Optimal Design of Cantilever-suspended Girders

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Cantilever-suspended girders are used frequently for bridges and low-rise, long-span buildings. Because this type of design calls for repetitive identical members, it is ideal for prefabricated construction. As a result, examples abound in structural steel. For various reasons, which include use of standard sections, economy and aesthetics, it is not uncommon for designers to specify the same size girders for all spans, particularly in buildings. In other words, prismatic construction is favored. This makes it important to arrive at an optimal design, i.e., to select a minimum-weight girder at the lowest possible design moment.

Optimization is possible when one determines the position of the hinge, at the end of the cantilever, in such a way the maximum design moment is minimized. If the full load (live plus dead) were placed over the affected span (to determine the maximum moments, M^+ and M^-, that are generated at the suspended portion and over the cantilever support, respectively) the optimal position of the hinge, as given by its distance to the cantilever support a would be obtained when $M^- = M^+$. General methods of analysis, including numerical iteration, are available in the literature[3] to determine both the optimal position of the hinge in a cantilever-suspended span and the design moment. The solutions to various typical cases are given in Fig. 1 for centrally (3-span) and side (2-span) suspended girders, respectively.

Page 2-129 of the 8th Edition of AISC *Manual of Steel Construction*[1] also provides comprehensive information, see Table 1, on maximum bending moments, cantilever lengths and other design properties of equal-span, cantilevered beams subjected to equally spaced, equal loads. (An identical table appears on p. 3-145 of the 1st Edition of AISC's *Load and Resistance Factor Design Manual of Steel Construction*.)[2] Although it is not stated explicitly in the AISC Table as such, it may be inferred, for the purpose of design, the loads P are not partial but full loads.

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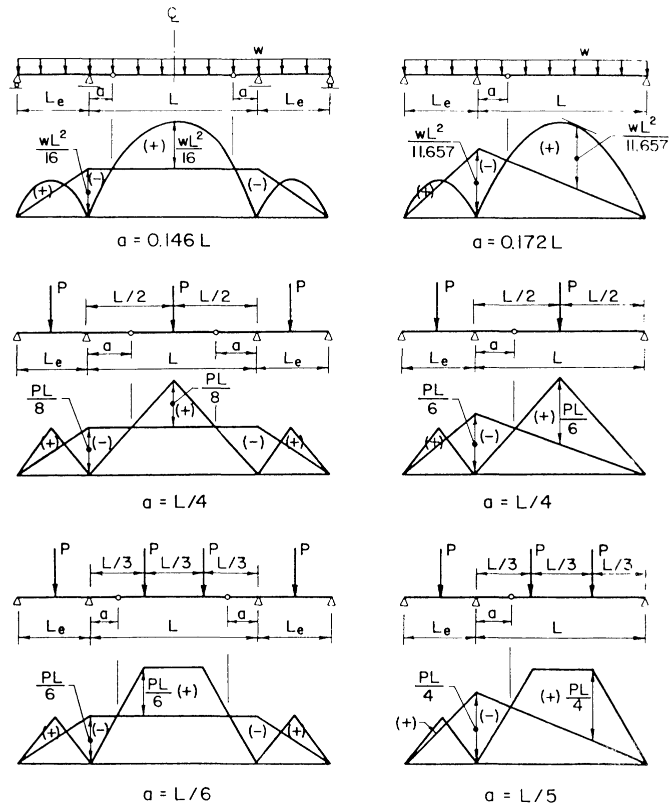


Fig. 1. Various cantilever-suspended beams

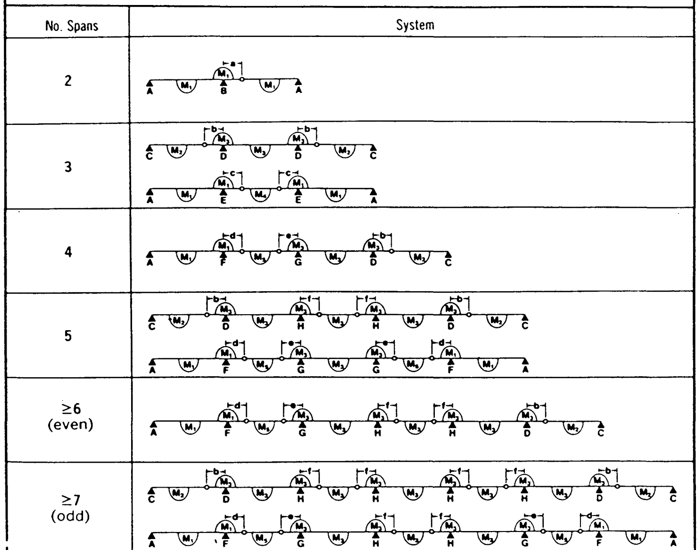
Note the solution of the problem, as shown in either Table 1 or Fig. 1, is based exclusively on an equal-load analysis of the cantilever-suspended span, without consideration to the effects of partial loading on the adjacent spans. Because the structure is statically determinate, the adjacent spans do not affect the value of maximum moments obtained in the cantilever-suspended span; placement of the live load on the adjacent spans does not affect this solution. Since designers usually select the cantilever-suspended span to be the longest and/or most heavily-loaded span, it may appear doubtless to them that larger moments (requiring heavier girders) cannot occur in adjacent spans. Because this premise is not always true, improper use of Table 1 or Fig. 1 may result in overstressed structures.

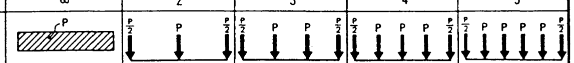
Table 1.

BEAM DIAGRAMS AND FORMULAS

Design properties of cantilevered beams

Equal loads, equally spaced



n		∞	2	3	4	5
Typical Span Loading						
Moments	M_1	0.086 x PL	0.167 x PL	0.250 x PL	0.333 x PL	0.429 x PL
	M_2	0.096 x PL	0.188 x PL	0.278 x PL	0.375 x PL	0.480 x PL
	M_3	0.063 x PL	0.125 x PL	0.167 x PL	0.250 x PL	0.300 x PL
	M_4	0.039 x PL	0.083 x PL	0.083 x PL	0.167 x PL	0.171 x PL
	M_5	0.051 x PL	0.104 x PL	0.139 x PL	0.208 x PL	0.249 x PL
Reactions	A	0.414 x P	0.833 x P	1.250 x P	1.667 x P	2.071 x P
	B	1.172 x P	2.333 x P	3.500 x P	4.667 x P	5.857 x P
	C	0.438 x P	0.875 x P	1.333 x P	1.750 x P	2.200 x P
	D	1.063 x P	2.125 x P	3.167 x P	4.250 x P	5.300 x P
	E	1.086 x P	2.167 x P	3.250 x P	4.333 x P	5.429 x P
	F	1.109 x P	2.208 x P	3.333 x P	4.417 x P	5.557 x P
	G	0.977 x P	1.958 x P	2.917 x P	3.917 x P	4.871 x P
	H	1.000 x P	2.000 x P	3.000 x P	4.000 x P	5.000 x P
Cantilever Dimensions	a	0.172 x L	0.250 x L	0.200 x L	0.182 x L	0.176 x L
	b	0.125 x L	0.200 x L	0.143 x L	0.143 x L	0.130 x L
	c	0.220 x L	0.333 x L	0.250 x L	0.222 x L	0.229 x L
	d	0.204 x L	0.308 x L	0.231 x L	0.211 x L	0.203 x L
	e	0.157 x L	0.273 x L	0.182 x L	0.176 x L	0.160 x L
	f	0.147 x L	0.250 x L	0.167 x L	0.167 x L	0.150 x L

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It is shown in this paper that optimal design of cantilevered-suspended spans may be governed by the maximum positive moment generated in an adjacent span. Thus, the designer must consider the effects of partial loading on maximum positive moments in the adjacent spans. Particularly for steel structures, where the ratio of dead load to full load (live plus dead) can be relatively small, true optimization may not be accomplished without recognition of the effects of partial loading. Two-span girders with a cantilever-suspended span are used for the sake of simplicity to illustrate optimal design and because they are more likely than multi-span girders to be affected by improper use of existing data.[1,2]

ANALYSIS

For a two-span girder subjected to a uniformly distributed load w the apparent optimal solution for the cantilever-suspended span, see Table 1 and Fig. 1, requires the hinge be located at a distance from the cantilever support given by:

$$a = 0.1716\ L \tag{1}$$

The apparent design moment M_{des} is as follows:

$$M_{des} = \frac{wL^2}{11.657} = 0.08579\ wL^2 = 0.6863\left(\frac{wL^2}{8}\right) \tag{2}$$

Note that L is the length of the span where the hinge is located. Usually, L is the longest or most heavily loaded span.

When the live-to-dead load ratio is large (as in the case of warehouse floor girders, or roof girders in regions of heavy snowfalls) it is possible the solutions given in Table 1 and Fig. 1 may lead to substantial underdesigning because the live load need not be placed on all spans simultaneously. Thus, the maximum positive moment in a fully loaded hingeless span may exceed the design moment given by Eq. 2 when the adjacent cantilever-suspended span is unloaded.

Consider first the design of a girder with two equal spans subjected to live load w_L that need not be placed over both spans simultaneously. The maximum negative and positive moments M^- and M^+ in the cantilever-suspended span are obtained when the latter span is under full load w, whether or not the hingeless span is fully loaded. The following is obtained:

$$M^- = \frac{waL}{2} \tag{3}$$

and

$$M^+ = \frac{w\ (L\text{-}a)^2}{8} \tag{4}$$

Solution of Eqs. 3 and 4 for the condition $M^- = M^+$ yields the values of distance a and design moment M_{des}, given by Eqs. 1 and 2, respectively. If now the designer proportioned the girder for M_{des}, overstressing would occur if the live load were placed exclusively on the hingeless span. The maximum positive moment M_h^+, generated over the fully loaded hingeless span (when the other span is subjected only to dead load) is given by:

$$M_h^+ = \frac{(wL - w_D a)^2}{8\ w} \tag{5}$$

Equations 3, 4 and 5 give the maximum values of M^-, M^+ and M_h^+, respectively, that can occur under any possible loading pattern.

It may be shown that M_h^+ exceeds the value of M_{des} (Eq. 2) for all cases where the live load exceeds zero. Thus, consider first writing these equations in dimensionless form, as follows:

$$\frac{M^-}{\frac{wL^2}{8}} = 4\,\alpha \tag{6}$$

$$\frac{M^+}{\frac{wL^2}{8}} = (1 - \alpha)^2 \tag{7}$$

and

$$\frac{M_h^+}{\frac{wL^2}{8}} = (1 - \omega\alpha)^2 \tag{8}$$

where $\alpha = a/L$ and $\omega = w_D/w$. Secondly, let these dimensionless moments be plotted against α (for various values of ω, in the case of Eq. 8), see Fig. 2. One notes that, only for the extreme case where $\omega = 1$ (no live load, or the arbitrary equivalent of considering the full load as permanent and thereby present at all times on both spans) would one get the solution given by Eqs. 1 and 2. This point is clearly indicated in Fig. 2 at $\alpha = a/L = 0.1716$ and $M^-/(wL^2/8) = M^+/(wL^2/8) = 0.6863$.

For the other extreme case where $\omega = 0$ (no dead load), one obtains $\alpha = a/L = 0.25$ and $M^- = M_h^+ = wL^2/8$; also, $M^+ = 0.5625\,(wL^2/8)$. It follows that the design moment is $M_{des} = wL^2/8$, which is 46% larger than the value given by Eq. 2. Note that only for $\omega = 0$ is the solution $\alpha = 0$ (two simply supported girders, i.e. $M^+ = M_h^+ = wL^2/8$, and $M^- = 0$) possible, and equivalent, to the cantilever-suspended girder. Note, however, the latter girder may still be preferable to the simply supported ones because of greater stiffness.

It is clear from Fig. 2, for any design that includes live load (i.e. whenever $\omega < 1$), the optimal solution is obtained by equating M^- and M_h^+. This requires solving for $\alpha = a/L$ in Eqs. 6 and 8. One obtains:

$$\alpha = \frac{a}{L} = \frac{1}{(2 + \omega) + 2\sqrt{1 + \omega}} \tag{9}$$

and

$$\frac{M_{des}}{\frac{wL^2}{8}} = \frac{4a}{L} = \frac{4}{(2 + \omega) + 2\sqrt{1 + \omega}} \tag{10}$$

The variation of these quantities with the load parameter ω is given in Table 2. Note that for the range $0 < \omega < 1$, the parameter $\alpha = a/L$ varies between 0.25 and 0.1716, while the quantity $M_{des}/(wL^2/8)$ varies between 1 and 0.6863.

Example 1

Consider design of a two equal-span girder where $L = 60$ ft, $w_D = 0.58$ k/ft, and $w_L = 0.90$ k/ft. It follows: $w = w_D + w_L = 0.58 + 0.90 = 1.48$ k/ft and $\omega = w_D/w = 0.3919$.

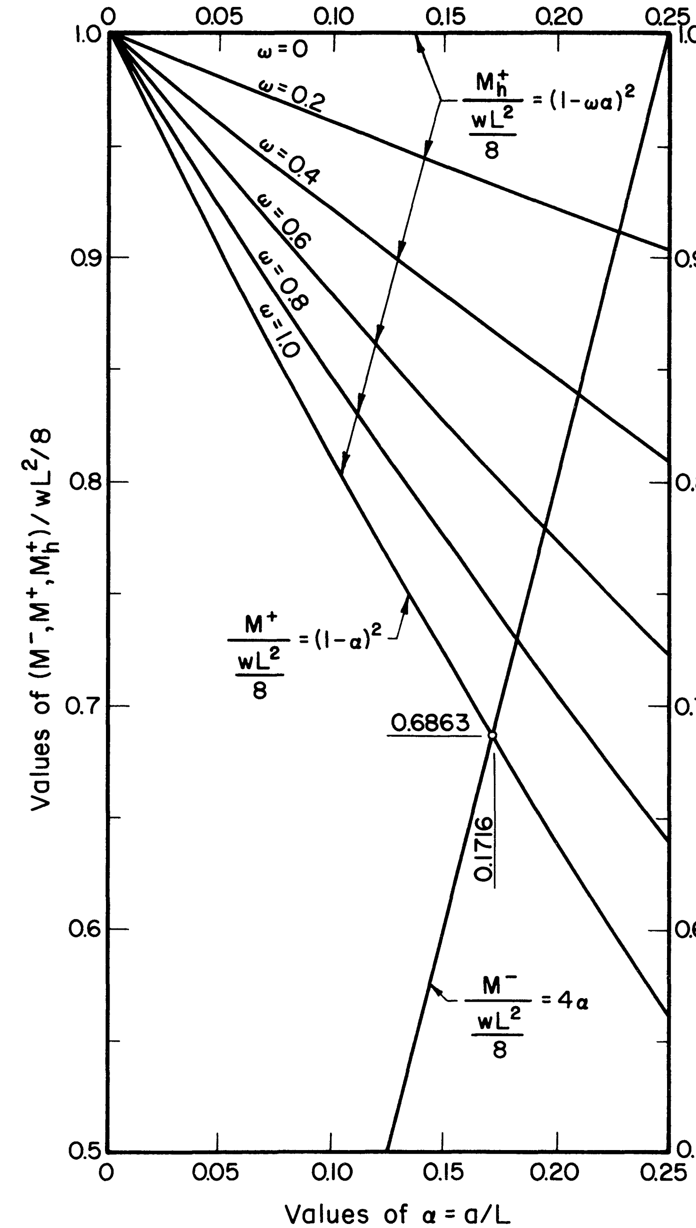

Fig. 2. Determination of design moment and hinge location in two equal-span, cantilever-suspended girders

Using Eqs. 9 and 10, determine:

$$\frac{a}{L} = \frac{1}{(2 + 0.3919) + 2\sqrt{(1 + 0.3919)}} = 0.21046$$

and

$$\frac{M_{des}}{\frac{wL^2}{8}} = \frac{4}{(2 + 0.3919) + 2\sqrt{(1 + 0.3919)}} = 0.84185$$

Table 2. Two Equal-span, Cantilever-suspended Girder. Variation of α and $M_{des}/(wL^2/8)$ with ω

$\omega = \frac{w_D}{w}$	$\alpha = \frac{a}{L}$	$\frac{M_{des}}{w\,L^2/8}$
0[a]	0.2500	1.0000
0.1	0.2382	0.9529
0.2	0.2277	0.9110
0.3	0.2183	0.8733
0.4	0.2098	0.8392
0.5[b]	0.2020	0.8082
0.6	0.1949	0.7798
0.7	0.1884	0.7536
0.8	0.1824	0.7295
0.9	0.1768	0.7071
1.0[c]	0.1716	0.6863

Note: w_D, w_L, and w are dead, live and full load, respectively; a is distance from central support to hinge, L is one span length and M_{des} is design moment.

[a] Case where $w_D = 0$ or, practically, where live load is very large in comparison to dead load.

[b] Case where $w_L = w_D$.

[c] Case where $w_L = 0$ or, practically, where live load is very small in comparison to dead load.

Thus, for $L = 60$ ft,

$$a = (0.21046)\,(60) = 12.63 \text{ ft, and}$$

$$M_{des} = 0.84185\,\frac{(1.48)\,(60)^2}{8} = 560.7 \text{ k/ft}$$

The values of the bending moments M^+, M^- and M_h^+, for all three possible loading patterns, are given in Table 3. Note the optimal design moment $M_{des} = 560.7$ k/ft occurs in at least one location for each loading pattern.

For the purpose of comparison, consider the girders designed instead using Eqs. 1 and 2, or from Table 1 and Fig. 1 directly. One would obtain $a = 10.30$ ft and $M_{des} = 457.1$ k/ft, the latter being the apparent design moment. The values of the bending moments M^+, M^- and M_h^+, for all three possible loading patterns are given in Table 4. Note the girder would be subjected to a maximum bending moment $M_h^+ = 579.40$ k/ft, that is much larger than the apparent design moment, if only the hingeless span (and not the cantilever-suspended span) were fully loaded. This value exceeds the apparent design moment (457.1 k/ft) by a substantial margin. Thus, the girder could be overstressed by almost 27% for that particular loading pattern. However, had the girder been properly proportioned for the maximum moment, i.e. 579.40 k/ft, it would have exceeded the optimal design (which only requires $M_{des} = 560.7$ k/ft) by 3%.

One may conclude a designer who used Eqs. 1 and 2 or Table 1 and Fig. 1 for selecting the cantilever length, would obtain a girder that could be overstressed in service because of the effects of partial loading. However, if after selecting the same cantilever length, the designer would subsequently check for the effects of partial loading, he would attain a safe, but not an optimal design.

Table 3. Ex. 1. Optimal Design. Bending Moments for All Three Loading Patterns

Bending moment	Full load on: Both spans k/ft	Cantilever-suspended span only k/ft	Hingeless span only k/ft
M^+	415.2	415.2	162.7
M^-	560.7*	560.7*	219.7
M_h^+	415.2	55.9	560.7*

* Design moment M_{des}

Table 4. Ex. 1. Incorrect Design. Bending Moments for All Three Loading Patterns

Bending moment	Full load on: Both spans k/ft	Cantilever-suspended span only k/ft	Hingeless span only k/ft
M^+	457.1*	457.1*	179.1
M^-	457.1*	457.1*	179.1
M_h^+	457.1*	82.5	579.4†

* Apparent design moment

† Exceeds apparent design moment by substantial margin

CASE OF TWO UNEQUAL SPANS

This study can be extended to the case of unequal adjacent spans. Take the short span L_e to be equal to βL, where L is the long span and $0 < \beta \leq 1$; the hinge is to be placed within the long span. Consider also that $\omega = w_D/w$, where w_D and w are the uniformly distributed dead and full loads, respectively. Similar analysis allows determination of the design moment M_{des} and hinge distance in the long span a for this case. The following is obtained:

$$\frac{a}{L} = \frac{\beta^2}{(2 + \omega) + 2\sqrt{1 + \omega}} \tag{11}$$

and

$$\frac{M_{des}}{\frac{wL^2}{8}} = \frac{4a}{L} = \frac{4\,\beta^2}{(2 + \omega) + 2\sqrt{1 + \omega}} \tag{12}$$

Note, for $\beta = 1$, Eqs. 11 and 12 render Eqs. 9 and 10, respectively.

It can be shown Eqs. 11 and 12 must be used for design only when two conditions given by: $\beta > 0.8284$ and $\omega < \beta(\beta - 0.8284)/0.1716$ are both satisfied. For $\beta < 0.8284$, and regardless of the value of ω, take $a = 0.1716\,L$ and $M_{des} = wL^2/11.657$ (from Eqs. 1 and 2, respectively) because the long cantilever-suspended span governs design. This is true for $\beta < 0.8284$, even if the live load were placed over the hingeless span while the other span were subjected to dead load only. Similarly, when the actual value of ω exceeds the quantity $\beta(\beta - 0.8284)/0.1716$, and regardless of the fact $\beta > 0.8284$, take $a = 0.1716\,L$ and $M_{des} = wL^2/11.657$ (from Eqs. 1 and 2, respectively) because, again, the long cantilever-suspended span governs design.

Example 2

Consider a two-span, cantilever-suspended girder such that $\beta = L_e/L = 0.9$ and $\omega = w_D/w = 0.3$. Determine the design values of a/L and $M_{des}/(wL^2/8)$.

Since $\beta = 0.9 > 0.8284$ and $\omega = 0.3 < (0.9)(0.9 - 0.8284)/0.1716 = 0.3754$, the hinge location and the design moment are governed by Eqs. 11 and 12, respectively. Thus,

$$\frac{a}{L} = \frac{(0.9)^2}{(2 + 0.3) + 2\sqrt{1 + 0.3}} = 0.17684$$

and

$$\frac{M_{des}}{\frac{wL^2}{8}} = 4 \times 0.17684 = 0.70737$$

The design moment is 3% larger than obtained using Eq. 2 $[M_{des}/(wL^2/8) = 0.6863]$. Note, had ω been larger than 0.3754, the design would have been governed instead by Eqs. 1 and 2, even when $\beta = 0.9 > 0.8284$. By the same token, had β been less than 0.8284, the design would have also been governed by Eqs. 1 and 2, regardless of the value of ω.

The values of dimensionless bending moments $M^+/(wL^2/8)$, $M^-/(wL^2/8)$ and $M_h^+/(wL^2/8)$, for all three possible loading patterns, are given in Table 5. Note that $M_{des}/(wL^2/8)$ occurs in at least one location for each loading pattern.

ADDITIONAL CONSIDERATIONS

Although the position of the hinge and the required design moment can be determined using this analysis, stiffness considerations must not be ignored. Compliance with deflection limitations under service loads is required for the cantilever-suspended span, as well as the side spans. While the position of the hinge need not be changed, the design of the girder should account for both M_{des} and EI requirements.

Lateral stability of the cantilever-suspended girders must be secured also, both during erection and under service conditions. Particular consideration should be given to the effects of negative moments because they generate compressive stresses on the bottom flange of the girder that contains the cantilever. Lateral instability may occur in this region for the length of girder situated between the tip of the cantilever and the farthest position of the inflection point at the side span. The latter point must be calculated by considering the hingeless span subjected only to dead load. In some cases, where β and/or ω are small, the whole length of the side span could be affected. A proper amount of lateral bracing may have to be provided.

Table 5. Ex. 2. Optimal Design. Bending Moments for All Three Loading Patterns

Bending moment	Full load on: Both spans k/ft	Cantilever-suspended span only k/ft	Hingeless span only k/ft
$M^+/(wL^2/8)$	0.6776	0.6776	0.2033
$M^-/(wL^2/8)$	0.7074*	0.7074*	0.2122
$M_h^+/(wL^2/8)$	0.4949	0.0180	0.7074*

* Dimensionless design moment $M_{des}/(wL^2/8)$

CONCLUSIONS

1. Cantilever-suspended girders designed without regard to the effects of the live load on adjacent side spans may be overstressed in certain cases by as much as 46%.
2. The table in AISC's Manuals[1,2] is based not on partial but on full loads. Improper use of this table may cause design of overstressed cantilever-suspended girders. A "designer beware" note to that effect would aid designers using this table.
3. Designers who select the cantilever length from the information contained in the AISC Design Manuals,[1,2] but who would actually calculate the effects of partial loading to determine the true design moment, will avoid overstressing. Their design, however, will not be optimal.
4. Formulas for optimal design of two-span, cantilever-suspended girders of equal span are given in the paper.
5. Optimal design of two-span, cantilever-suspended girders of unequal span length may be performed, without considering the effects of partial loading, when either β (the ratio of short-to-long span lengths) is less than 0.8284, or ω (the ratio of dead-to-full load) is larger than the quantity $\beta(\beta - 0.8284)/0.1716$. For such cases, use $a = 0.1716\,L$ and $M_{des} = wL^2/11.657$, be-

cause the long cantilever-suspended span governs design.

6. For all other cases of design of two-span, cantilever-suspended girders of unequal span length, i.e. when both $\beta > 0.8284$ and $\omega < \beta (\beta - 0.8284)/0.1716$, formulas for optimal design are given in the paper to determine the actual design moment and position of the hinge.
7. Design of cantilever-suspended girders, that is based on optimization of bending strength, must be carefully checked for stiffness and lateral stability.

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APPENDIX 1

The following symbols are used in this paper:

a = Distance of hinge to cantilever support
L = Length of cantilever-suspended span between supports
L_e = Length of side span between supports
M^- = Maximum negative moment, over cantilever support
M^+ = Maximum positive moment, in suspended span
M_h^+ = Maximum positive moment, in hingeless span
M_{des} = Design moment
w = Full load
w_D = Dead load
w_L = Live load
α = a/L
β = L_e/L
ω = w_D/w