A Practical P-Delta Analysis Method for Type FR and PR Frames

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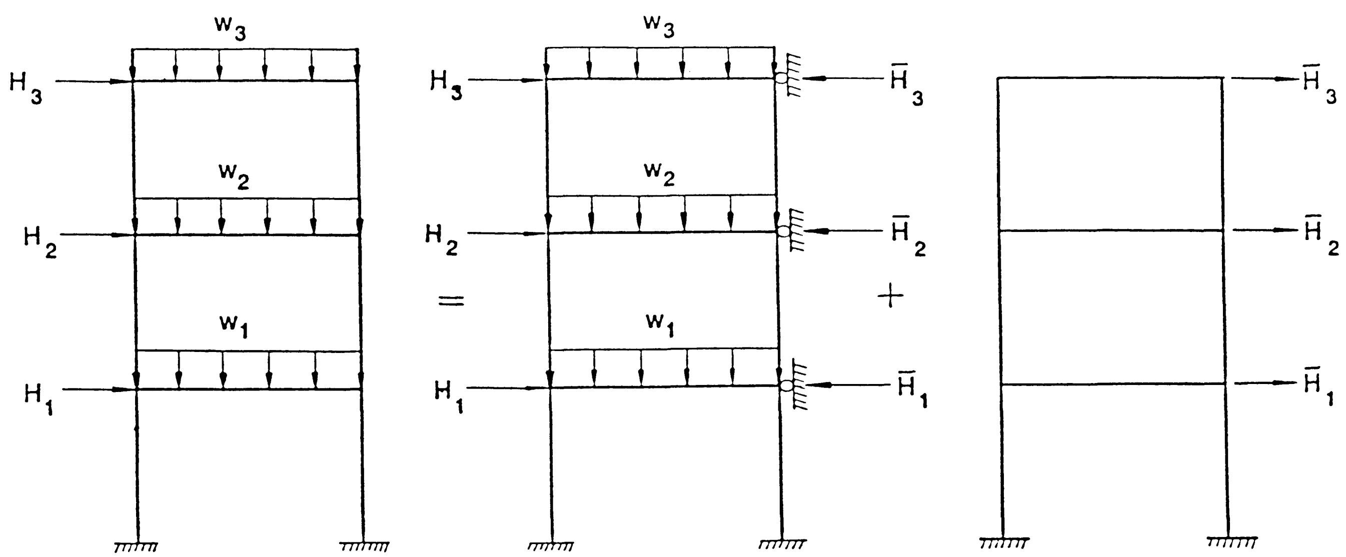
In the stability design of frameworks, attention must be given to the additional moment developed as a result of member (P-δ) and frame (P-Δ) amplification effects from compressive axial loads. These secondary moments may have a deleterious effect on the strength and stability of the frame and so they must be reckoned with in the analysis and design processes. These are various approaches ranging from very simple to rather rigorous by which a designer can employ to account for these second-order effects. In particular, the AISC/LRFD Specification[1] adopts the so-called moment magnifier method in which moment amplification factors B_1 and B_2 are introduced to account for the additional moments due to P-δ and P-Δ effects, respectively, as an alternative to a complete second-order elastic analysis. In using this approximate method, the designer must perform two separate first order analyses: A non-sway analysis in which artificial supports are provided to the frame to prevent it from displacing laterally; and a sway analysis in which the frame is allowed to displace laterally (Fig. 1). The first analysis gives M_{nt} that is multiplied by B_1 to account for the P-δ effect and the second analysis gives M_{1t} that is multiplied by B_2 to account for the P-Δ effect. These two magnified moments are then added algebraically to obtain the design moment (required flexural strength) M_u for the member in question. The advantage of this method is that it is straightforward and can be implemented easily in design since only first-order analysis is required. The disadvantage of this



Original frame *Non-sway frame analysis for* M_{nt} *Sway frame Analysis for* M_{Lt}

Fig. 1. AISC/LRFD procedure for first-order frame analysis

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method is that it is applicable only to rectangular rigid (Type FR) frames. This restriction arises as a result of the assumptions and simplifications used in the structural model in deriving the expression for B_2. Another disadvantage is that the additional moments arising from the P-δ and P-Δ effects may not be synergistic, since they necessarily do not occur at the same location. The use of this method will, therefore, in some cases overestimate the required strength for the member leading to an uneconomical design. Furthermore, for multi-story, multi-bay frames, there is the question of where to place the artificial supports for the non-sway analysis, since placement of these supports at different locations will lead to slightly different but noticeable results.

The aforementioned disadvantages can be circumvented if one uses the preferred method contained in the LRFD Specification by performing a direct second-order elastic analysis to obtain M_u. In using this direct approach, the designer often relies on computer programs capable of performing second-order structural analysis that account for both the deterioration of member flexural rigidity under axial compression (P-δ) and equilibrium of the deflected structure (P-Δ).

Although second-order structural analysis programs are available in the market, extreme care must be exercised in using these programs. Special attention must be paid to such factors as modeling of the structure and setting tolerance limits for the analysis. It is important that a designer possess some basic knowledge of nonlinear analysis and behavior of the structure before any attempt is made to accept the results of the analysis.

In this paper, a simple and practical method of second-order frame analysis using first-order structural analysis technique is proposed. The proposed method accounts for the P-δ and P-Δ effects by the use of a fictitious or pseudo lateral load. This method has the advantage over other existing methods in that no special consideration is required for the modeling of the structure—and the approach is applicable not only to rectangular rigid (Type FR) frames but also to non-rectangular, semi-rigid (Type PR) frames. As demonstrated in subsequent sections, the approach is based on a firm theoretical background and provides good results for frames subject to normal loading conditions.

THEORETICAL BASIS OF THE METHOD

Consider the beam-column of flexural rigidity EI with no relative joint translation subjected to an axial force of P, end moments M_A, M_B and an arbitrary lateral load of w as shown in Fig. 2. The differential equation governing the small displacement behavior of this member is given by (2a).

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = w \tag{1}$$

Upon rearrangement, Eq. 1 can be written as

$$EI\frac{d^4y}{dx^4} = w + w' \tag{2a}$$

where

$$w' = -P\frac{d^2y}{dx^2} \tag{2b}$$

If we compare Eq. 2a with the differential equation of a beam (Fig. 3),

$$EI\frac{d^4y}{dx^4} = w \tag{3}$$

it is seen a beam-column differs from a beam by only the extra term w' given by Eq. 2b. Based on this observation, we can, therefore, account for the beam-column effect by applying a pseudo lateral in-span load of $w' = -P(d^2y/dx^2)$ to the member (Fig. 4). In other words, the beam-column shown in Fig. 2 can be analyzed as a beam shown in Fig. 4 provided that a fictitious lateral load w' equal to the negative product of the axial force P and the second derivative of the displacement d^2y/dx^2 is applied to the member.

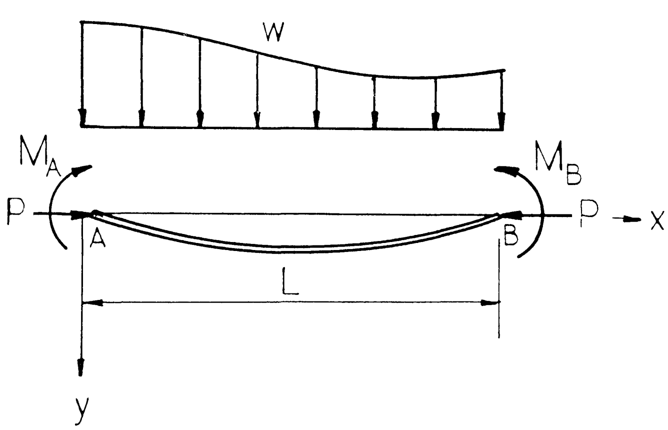


Fig. 2. Beam-column

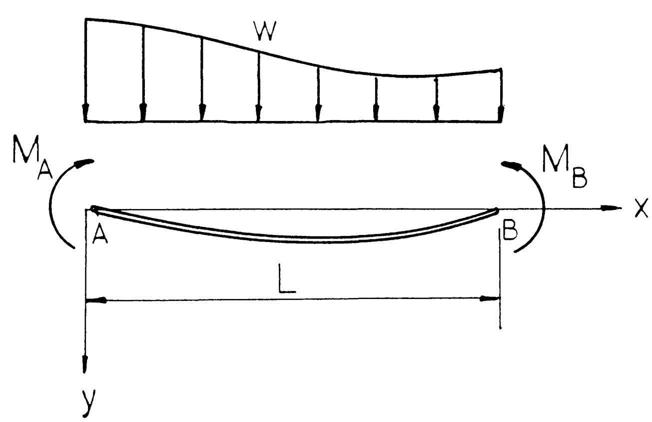


Fig. 3. Beam

If small displacements are assumed, the following relationship holds

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} \tag{4}$$

where M is the bending moment (considering second-order effects) in the member.

In view of Eq. 4, the fictitious or pseudo lateral load expressed in Eq. 2b can now be written as

$$w' = P\left(\frac{M}{EI}\right) = \left(\frac{P}{EI}\right)\mathrm{M} \tag{5}$$

Equation 5 indicates the pseudo lateral load can be obtained from the second-order moment diagram by scaling it down by a factor of P/EI. Since P and M are not known in advance for a member in a structure, the exact value of w' cannot be evaluated directly. However, as a first approximation, the first-order values for P and M can be used. Using these values, an approximate value for w' can be evaluated using

$$w'_1 = \left(\frac{P_1}{EI}\right) M_1 \tag{6}$$

in which the subscript 1 refers to the first cycle of calculation. By loading the member with this pseudo lateral load together with the actual loadings, a first-order analysis again can be performed on the structure from which a better estimate of P and M can be obtained. Using these new values P and M, a better approximation for w' can be evaluated. Using this updated value of w' and the actual loadings, another first-order analysis can be performed to obtain yet better values for P and M. Thus, an expression for the pseudo lateral load at the i-th cycle of calculation can be written as

$$w'_i = \left(\frac{P_i}{EI}\right) M_i \tag{7}$$

If the member in question is subject to relative joint translation (Fig. 5a), a secondary overturning moment equals the product of the axial force P and the relative joint displacement Δ induced in the member. This so-called P-Δ moment traditionally is taken into account by replacing the axial force P by a pair of equal and opposite

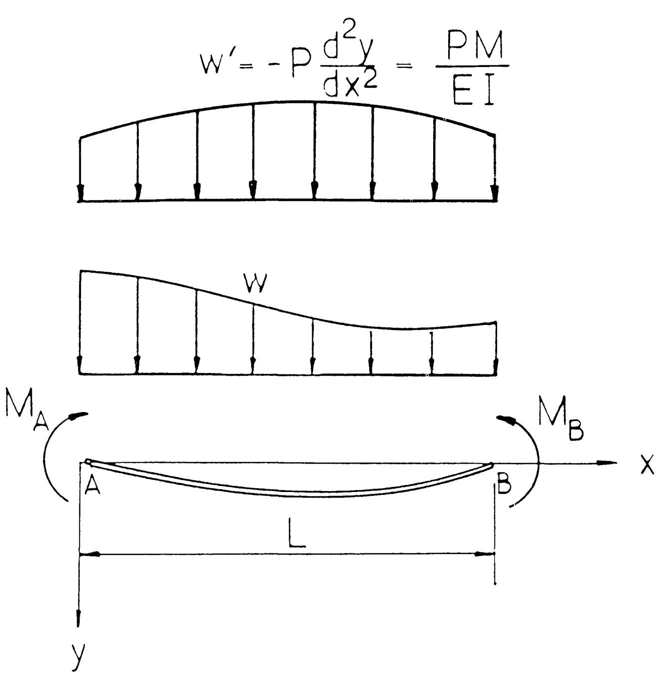


Fig. 4. Beam with pseudo-lateral, in-span load

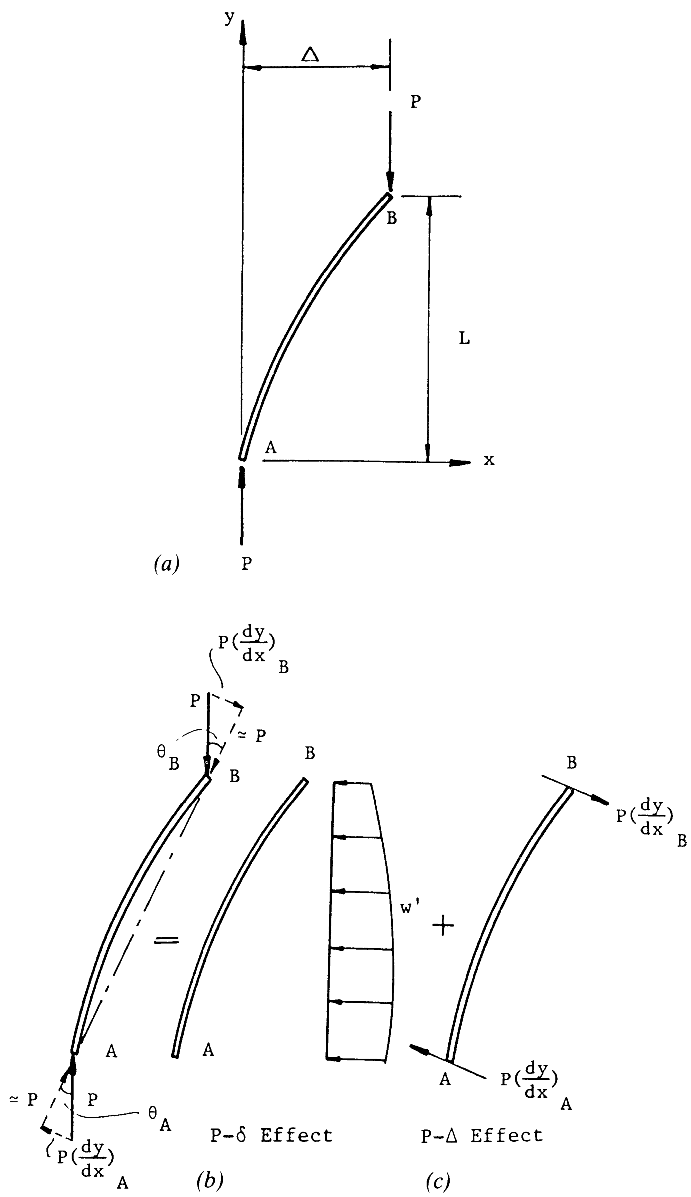


Fig. 5. Beam-column with relative joint translation

shear forces equal $P\Delta/L$ (L is the length of the member) acting at the ends of the member. Strictly speaking, this approach is not exactly correct. The pitfall lies in the fact that by subjecting the member to a pair of end shears of $P\Delta/L$, only the P-Δ effect can be accounted for, whereas the P-δ effect, which is also present in the member cannot be accounted for. In what follows, it will be shown that with slight modification, the proposed pseudo load method can also be applied to members with relative joint translation. The method accounts for both the P-δ and P-Δ effects simultaneously for these members.

To illustrate the rationale behind the method, let's consider Fig. 5b in which the axial force P acting on the member has been decomposed into components. The component that acts along the chord (dashed-dotted line) of the members is given by

$$P \cos \theta_A \approx P \text{ at the } A\text{-th end} \tag{8a}$$

and

$$P \cos \theta_B \approx P \text{ at the } B\text{-th end} \tag{8b}$$

respectively, and the component that acts perpendicular to the chord is given by

$$P \sin \theta_A \approx P\,\theta_A \approx P\left(\frac{dy}{dx}\right)_A \text{ at the } A\text{-th end} \tag{9a}$$

and

$$P \sin \theta_B \approx P\,\theta_B \approx P\left(\frac{dy}{dx}\right)_B \text{ at the } B\text{-th end} \tag{9b}$$

respectively. The component of P that acts along the member chord (i.e. Eqs. 8a and 8b) gives rise to the P-δ effect and the component of P that acts perpendicular to the member chord (i.e. Eqs. 9a and 9b) contributes to the P-Δ effect of the member. It has been shown earlier the P-δ effect can be accounted for by applying the pseudo lateral load of w' to the member. To account for the P-Δ effect, what we need to do is to apply a pair of pseudo shear forces equal $P(dy/dx)_A$ and $P(dy/dx)_B$ at the A-th and B-th ends of the member, respectively. This is shown in Fig. 5c. Note that by subjecting the member to a pseudo lateral load of w' in conjunction with the pseudo end shears, both the P-δ and P-Δ effects can be accounted for simultaneously.

Since the end slopes $(dy/dx)_A$ and $(dy/dx)_B$ are not known in advance, it is necessary to use approximate values from a first-order analysis. These values can be improved upon in subsequent cycles of analyses.

It is important to note the directions of these pseudo loads. The pseudo lateral load must be applied in the direction to amplify the displacement of the member with respect to its chord, while the pseudo end shears must be applied in the direction to cause the member to rotate in the same sense as the P-Δ moment.

If more than one member is connected at a joint, the pseudo end shears for these members must be combined to obtain a psuedo joint load to be applied to the structure. For instance, if two columns meet at a joint, the pseudo joint load H' is obtained from

$$H' = \left[P\left(\frac{dy}{dx}\right)_B\right]_{\text{lower story}} - \left[P\left(\frac{dy}{dx}\right)_A\right]_{\text{upper story}} \tag{10}$$

This pseudo joint load must be applied in such a direction as to cause an increase in deflection of the structure from its original configuration.

PSEUDO LATERAL LOAD METHOD

Member With No Relative Joint Translation

To demonstrate how the pseudo lateral load can account for the member instability effect, cosider the beam-column in Fig. 6. The member is subjected to an axial force of P and a concentrated mid-span lateral load of Q. Under the load Q, the member will deflect. The axial force will act through this lateral displacement creating additional deflection and moment. This phenomenon is referred to as the P-δ effect. The maximum moment of this member occurs at mid span and has a theoretical value (2) of

$$M_{\max} = \left(\frac{tan \frac{kL}{2}}{\frac{kL}{2}} \right) M_o \tag{11}$$

where M_o is the first-order moment given by $QL/4$ and $k = \sqrt{P/EI}$. It is now desirable to obtain an approximate expression for $M_{\max}$ using the proposed approach and compare it with Eq. 11.

To begin, let's ignore the axial load P and analyze the member subjected to Q only using first-order analysis technique. The result of this analysis is in Fig. 7a. Next, scale the moment diagram of Fig. 7a by the factor P/EI to obtain the pseudo lateral load as shown in Fig. 7b. Load the member by this pseudo lateral load together with the actual lateral load, analyze the beam again using first-order analysis. The moment at midspan (i.e. $M_{\max}$) now becomes

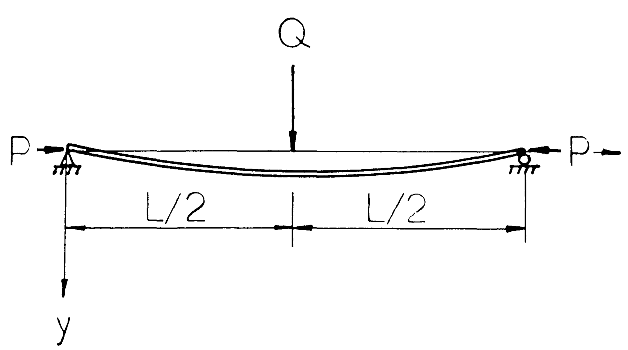


Fig. 6. Beam-column subjected to concentrated mid-span load

$$M_{\max} \approx \left[1+\frac{1}{12}(kL)^2\right] M_o \tag{12}$$

Table 1 shows a comparison of the theoretical value for $M_{\max}$ expressed in Eq. 11 with the approximate value given in Eq. 12 for the range of kL commonly encountered in nonsway frames. It can be seen that reasonably good approximations are obtained for small kL values. For large kL values, the approximate values begin to deviate from the exact values. This derivation becomes more pronounced as kL increases. To obtain a better approximation, a second iteration is performed. The moment diagram obtained from the first iteration is scaled by the factor P/EI, the resulting pseudo lateral load together with the actual lateral load are then loaded on the member as shown in Fig. 7c. Under these loadings, it can be shown by elementary structural analysis that the mid-span moment ($M_{\max}$) is given by

$$M_{\max} \approx \left[1+\frac{1}{12}(kL)^2+\frac{1}{120}(kL)^4\right] M_o \tag{13}$$

A comparison of Eq. 13 with Eq. 11 is shown on Table 1. As shown, Eq. 13 gives a better approximation than Eq.

Table 1. Comparison of $(M_{max}/M_o)_{exact}$ with $(M_{max}/M_o)_{approx.}$ for Supported Beam-Column

kL	M_{max}/M_o		
	Exact Eq. 11	Approximate Eq. 12 (one cycle)	 Eq. 13 (two cycles)
0.20	1.003	1.003	1.003
0.40	1.014	1.013	1.014
0.60	1.031	1.030	1.031
0.80	1.057	1.053	1.057
1.00	1.093	1.083	1.092
1.20	1.140	1.120	1.137
1.40	1.203	1.163	1.195
1.60	1.287	1.213	1.268
1.80	1.400	1.270	1.357
2.00	1.557	1.333	1.467

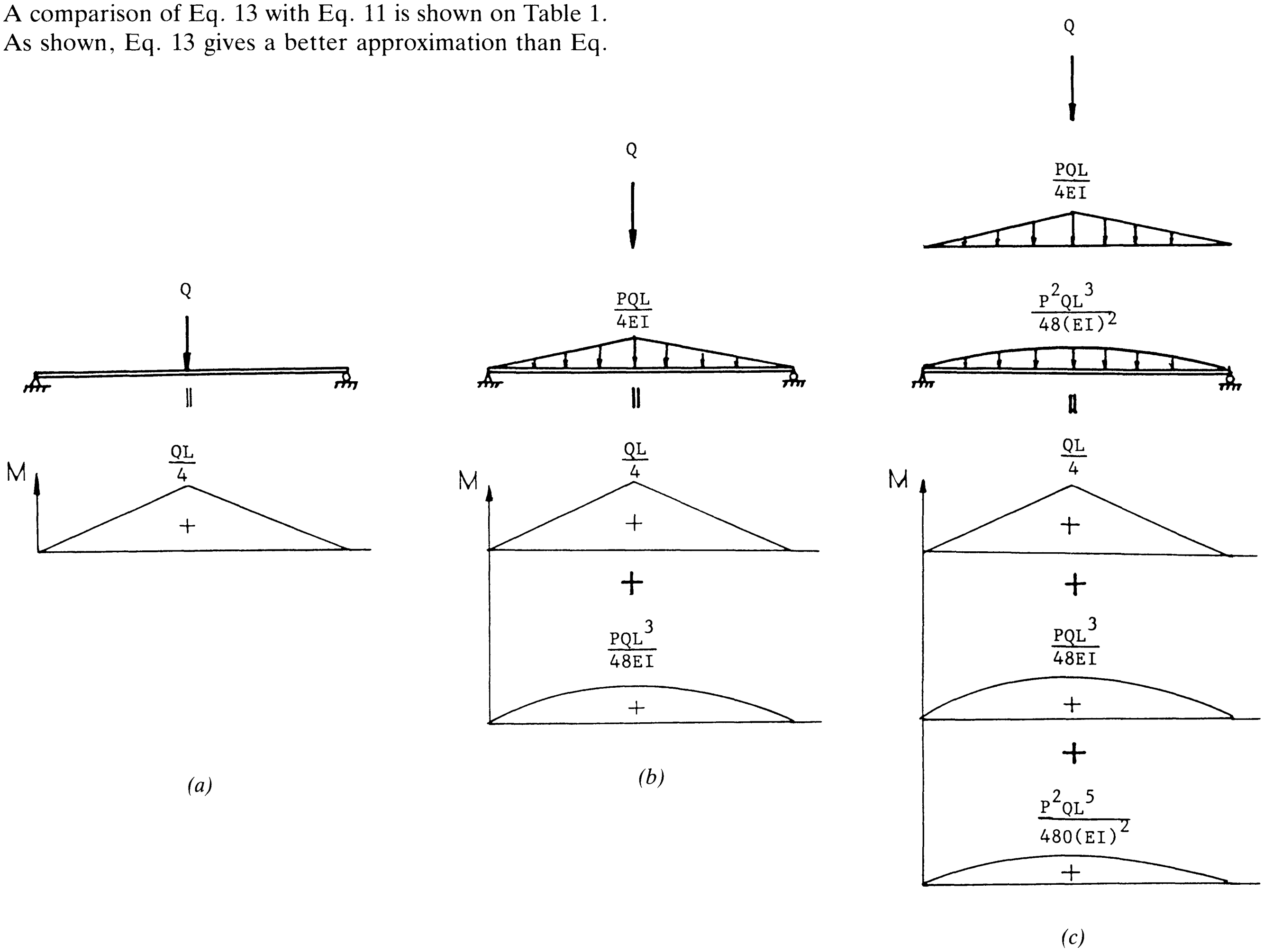


Fig. 7. P-δ analysis by the pseudo-lateral load method

12 to Eq. 11, especially for higher kL values. This process can be continued further until the desired accuracy is obtained. This is not attempted here. Instead, it will be shown in the forthcoming that there is a physical significance to the pseudo lateral load used in the proposed method.

If we expand Eq. 11 using Taylor series expansion, we obtain

$$M_{\max} = \left[1 + \frac{1}{3}\left(\frac{kL}{2}\right)^2 + \frac{2}{15}\left(\frac{kL}{2}\right)^4 + \ldots\ldots\right] M_o$$

or

$$M_{\max} = \left[1 + \frac{1}{12}(kL)^2 + \frac{1}{120}(kL)^4 + \ldots\ldots\right] M_o \qquad (14)$$

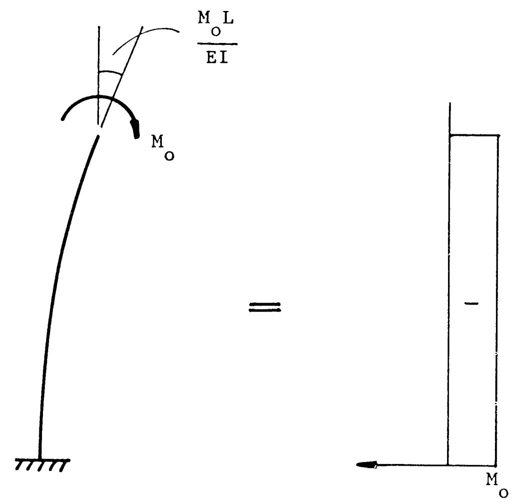
Upon comparison of the above equation with Eqs. 12 and 13, it can be seen readily that by successive application of the pseudo lateral load, additional terms in the Taylor Series expression of the theoretical value of $M_{\max}$ will be obtained. Thus, for each cycle of calculation, a better approximation to $M_{\max}$ will be obtained, since one more term in the series will be generated. For ordinary structures under normal loading conditions, the value of kL will be small. Convergence of the series will be fast and so one iteration (two cycles of calculations) will usually suffice.

Member With Relative Joint Translation

Consider the cantilever beam-column shown in Fig. 8. This member is subjected to a concentrated moment of M_o and a vertical force of P at its free end. Under the action of M_o, the members will deflect sidesway. The vertical force P will then act through this sidesway displacement creating additional sidesway and moment. This phenomenon is referred to as the P-Δ effect. Note that in addition to this P-Δ effect, the P-δ effect is also present since the member also deflects with respect to its chord. Using structural stability theory, it can readily be shown that the theoretical moment at the fixed-end of the member is given by

$$M_F = (\sec kL)\ M_o \qquad (15)$$

where $k = \sqrt{P/EI}$



(a)

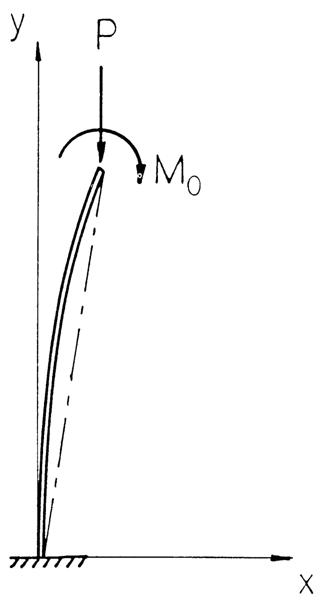
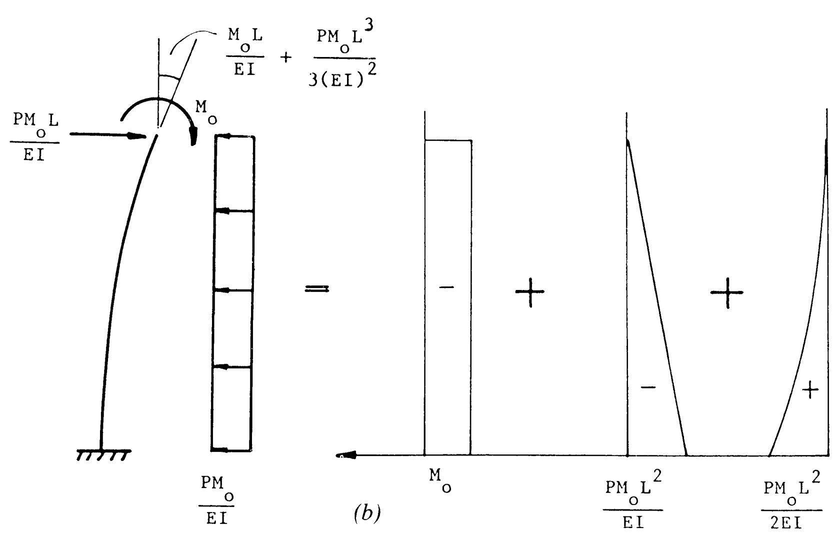


Fig. 8. Cantilever beam-column



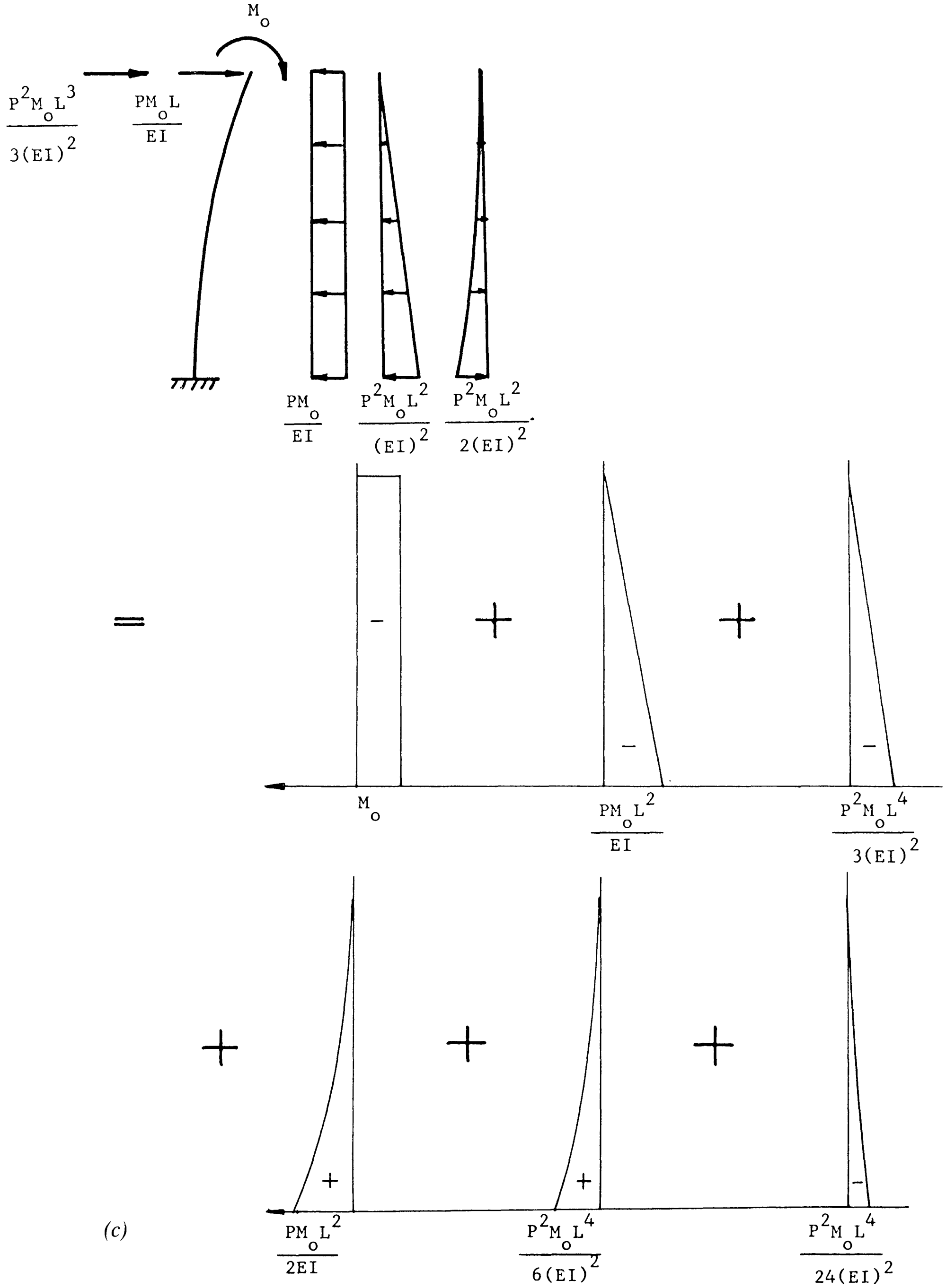
(b)

Fig. 9. P-δ analysis by pseudo-lateral load method

To obtain an approximate value for M_F using the proposed method, the member is analyzed as a beam with the concentrated moment M_o acting alone as shown in Fig. 9a. The result of this analysis indicates that $(dy/dy)_A = 0$ and $(dy/dx)_B = M_oL/EI$. Consequently, in order to account for the P-Δ effect, an end shear of PM_oL/EI is applied to the member. To account for the P-δ effect, the moment diagram of Fig. 9a is scaled by the factor P/EI to obtain a pseudo lateral load of $w_1' = PM_o/EI$. These pseudo loads are shown acting on the member in conjunction with the real load M_o in Fig. 9b. Under these loadings, M_F is calculated to be

$$M_F \approx \left[1 + \frac{1}{2}(kL)^2\right]M_o \tag{16}$$

A comparison of this approximate value of M_F with the exact value expressed in Eq. 15 for a range of kL typical of sway frames is presented in Table 2. Good agreements are observed at low kL values. For high kL values, the approximate value deviates from the exact value. However,



(c)

Fig. 9. (continued)

Table 2. Comparison of $(M_F/M_o)_{exact}$ with $(M_F/M_o)_{approx.}$ for Cantilever Beam-Column

kL	M_{max}/M_o		
	Exact Eq. 15	Approximate Eq. 16 (one cycle)	Approximate Eq. 17 (two cycles)
0.20	1.020	1.020	1.020
0.40	1.086	1.080	1.085
0.60	1.212	1.180	1.207
0.80	1.435	1.320	1.405
1.00	1.851	1.500	1.708

improvement can be made by performing another cycle of calculation. This new cycle of calculation is depicted in Fig. 9c. The moment at the fixed-end is now

$$M_F \approx \left[1 + \frac{1}{2}(kL)^2 + \frac{5}{24}(kL)^4\right] M_o \tag{17}$$

It can be seen in Table 2 that Eq. 17 gives a better approximation of M_F than Eq. 16.

A reader should recognize Eqs. 16 and 17 are the Taylor Series expansion of Eq. 15. Hence, the pseudo loads for sway members bear the same physical significance as that for non-sway members in that each set of new pseudo loads represents an additional term in the power series expansion of the exact solution.

APPLICABILITY OF THE METHOD

In the preceeding section, two simple examples were shown to demonstrate the use of the proposed method. It should be mentioned that the method can easily be extended to structures composed of an assemblage of frame members. The steps that need to be followed are:

1. Perform a first-order analysis on the structure.
2. Construct bending moment diagram for each and every member of the structure.
3. Obtain pseudo lateral in-span loads for all the members of the structure by scaling the moment diagram of the member in question by the factor P/EI where P is the axial force and EI is the flexural rigidity of the member. For members subjected to sway, additional end shears and, hence, pseudo joint loads must be calculated and applied to the joints of the structure according to Eqs. 9a, 9b and 10.
4. Load the structure by these pseudo loads together with the actual loadings. Perform another first-order analysis on the structure.
5. Repeat steps 2 to 4 if desired.

By following this procedure, secondary moments which must be accounted for in design can be obtained by first-order analysis technique. In addition, since no assumption regarding the structure geometry and member end conditions were made in the derivation of the method, the method thus is applicable not only to rectangular Type *FR* frames but also to non-rectangular Type *PR* frames. This represents an obvious advantage over the moment magnifier method contained in the current LRFD Specification by which only rectangular Type *FR* frames can be handled.

In using the proposed method, it is necessary for the designer to carry out first-order frame analysis on the structure. A number of first-order frame analysis schemes are available in the literature. Examples are: the slope-deflection method, moment distribution method and matrix stiffness method. In all these methods, the analyst is required to obtain values for the fixed-end moments to account for any in-span loads that might be present in the member. Since in-span member loads are always present in the pseudo lateral load method of frame analysis, a brief discussion of these fixed-end forces is necessary and will be presented in the following section.

FIXED-END FORCES

Table 3a lists the general expressions for the fixed-end forces of several pseudo lateral load distributions that are most commonly encountered. Recall the pseudo lateral load is obtained from the moment diagram of a first-order analysis, it follows that the order of the pseudo lateral load expression will be two higher than that of the real load. Thus, Case 1 in Table 3a represents the pseudo lateral load for a member subjected to end moments only. Case 2 represents the pseudo lateral load for a member subjected to a concentrated lateral load and end moments. Case 3 represents the pseudo lateral load for a member subjected to a uniformly distributed lateral load and end moments.

If a member is subjected to a real lateral load whose order is higher than a uniformly distributed load (i.e. zero order), the moment expression will have an order higher than two. For such cases, it may be preferable to obtain an approximate rather than an exact expression for the fixed-end moments of the resulting pseudo lateral load. This is shown as Case 4 in Table 3b. The member is first divided into n segments. An equivalent concentrated load is then calculated for each segment from the equation

$$Q_i = \frac{w_i' L}{n} \tag{18}$$

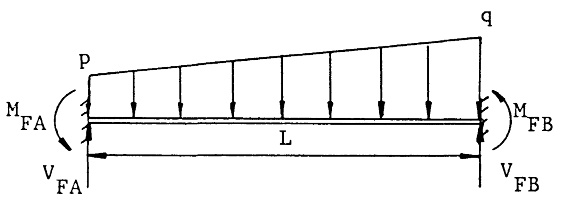
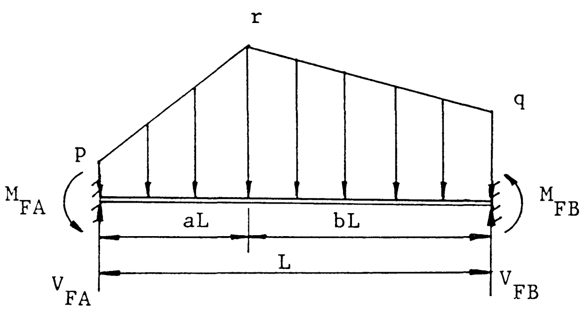
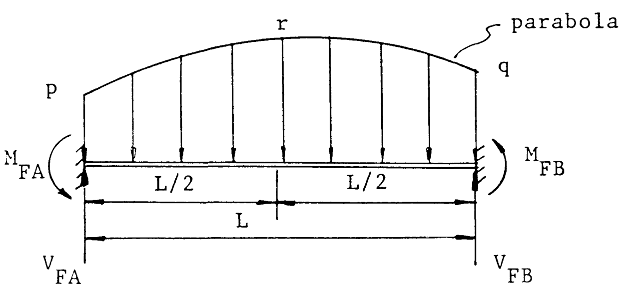
In Eq. 18, w_i' is the value of w' at point i. Once Q_i is calculated, the fixed-end forces can be evaluated from the given expressions.

The expressions shown in Tables 3a and 3b are applicable only if rigid connections are present at both ends of the members. If semi-rigid connections exist at either end, the fixed-end forces must be modified to account for the presence of these connections. Denoting R_{kA} and R_{kB} as the connection stiffness at the A-th and B-th ends of the mem-

Table 3a. Fixed-end Forces

$$M_{FA} = \frac{[(6 - \alpha_1)p + (4 - \alpha_1)q + 2\alpha_1 r]\, L^2}{120}, \quad M_{FB} = \frac{-[(4 - \alpha_2)p + (6 - \alpha_2)q + 2\,\alpha_2 r]\, L^2}{120}$$

$$V_{FA} = \frac{[(4 - \alpha_3)p + (2 - \alpha_3)q + 2\alpha_3 r]\, L}{12} + \frac{M_{FA} + M_{FB}}{L}, \quad V_{FB} = \frac{[(2 - \alpha_4)p + (4 - \alpha_4)q + 2\alpha_4 r]\, L}{12} - \frac{M_{FA} + M_{FB}}{L}$$

	Load Case	Values of α's
1		$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$
2		$\alpha_1 = 2a^2\,(10 - 15a + 6a^2) + 3b^3\,(5 - 4b)$ $\alpha_2 = 3a^3\,(5 - 4a) + 2b^2\,(10 - 15b + 6b^2)$ $\alpha_3 = a^2 + 3ab + 2b^2$ $a_4 = 2a^2 + 3ab + b^2$
3		$a_1 = \alpha_2 = 4$ $\alpha_3 = \alpha_4 = 2$

bers respectively, the fixed-end moments for the member are modified to (3)

$$M^c_{FA} = \frac{\left(1 + \dfrac{4EI}{LR_{kB}}\right) M_{FA} - \dfrac{2EI}{LR_{kB}} M_{FB}}{\left(1 + \dfrac{4EI}{LR_{kA}}\right)\left(1 + \dfrac{EI}{LR_{kB}}\right) - \dfrac{4}{R_{kA}R_{kB}}\left(\dfrac{EI}{L}\right)^2} \tag{19a}$$

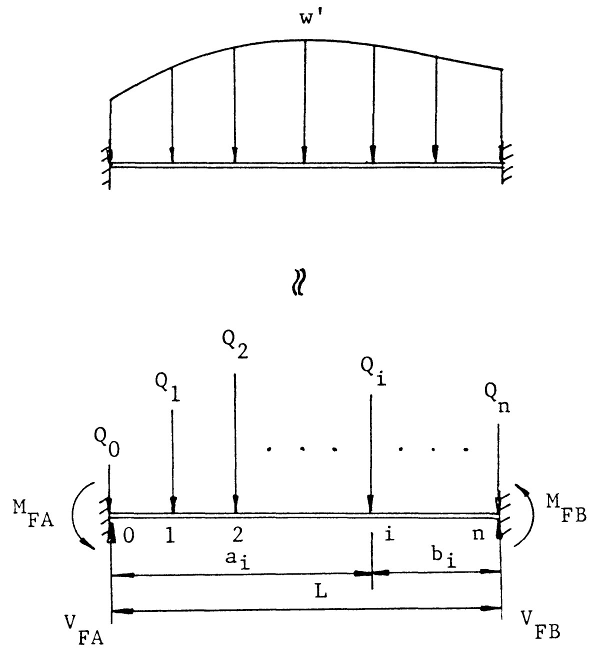
$$M^c_{FB} = \frac{-\dfrac{2EI}{LR_{kA}} M_{FA} + \left(1 + \dfrac{4EI}{LR_{kA}}\right) M_{FB}}{\left(1 + \dfrac{4EI}{LR_{kA}}\right)\left(1 + \dfrac{4EI}{LR_{kB}}\right) - \dfrac{4}{R_{kA}R_{kB}}\left(\dfrac{EI}{L}\right)^2} \tag{19b}$$

In these expressions, M_{FA} and M_{FB} are the fixed-end moments of the member with rigid connections and M^c_{FA} and M^c_{FB} are the fixed-end moments of the members with semi-rigid (*PR*) connections. The subscripts *A* and *B* refer to the *A*-th and *B*-th ends of the member, respectively. Once M^c_{FA} and M^c_{FB} are calculated, the modified fixed-end shears V^c_{FA} and V^c_{FB} can be evaluated by considering member equilibrium.

NUMERICAL EXAMPLES

In this section, several frame examples will be given to demonstrate the validity of the proposed method.

Table 3b. Fixed-end Forces

	Load Case	Fixed-end Forces
4		$M_{FA} = \dfrac{\Sigma Q_i\, a_i b_i^2}{L^2}$, $M_{FB} = \dfrac{\Sigma Q_i\, a_i^2 b_i}{L^2}$ $V_{FA} = \dfrac{\Sigma Q_i (3a_i + b_i) b_i^2}{L^3}$, $V_{FB} = \dfrac{\Sigma Q_i a_i^2 (a_i + 3b_i)}{L^3}$

Example 1

Figure 10a shows the geometry and loading conditions of a pinned-base portal frame. All members of the frame are made of *W*12x65 sections. The loadings shown are factored loads. The frame is braced against out-of-plane bending. It is desired to determine the maximum moment in the column and the beam considering second-order effects.

To proceed with the proposed method, a first-order analysis of the frame is performed from which value of the axial force *P* and bending moment *M* are obtained for each individual member. Next, obtain the pseudo lateral in-span load for each member by scaling the moment diagram down by the factor *P*/*EI*. For members that experience relative joint translations (i.e. the columns), additional end shears equal to the product of *P* and the end slope must be calculated and transformed into pseudo joint loads. Load the frame with these pseudo loads together with the real loads as shown in Fig. 10b. Perform another first-order analysis on the frame. The results of this analysis are presented in Table 4 in which maximum moments in the column and the beam are shown for various methods of analyses.

As can be seen, the proposed method gives an excellent approximation to the more exact method of second-order frame analysis with just one iteration. For the purpose of comparison, the moment magnifier method (B_1 and B_2 method) recommended in the LRFD Specification is also presented. It can be seen that for this simple portal frame the results obtained using the proposed method are very comparable to those obtained using the LRFD moment magnifier method.

Example 2

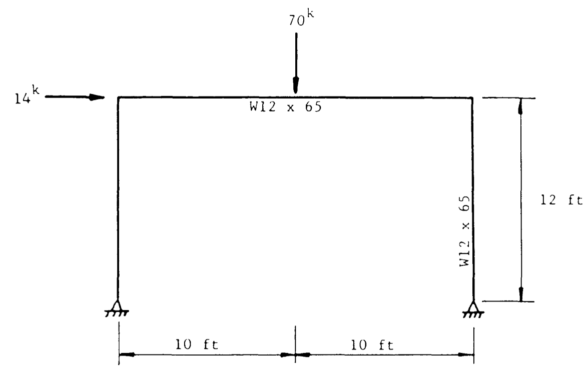
In this example, a three-story frame shown in Fig. 11a will be analyzed. As in the preceding example, all loads shown are factored loads and the frame is braced against out-of-plane bending. Following the same procedure as before, a set of pseudo lateral in-span loads is obtained by scaling the first-order moment diagrams by the factor *P*/*EI* and a set of pseudo joint loads is obtained by first calculating the end shears according to Eqs. 9a and 9b and then transforming them to joint loads according to Eq. 10. Using these pseudo loadings in conjunction with the real loadings Fig. 11b, another first-order analysis is performed on the frame. The results for the maximum column moments obtained using the proposed method and several other methods are shown in Table 5.

Table 4. Moment in Leeward Column and Beam of the Simple Portal Frame (kip-in)

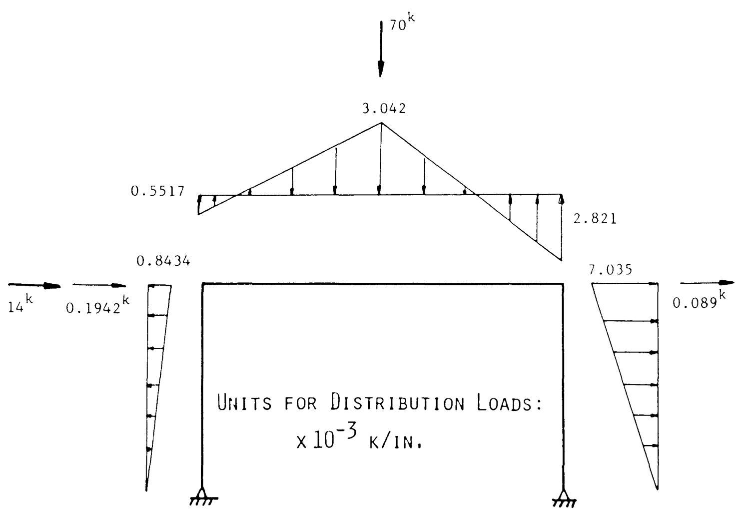
Method of Analysis	Column	Beam
First-order Analysis	2506	2702
Second-order Analysis	2537	2711
Proposed Method (1 iteration)	2537	2710
B_1, B_2 Method	2535	2710

Table 5. Moment in Leeward Columns of the Three-story Frame (kip-in)

Method of Analysis	1st Story	2nd Story	3rd Story
First-order Analysis	833	686	576
Second-order Analysis	845	695	579
Proposed Method (1 iteration)	844	695	579
B_1, B_2 Method	963	717	587

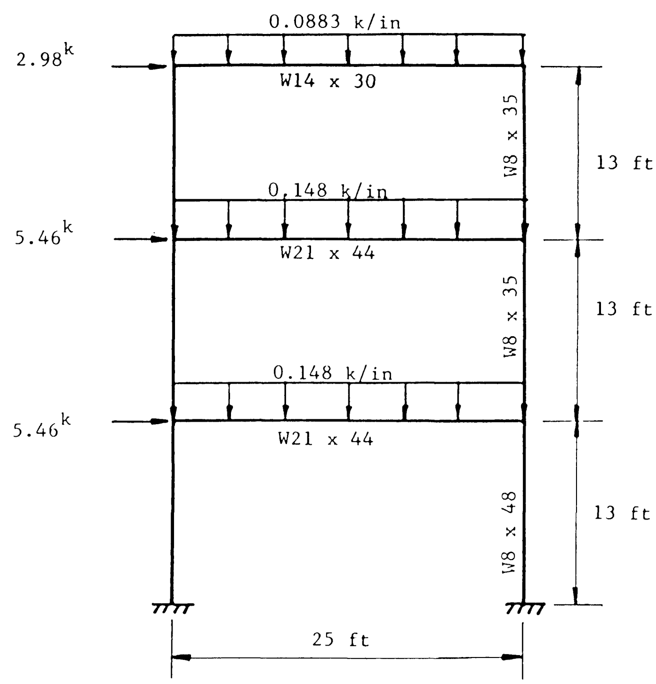


(a) Frame geometry and loading



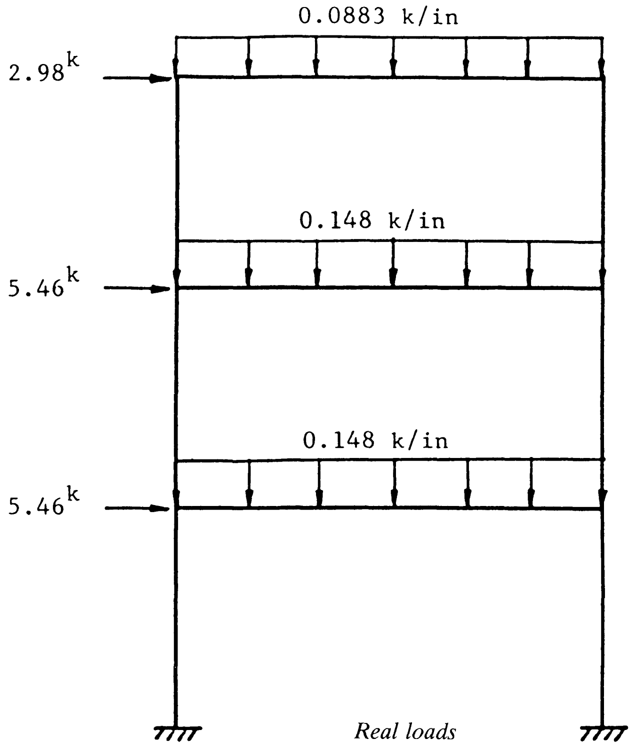
(b) Frame subjected to real and pseudo loads

Fig. 10. Numerical example—simple portal frame



(a) Frame geometry and loading

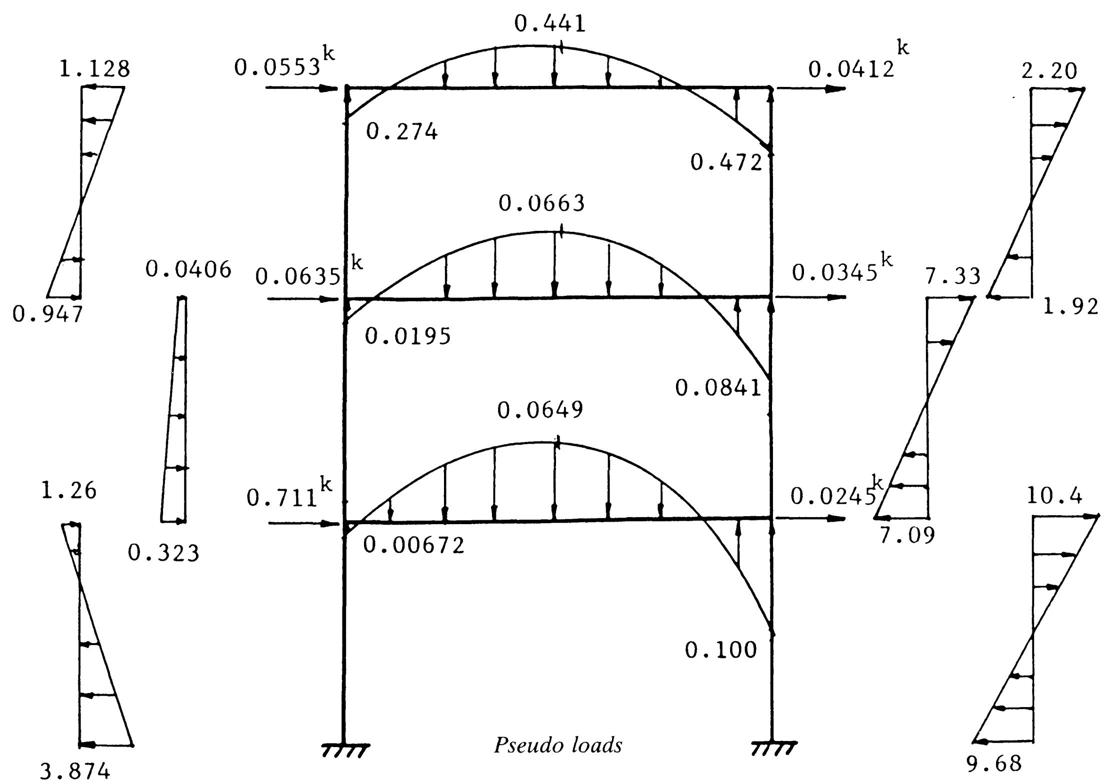
Fig. 11. Numerical example—three-story frame



Although only one iteration is used, the proposed method gives excellent results when compared to a second-order analysis. The LRFD moment magnifier method, however, overestimates the column moments. The overestimation is rather pronounced for the first- and second-story column moments. This conservatism is attributed to the fact that the maximum nonsway moment and the maximum sway moment do not coincide at the same location. For instance, for the first- and second-story columns, $(M_{nt})_{max}$ and $(M_{lt})_{max}$ occur at different ends of the member. Depending on the degree of conservatism, the LRFD moment magnifier method could lead to a less economical design.

Example 3

In the preceding examples, the frames analyzed were all rectangular in geometry and the connections were assumed to be rigid. To demonstrate the proposed method is also applicable to non-rectangular, semi-rigid frames, the gable frame shown in Fig. 12 will be analyzed. All members of the frames are made of W8×40 sections. The frame is assumed to be braced against out-of-plane bending. While the connection at the apex of the frame is assumed to be rigid, the connections at the eaves of the frame are modeled as semi-rigid. The connection stiffness



(Units for distributed loads: × 10⁻³ K/in.)

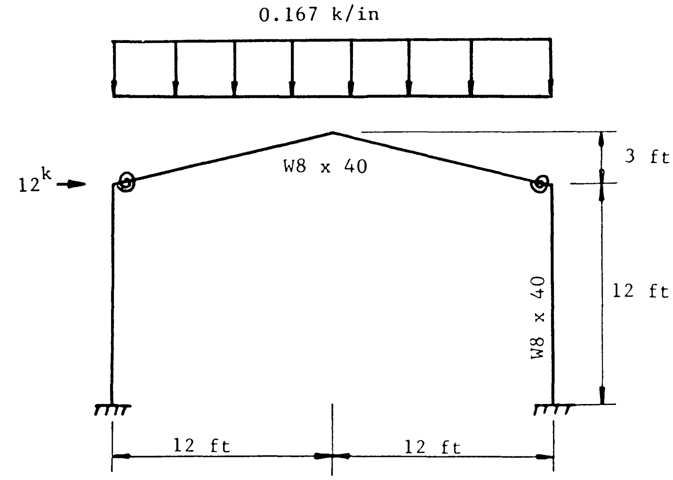
(b) Frame subjected to real and pseudo loads

Fig. 11. (continued)

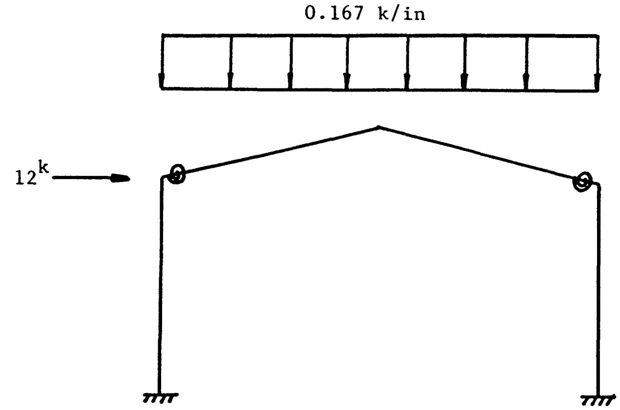
Table 6. Moment in Girder and Leeward Column of the Semi-rigid Gable Frame (kip-in)

Method of Analysis	Girder	Column
First-order Analysis	869	1167
Second-order Analysis	890	1187
Proposed Method (1 iteration)	890	1186

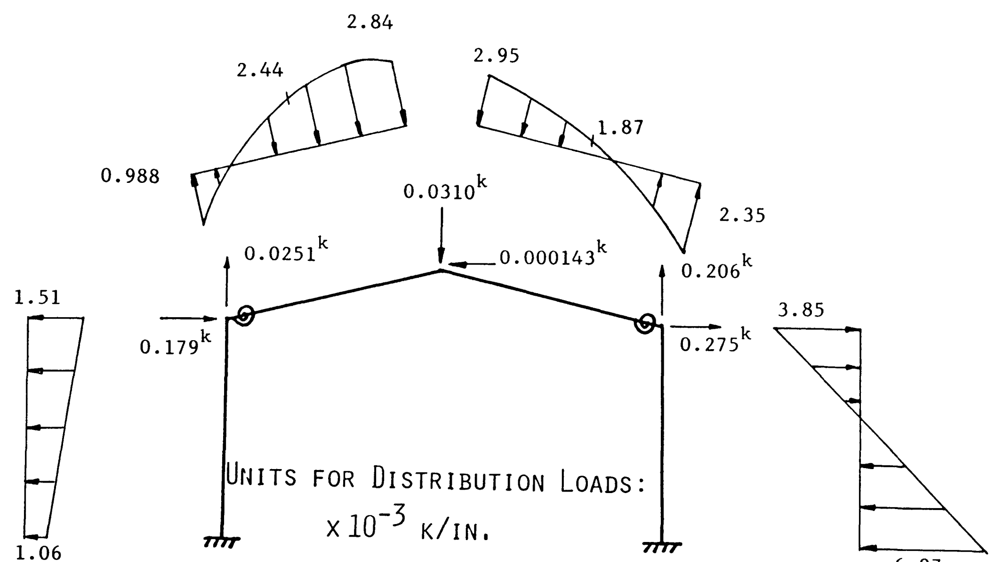
assumed for these connections is R_k = 28525 kip-in./rad. The frame is analyzed using a semi-rigid frame program developed by the author. The results obtained for the maximum moment in the girder and the column using several different analysis schemes are presented in Table 6. It can be seen that even with just one iteration, the proposed method gives very good results for the maximum girder and column moments. The LRFD moment magnifier method is not used for this example, since the magnification factor B_2 is applicable only to rectangular rigid frames.



(a) Frame geometry and loading



Real loads



Pseudo loads

(b) Frame subjected to real and pseudo loads

Fig. 12. Numerical example—gable frame

SUMMARY AND CONCLUSIONS

A simple and effective method of second-order frame analyses was presented in this paper. The method accounts for the P-δ and P-Δ effects by the use of pseudo loads obtained from a first-order analysis of the frame. The method is quite versatile in that it is applicable not only to rectangular Type FR frames but also to nonrectangular Type PR frames. The validity of the method has been demonstrated and it is therefore recommended for general use.

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NOMENCLATURE

B_1	=Nonsway moment amplification factor
B_2	=Sway moment amplification factor
E	=Modulus of elasticity
H'	=Pseudo joint load
I	=Moment of inertia
k	$=\sqrt{P/EI}$
L	=Length
M	=Moment
M_F	=Fixed-end moment
M_F^c	=Modified fixed-end moment for the presence of semi-rigid connection
$M_{\max}$	=Maximum moment
M_o	=First-order moment
P	=Axial force
Q	=Transverse force
R_k	=Connection stiffness
V_f	=Fixed-end shear
V_f^c	=Modified fixed-end shear for the presence of semi-rigid connection
w	=Distributed load
w'	=Pseudo lateral in-span load
δ	=Deflection of member with respect to its chord
Δ	=Deflection of member with respect to its initial position