cos(kL))/6.)$$
$$\mathrm{DENOM}=(\sin(kL)-kL)(k\sin(kL))+(k(1.-\cos(kL))^2)$$
$$M_R=\frac{qEI}{P}+M_L-\frac{\sin(kL)}{\mathrm{DENOM}}(0.5qL(1.-\cos(kL))-kqL^2(\sin(kL))/6.) \qquad (8)$$

where:

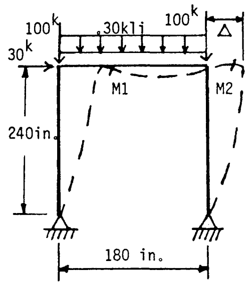
M_L = Fixed end moment left end (counterclockwise positive, M_L will be positive for downward q)
M_R = Fixed end moment right end (clockwise positive, M_R will be positive for downward q)
q = Value of triangular load at right end (down positive)

Other variables as defined in Eq. 7.



A = 30in^2 , E = 29000ksi , I = 1000in^4 All members

	Δ in.	M in-k	Cycles Beyond Linear
Linear	.715	1440	--
Pseudo Load[1,2]	.995	1800	1
Stiffness Mod.[1]	1.18	2030	1
Stiffness Mod.	1.18	2030	1
"Exact"	1.18	2030.	--



	Δ in.	M1 in-k	M2 in-k	Cycles Beyond Linear
Linear	3.30	3170	-4030	--
Pseudo Load[1,2]	3.70	3610	-4440	1
Stiffness Mod.[1]	3.80	3670	-4490	2
Stiffness Mod.	3.80	3670	-4490	2

[1] Does not include effect of axial load on fixed end actions.
[2] Convergence based on change in member axial force

Fig. 8. Plane frame examples.

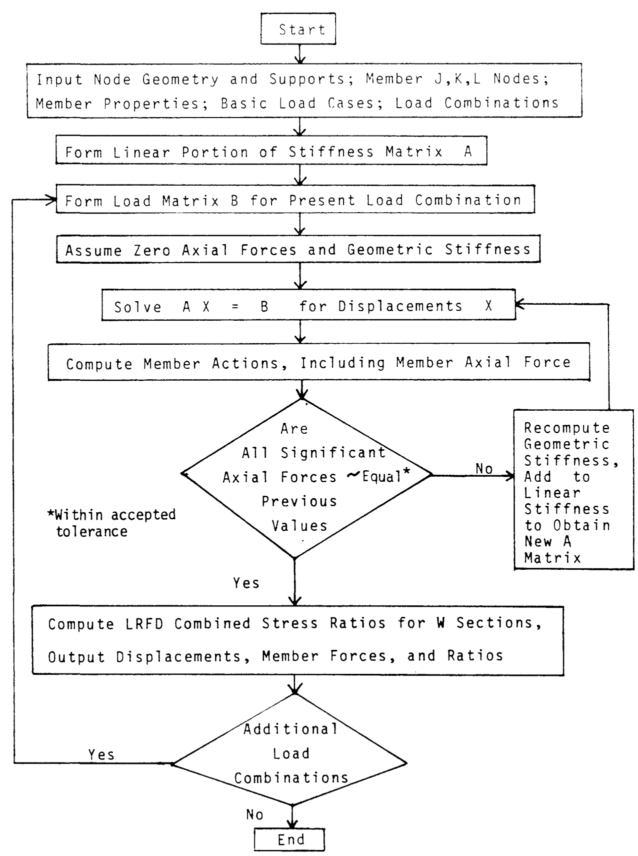


Fig. 9. SPFNL flowchart.

For the stiffness modification approach, revised fixed end actions based on the latest axial forces are computed in each cycle and used with unchanged joint loads to recreate the global matrix *B*. After convergence, the latest fixed end actions are used to compute the member end actions in Eq. 4. For the pseudo load method, changes in fixed end actions would have to be added to the incremental load derived from the geometric stiffness matrix. It should be noted that several of the commercially available second-order elastic analysis programs use the pseudo load method but do not include second-order elastic effects on fixed end actions. A comparison of results from the several methods and options described is given in Fig. 8 for two simple examples.

The dependency of the global stiffness matrix *A* on member axial forces makes the equations $AX = B$ nonlinear. The displacements due to the sum of two sets of applied loads are not equal to the sum of the displacements due to the load sets applied separately. For the stiffness modification method, this means that for each load combination desired, the applied load matrix *B* and global stiffness matrix *A* must be recreated, and the equations $AX = B$ resolved using the iterative procedure described above. For the pseudo load method, it means that the basic load cases must be combined as desired before computation and analysis of the pseudo loads (i.e., deflections and member end actions of the basic load cases should not be factored and combined). Load factors, if required, must be applied prior to the analysis for either method.

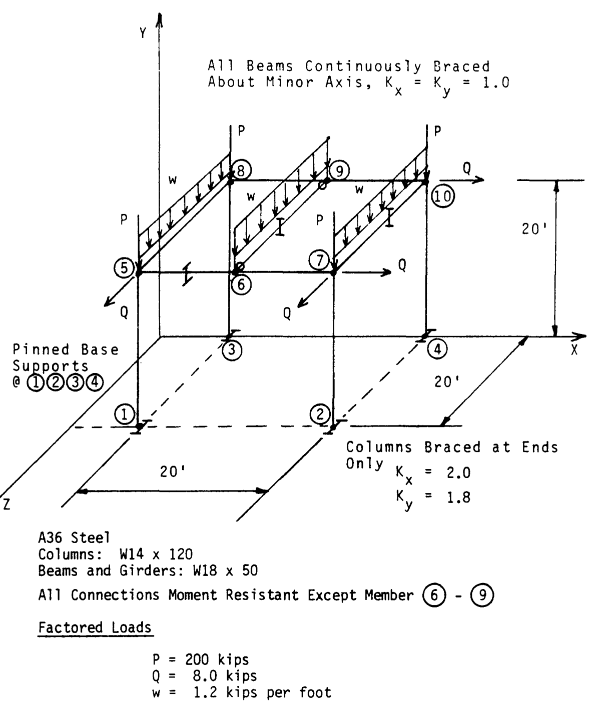


Fig. 10. SPFNL example problem.

PROGRAM SPFNL

A FORTRAN program SPFNL (SPace Frame, NonLinear) was written to perform an automated second-order elastic analysis of space frames and to use the results for checking steel wide flanged beam columns in accordance with the AISC LRFD. A flow chart for this program is given in Fig. 9. The stiffness modification method is used. At present, effects of axial force on fixed end actions are not included, however they are negligible for the example problem included.

LRFD code checking is by Formulae H-la (Eq. 1) and H-lb. Properties of wide flange shapes needed are read from a data table included in the program. Member yield stress, effective column lengths about two axes and compression flange unbraced length for weak axis buckling are input. Both ends of each wide flange member are checked. P_u, M_{ux}, M_{uy} are obtained directly from the automated P-Delta analysis. The nominal axial resistance P_n for a beam column in tension equals the product of the material yield stress and the member gross cross sectional area. The resistance factor ϕ is 0.9. For beam columns in compression, P_n is the product of the gross area and the critical buckling stress F_{cr}; ϕ is 0.85. The nominal flexural strength about the strong axis, M_{nx}, is dependent upon the unbraced length of the compression flange and is computed in accordance with provisions in Chapter F of the LRFD specification. The nominal flexural strength about the weak axis, M_{ny}, is equal to the weak axis plastic moment capacity, whatever the compression side bracing may be. The resistance factor for bending, ϕ_b, is 0.9.

Member ②-⑦ @ ⑦	Linear Analysis	Automated P-Delta Analysis
Axial Force,kips	234 C	241 C
Major Axis Shear,kips	5.42	5.26
Major Axis Moment,in-kips	1300	1520
Minor Axis Shear,kips	4.70	4.40
Minor Axis Moment,in-kips	1130	1770
Deflection X Direction,in.	1.68	2.96
Deflection Z Direction,in.	0.86	1.07
LRFD Combined Stress Ratio	1.03	1.12

Member ⑦-⑩ @ ⑦	Linear Analysis	Automated P-Delta Analysis
Axial Force,kips	1.4 C	1.4 C
Major Axis Shear,kips	20.0	21.8
Major Axis Moment,in-kips	1300	1520

Fig. 11. SPFNL selected output for example problem.

An example problem solved using SPFNL is shown in Fig. 10. Selected output is given in Fig. 11. Two cycles beyond the linear analysis were required for convergence. It can also be noted from Fig. 10 that the axial force increase in member 2-7 due to second-order elastic effects was very small compared to the moment increases.

CONCLUSION

With the availability of computers and suitable software, the inclusion of second-order elastic effects can be automated. This is much simpler and more accurate than using the approximate moment magnification factors given by the various specifications. The price of this improvement is the greater execution time required for the computer analysis. However, with the continued advancements in hardware this price is becoming insignificant for larger and larger structures.

REFERENCES

1. Clough, R. W., and J. Penzien, *Dynamics of Structures* (Chapter 11), New York: McGraw-Hill, 1975.
2. Vanderbilt, M. D., *Matrix Structural Analysis* (Chapter 8), New York: Quantum Publishers, 1974.
3. American Institute of Steel Construction, *Load and Resistance Factor Design Specification for Structural Steel Buildings,* Chicago: AISC, 1986.
4. ———, *Specification for Structural Steel Buildings—Allowable Stress Design and Plastic Design,* Chicago: AISC, 1989.
5. Timoshenko, S. P., and J. M. Gere, *Theory of Elastic Stability* (Chapter 1), New York: McGraw-Hill, 1961.