Plastic Analysis of Pinned-base Haunched Gable Frames

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Summary

The plastic analysis of pinned-base haunched gable frames, subjected to distributed vertical and horizontal loadings, has been carried out in a manner amenable to hand calculation. The analysis is based on the simple plastic theory and has resulted in analytical expressions for the plastic moment requirement of the rafter. The analysis is further facilitated by assembling the resulting expressions on non-dimensionalized charts. The method of analysis, a typical set of charts and an example demonstrating the method are presented in this paper.

INTRODUCTION

Pinned-base haunched gable frames are extensively used in light to moderately heavy industries. In practice, these frames are normally designed elastically. The extensive array of available design aids for the elastic analysis and design of such frames on one hand and the lack of such tools for the plastic analysis and design on the other hand have discouraged the practicing engineer from using the latter approach. With the advent of limit state design, plastic design is expected to become more popular among structural engineers, especially if they are provided with simple design aids, possibly amenable to hand calculations. One such method for the plastic analysis of single-span, pinned-base haunched gable frames has been developed and is presented here.

Plastic analysis of gable frames has been addressed by a number of researchers in the past. Ketter was the first to produce charts pertaining to the rapid design of single and multi-span, pinned-base gable frames composed of members with uniform cross-section.[1] Columns were assumed free from plastic hinges and the loading was composed of a uniformly distributed vertical load and some horizontal loads. This work was later presented by the American Institute of Steel Construction.[2] Harrison described a procedure for the plastic analysis of single-bay gable frames in a manner suitable for digital computation.[3] Vertical uniformly distributed loading as well as wind loading in accordance with the Australian S.A.A. Interim Code 350 were considered. As in Ref. 1, uniformity of member cross section was assumed, but plastic hinge formation in the columns as well as the rafters was allowed. The results were presented in chart form. Single-span, pinned-base haunched gable frames were studied by Manolis and Beskos.[4] A uniformly distributed vertical load in conjunction with a concentrated horizontal load applied at the eave were considered. Haunches were assumed both in the rafters and the columns. This method, although it allows for consideration of various rafter-to-columns plastic moment ratios, is not readily extensible to the analysis of multiple span frames.

In the work to be presented here, haunched frames subject to the combined effect of uniformly distributed vertical load and some arbitrary lateral loads are considered. It will be assumed the columns and the haunches are free from plastic hinges. This simplifying assumption leads to economical solutions in frames with practical geometry and loading. The assumption also paves the way for an easy extension of the method to include the plastic analysis of multiple span haunched frames. The mechanism approach is used here for deriving analytical expressions for the non-dimensionalized plastic moment requirement of the rafter. An expression is derived for each possible failure mechanism. Also, expressions are developed for the location of the plastic hinges in various mechanisms. Design aids in the form of plots are developed based on the analytical expressions. Some typical plots are given in Appendix B. Using these plots, a typical gable frame skeleton is analyzed plastically and then designed based on the AISC Code.[5] The design is compared to the results obtained using the allowable stress approach based on the same code.

ANALYTICAL EXPRESSIONS

Consider the single-span, pinned-base gable frame of Fig. 1. The span is L, the height of the column is aL, the total rise of the rafters is bL and the horizontal projection of the

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haunch is cL. The frame is subjected to a uniformly distributed vertical load w. The effect of horizontal loads on the windward and the leeward sides is replaced by the equivalent overturning moments $A\,wL^2/2$ and $D\,wL^2/2$ respectively. Since the columns and haunches are assumed to be free of plastic hinges, the above replacement does not affect the virtual work expressions for the loads applied below the points C and D. For the loads applied above these points, it can be shown that the effect of that replacement is conservative.[1]

There are five possible plastic hinge locations (Fig. 1) and the system is one degree indeterminate. Therefore there are four independent mechanisms. The chosen independent mechanisms are shown in Fig. 2. Proceeding then to determine the plastic moment requirement of the rafters for each of the selected independent mechanisms, the mechanism "b" is considered first. If the frame is assumed to undergo the virtual displacement shown in Fig. 3, the following expressions can be derived from the geometry. The parameters are as defined in Fig. 3.

$$x = \frac{\alpha(a + 2cb)}{a(c + \alpha) + 4\alpha cb} L \tag{1}$$

$$y = \frac{(a + 2cb)(a + 2\alpha b)}{c(a + 2\alpha b) + \alpha(a + 2cb)} \tag{2}$$

$$y_1 = \frac{(a + 2cb)[(a + 2\alpha b)(1 - c) - \alpha(a + 2cb)]}{c(a + 2\alpha b) + \alpha(a + 2cb)} L \tag{3}$$

$$y_2 = \frac{(a + 2\alpha b)[(a + 2cb)(1 - \alpha) - c(a + 2\alpha b)]}{c(a + 2\alpha b) + \alpha(a + 2cb)} L \tag{4}$$

$$e = x - (1 + \alpha - c)\frac{L}{2}$$

$$= \frac{2 - (1 + \alpha - c)(1 + \frac{c}{a} \cdot U)}{2(1 + \frac{c}{\alpha} \cdot U)} L \tag{5}$$

$$\theta_{IC} = \frac{(a + 2bc)L}{y_1} \theta = \frac{\alpha + c\,U}{V} \theta \tag{6}$$

$$\theta_A = \frac{y_2}{(a + 2\alpha b)L} \theta_{IC} = \frac{1 - \alpha - c\,U}{V} \theta \tag{7}$$

$$\theta_2 = \theta_{IC} + \theta_A = \frac{1}{V} \theta \tag{8}$$

$$\theta_5 = \theta_{IC} + \theta = \frac{U}{V} \theta \tag{9}$$

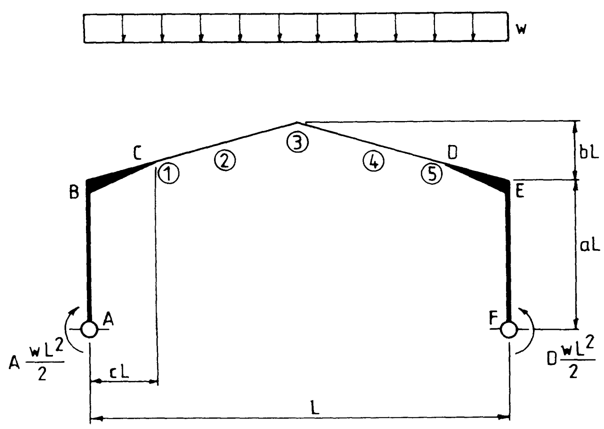


Fig. 1. Loading and location of possible plastic hinges

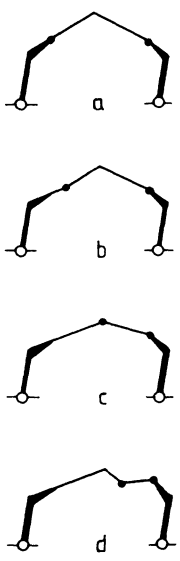


Fig. 2. Independent mechanisms

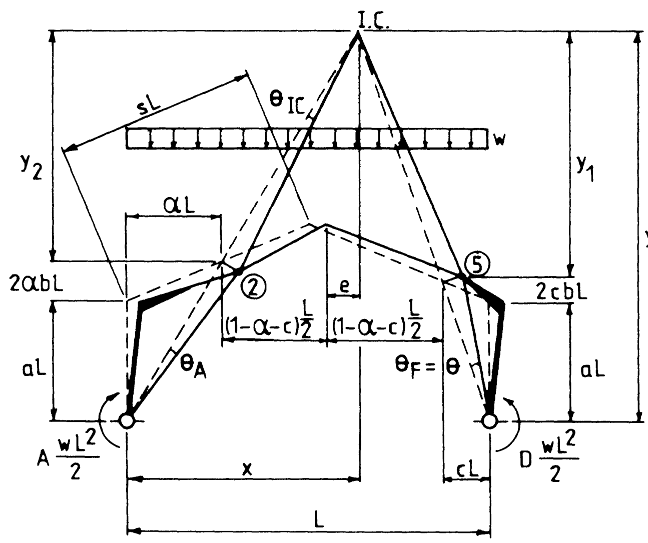


Fig. 3. Failure mechanism "b"

where:

$$U = \frac{1 + 2(b/a)\alpha}{1 + 2(b/a)c}$$

$$V = (1 - c)\,U - \alpha$$

The contribution to external work done by the loads of Segments A-2, 2-5 and 5-F in Fig. 3 are then given by

$$w_{A\text{-}2} = \frac{wL^2}{2}(A + \alpha^2)\,\theta_A \tag{10}$$

$$w_{2\text{-}5} = (1 - \alpha - c)\,wL\,e\,\theta_{IC} \tag{11}$$

$$w_{5\text{-}F} = -\frac{wL^2}{2}(c^2 + D)\,\theta \tag{12}$$

The work expression associated with the assumed virtual displacement of Fig. 3 can now be rewritten as

$$w_{A\text{-}2} + w_{2\text{-}5} + w_{5\text{-}F} = M_p(\theta_2 + \theta_5) \tag{13}$$

Substitution of Formulas 7 and 6 into 10 and 11 and 8, 9 and 10 to 12 into 13 (after some lengthy simplification and rearrangements) results in the non-dimensionalized plastic moment requirement of the rafters for the failure mechanism "b."

$$\frac{M_p}{wL^2} = \frac{1}{2(1 + U)}\{(1 - \alpha)(A + \alpha) + D\alpha - [c(A - D + 1 - c) + D]\,U\} \tag{14}$$

Formula 14 is in terms of the unknown distance αL to the plastic hinge in the rafter. The correct value of the independent variable α is found from the expression

$$\frac{\partial M_p}{\partial \alpha} = 0 \tag{15}$$

which leads to

$$\alpha = \frac{1}{\frac{b}{a}}\left\{\left[\left(1 + \frac{b}{a}c\right)^2 - \frac{b}{a}\left((1 + 2\frac{b}{a}c)(A - D) + \frac{b}{a}(A + D) - \frac{b}{a}c^2 - 1\right)\right]^{1/2} - \left(1 + \frac{b}{a}c\right)\right\} \text{ for } \frac{b}{a} > 0 \tag{16}$$

and

$$\alpha = \frac{1 - A + D}{2} \text{ for } \frac{b}{a} = 0$$

The solution for mechanism "a" is derived as a special case of mechanism "b" by setting $\alpha = c$.

$$\frac{M_p}{wL^2} = \frac{1}{4}(1 - 2c)(A - D) \tag{17}$$

Similarly the solution for mechanism "c" is derived by letting $\alpha = 1/2$:

$$\frac{M_p}{wL^2} = \frac{1}{4\left(1 + \frac{1 + b/a}{1 + 2(b/a)c}\right)}\left\{A + \frac{1}{2} + D - 2[c(A - D + 1 - c) + D]\frac{1 + b/a}{1 + 2(b/a)c}\right\} \tag{18}$$

The expression for the plastic moment requirement of failure mechanism "d" is derived in a manner similar to that given for mechanism "b." The final equations of the above mentioned mechanisms and of the other possible mechanisms (which can be obtained by combining the independent mechanisms) are given in Appendix A.

Having the plastic moment requirements of the rafter for all possible mechanisms, it is then necessary to determine the largest plastic moment value. This will in general depend on the values of b/a, c, A and D under consideration. The range of applicability of each failure mechanism can therefore be determined by assigning various values to these parameters and solving for each of the equations. Carrying out such a procedure, it is observed that only mechanisms "1," "2" and "6" govern the solutions. The resulting values can be assembled in graphical form where the non-dimensionalized plastic moment requirement of the rafter M_p/wL^2 is shown relative to A and D for a given b/a and c. A typical set of plots for $b/a = 0.3$ and $c = 0.10$ is given in Appendix B. The corresponding α values are also given in graphical form. The complete set of plots which includes $b/a = 0, 0.1, 0.2, \ldots, 1.0$ and $c = 0.0, 0.05, 0.10, \ldots, 0.25$ has been given in Ref. 6.

Note that for values of $A = D$, the failure mechanisms would be symmetrical with hinges forming at points 1, 2, 4 and 5, resulting in an over-collapsed mechanism. It can be shown that the solution of this mechanism is the same as for mechanism "b" when $A = D$.

ECONOMY

The effect of the assumption that the columns and haunches are free from plastic hinges on the economy of the frame is investigated by performing a parametric study using the method of Ref. 4 and comparing the results with those obtained using the proposed method. Ref. 4 considers the possibility of the plastic hinge formation in the column and enables the plastic analysis of frames with various rafter-to-column plastic moment ratios.

Although many factors influence economical design, the criterion used in this paper is the least weight. Other factors are assumed to remain unchanged. It is also assumed that a one-to-one correspondence exists between weight and plastic modulus, or M_p.[1] The plastic moment in the haunched portion of the rafter is taken to be uniform and equal to the plastic moment requirement at the eave (the maximum value). This assumption magnifies the undesirable effect of the assumption that haunches are free from plastic hinges on the economy of the frame.

Typical results of the above study are shown in Fig. 4, where the sum of the plastic moment requirement of the frame (the integral of M_p along the columns and rafters) are shown versus the rafter-to-column plastic moment ratio ($k = M_{p_r}/M_{p_c}$) and c. The hollow circles denote values obtained using the proposed method and the lines illustrate the solutions obtained using the method of Ref. 4 (the dots indicate the calculated points). It can be seen that, for the practical range of the haunch parameter ($c \leq 0.15$), the analysis based on the proposed method results in designs in the vicinity of the least weight.

EXAMPLE

A pinned-base steel gable frame spaced 19 ft-8.2 in. (6 m) with a span of 92 ft (23 m), a column height of 26 ft-3 in. (8 m) and a total rise of 7 ft-10.5 in. (2.4 m) containing a 22-kip (10-ton) overhead travelling crane will be designed. A haunch parameter $c = 0.1$ is assumed. This information, and other loadings considered, are shown in Fig. 5. This loading combination is assumed to be the critical one and therefore the design is based on this loading case only. The loads of Fig. 5 include a load factor of 1.7. With the geometrical parameters $a/b = 0.3$ and $c = 0.1$ and the loading parameters $A = 0.34$ and $D = -0.005$, the plastic moment requirement of the rafter and the plastic hinge location parameter, α, are found from the plots of Appendix B or Equations 14 and 16.

$$\frac{M_p}{wL^2} = 0.0745$$

$$\alpha = 0.254$$

The resulting axial force, shear force and bending moment diagrams are shown in Fig. 6. The same diagrams are given in Fig. 7 for the elastic analysis of the frame when subject to factored loads (factored loads are used in order to make a direct comparison possible).

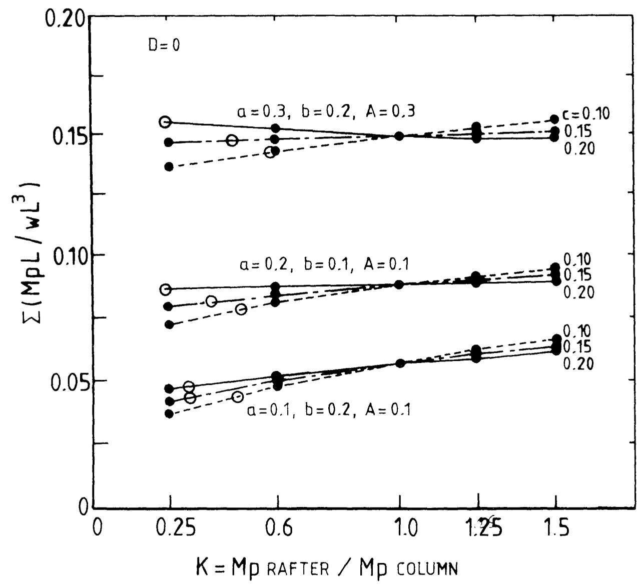


Fig. 4. Comparison of proposed method with method of Ref. 4

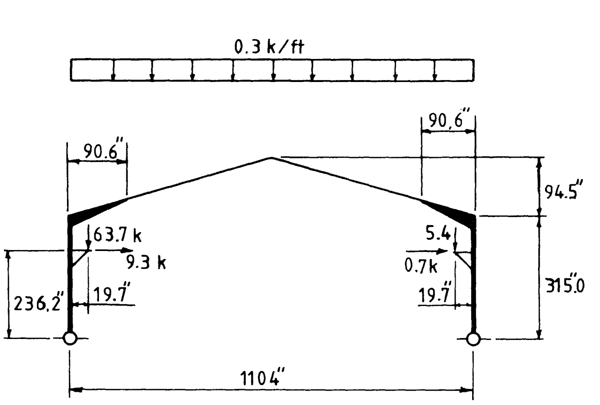


Fig. 5. Geometry and loading of example frame

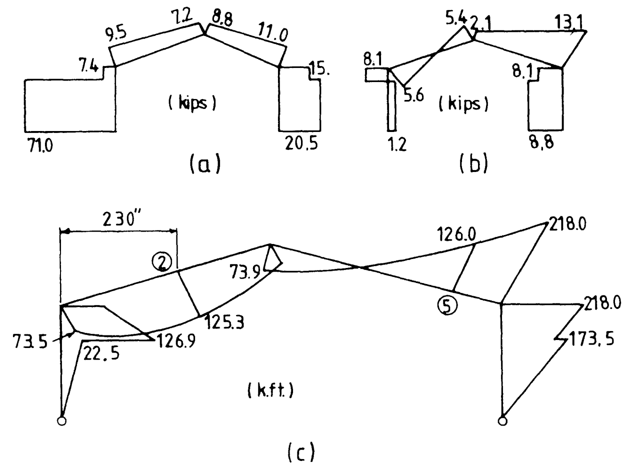


Fig. 6. (a) Axial force diagram, (b) shear force diagram and (c) bending moment diagram

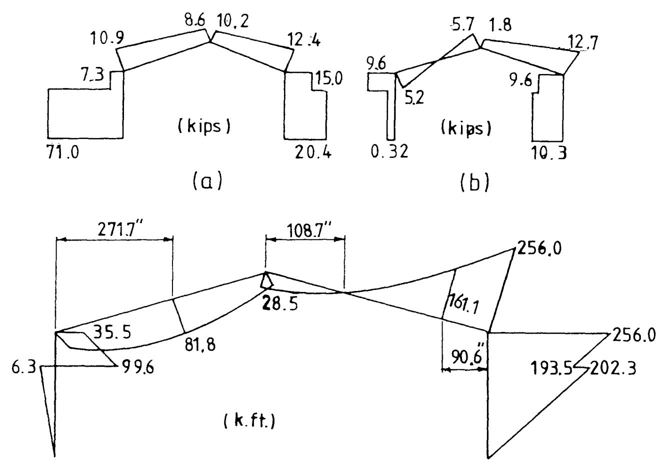


Fig. 7. (a) Axial force diagram, (b) shear force diagram and (c) bending moment diagram

It is assumed the rafters are provided with sufficient lateral supports and the columns are restrained against both column and lateral buckling at the base, the crane and the eave levels only. The frame is designed using both the plastic design and allowable stress design provisions of the AISC Code.[5] The results are summarized in Table 1. In this frame, because of the larger unbraced length, the portion of the leeward column below the crane level governed the design of the columns. Hence the portions of the columns between the crane and the eave level possess some reserve capacity. This ensures the plastic hinge would form in the rafter as assumed in the analysis. Therefore the lateral bracing requirement for regions adjacent to a plastic hinge does not have to be observed in the column. Care should be taken when the possibility exists of a shift in the location of plastic hinge from the rafter to the column. In such cases proper lateral bracing should be provided in the column.

Note in Table 1, the plastic design has led to about 22% savings in weight. It should be noted that these designs were only based on strength considerations. Satisfying the serviceability limit state would have led to identical designs for this particular example where the lateral deflection at the crane level must be kept within some specified limit. In situations where the frame is not deflection-sensitive, however, plastic design can be used to economic advantage. In other situations, the plastic design aids of the type given in Appendix B can result in rapid analysis and hence design of such frames, provided that lengthy deflection calculations be avoided. To accomplish this objective, charts are provided for deflection calculation of gable frames subjected to various loads.

DEFLECTION ESTIMATES

The existing methods of calculating deflections in plastically designed structures, which use the resulting bending moments at failure, can be categorized into two groups, the approximate and the accurate.

The approximate methods such as the last hinge method of Neal[7] and the Heyman's approach[8] (in which he reduces the "bending moment diagram at failure" by the load factor and treats it as the working moment diagram), besides being tedious, are erroneous and could lead to errors (relative to elastic analysis) in the range of 100 and 50% respectively. The accurate methods such as the correction approach[9] and the direct method[10] of Melchers (in which the statically admissible but elastically non-compatible bending moment diagram at failure is used in conjunction with a statically admissible and elastically compatible bending moment diagram corresponding to the unit dummy load) would require elastic analysis of the structure. This means that a plastically analysed frame needs to be elastically analysed also.

Table 1. Summary of Designs

Design	Rafters	Columns	Haunches Cut from	Frame Weight (kips) *kN*
Plastic	W16 × 26	W18 × 55	W16 × 16	1.55 (6.92)
Allowable Stress	W16 × 36	W18 × 60	W16 × 36	1.89 (8.42)

To save the practical engineer from such two-fold analysis, non-dimensional expressions for the deflection of gable frames subjected to various loads have been developed and presented as plots of non-dimensionalized deflection versus stiffness ratio $k = a I_r / s I_c$ vs. b/a (see Appendix C). I_r and I_c are the moments of inertia of the rafter and column respectively and s is as defined in Fig. 3. The calculation of deflection for gable frames subjected to a combination of various loads reduces, then, to the superposition of the deflections induced by individual loads which are readily read off the charts.

The charts developed to date have all been based on the assumption that the rafters are uniform in cross-section. The effect of haunches on the deflection of two typical frames has been studied. It was found, as expected, that the haunches of practical length ($0.1 < c \leq 0.15$) have appreciable effect on deflections. In one of the sample frames, the haunches with $c = 0.15$ reduced the deflection by 40%. Currently charts are being developed for haunched gable frames. However, until such charts are available the charts developed for uniform rafter section could be used to provide conservative estimates of deflections in haunched frames. Two typical plots are given in Appendix C. A complete set of plots has been given in Ref. 11.

CONCLUSION

Expressions which provide the required plastic moment of the rafters as a function of geometrical and loading parameters in pinned-base, haunched gable frames were presented. With the aid of these expressions, the plastic analysis (and thereby design) of such frames is rapidly accomplished. The analysis was further facilitated by assembling the numerical values of the expressions on chart form. Typical sample charts were also provided.

Only uniformly distributed vertical load was considered in conjunction with arbitrary horizontal loads. This is suitable for situations where the vertical dead plus live loads are much larger than the vertical component of wind load on the roof. Work is underway for deriving similar expressions for cases where the intensity of wind pressure causes appreciable non-uniformity in roof loading. The present work is also being extended to include the analysis of multi-span frames.

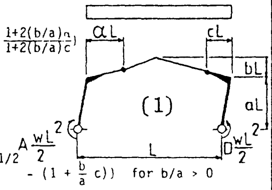
To avoid a separate elastic analysis for the serviceability check, possibility of developing charts for deflection calculations under working loads was explored. Charts which

APPENDIX A

$$\frac{M_p}{wL^2} = \frac{1}{2\left(1 + \frac{1+2(b/a)\alpha}{1+2(b/a)c}\right)} \left((1-\alpha)(A+\alpha) + D\alpha - (c(A - D + 1 - c) + D)\frac{1+2(b/a)\alpha}{1+2(b/a)c}\right)$$

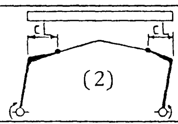
$$\alpha = (1 - A + D)/2 \quad \text{for } b/a = 0$$

$$\alpha = \frac{1}{\frac{b}{a}}\left\{\left((1 + \frac{b}{a}c)^2 - \frac{b}{a}\left((1 + 2\frac{b}{a}c)(A - D) + \frac{b}{a}(A + D) - \frac{b}{a}c^2 - 1\right)\right)^{1/2} - (1 + \frac{b}{a}c)\right\} \quad \text{for } b/a > 0$$



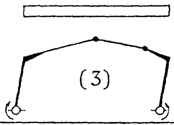
A special case of (1) when $\alpha = c$

$$\frac{M}{wL^2} = \frac{1}{4}(1 - 2c)(A - D)$$



A special case of (1) when $\alpha = \frac{1}{2}$

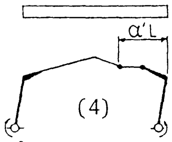
$$\frac{M_p}{wL^2} = \frac{1}{4\left(1 + \frac{1-b/a}{1+2(b/a)c}\right)}\left\{A + \frac{1}{2} + D - 2\,(c(A - D) + 1 - c) + D)\frac{1+b/a}{1+2(b/a)c}\right\}$$



$$\frac{M_p}{wL^2} = \frac{1}{2\left(1 + \frac{1+2(b/a)\alpha}{1+2(b/a)c}\right.}\left(\alpha'(1 - \alpha' + A - D) - c(1 - c + A - D)\frac{1+2(b/a)\alpha'}{1+2(b/a)c} + D\left(1 - \frac{1+2(b/a)\alpha'}{1+2(b/a)c}\right)\right)$$

$$\alpha' = (1 + A - D)/2 \quad \text{for } \frac{b}{a} = 0$$

$$\alpha' = \frac{1}{\frac{b}{a}}\left\{\left((1 + \frac{b}{a}c)^2 + \frac{b}{a}(1 + A - D(1 + 2\frac{b}{a}) + \frac{b}{a}c^2)\right)^{1/2} - (1 + \frac{b}{a}c)\right\} \quad \text{for } \frac{b}{a} > 0$$

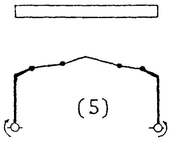


A special case of (1) when $A = D$

$$\frac{M_p}{wL^2} = \frac{1}{2\left(1 + \frac{1+2(b/a)\alpha}{1+2(b/a)c}\right)}\left(\alpha(1 - \alpha) - (c - c^2 + A)\frac{1+2(b/a)\alpha}{1+2(b/a)c} + A\right)$$

$$\alpha = \frac{1}{2} \quad \text{for } \frac{b}{a} = 0$$

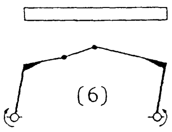
$$\alpha = \frac{1}{\frac{b}{a}}\left\{\left((1 + \frac{b}{a}c)^2 - \frac{b}{a}(2\frac{b}{a}A - \frac{b}{a}c^2 - 1)\right)^{1/2} - (1 + \frac{b}{a}c)\right\} \quad \text{for } \frac{b}{a} > 0$$



$$\frac{M_p}{wL^2} = \frac{1}{2\left(1 + \frac{1+2(b/a)\alpha}{1+b/a}\right)}\left((1 - \alpha)(A + \alpha) + D\alpha - \frac{1}{2}(A + D + \frac{1}{2})\frac{1+2(b/a)\alpha}{1+b/a}\right)$$

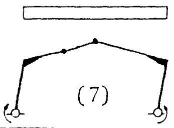
$$\alpha = (1 - A + D)/2 \quad \text{for } \frac{b}{a} = 0$$

$$\alpha = \frac{1}{\frac{b}{a}}\left\{\left((1 + \frac{b}{2a})^2 - \frac{b}{a}(A - D + 2\frac{b}{a}A - \frac{b}{4a} - 1)\right)^{1/2} - (1 + \frac{b}{2a})\right\} \quad \text{for } \frac{b}{a} > 0$$



A special case of (6) when $\alpha = c$

$$\frac{M_p}{wL^2} = \frac{1}{2\left(1 + \frac{1+2(b/a)c}{1+b/a}\right)}\left((1 - c)(A + C) + Dc - \frac{1}{2}(A + D + \frac{1}{2})\frac{1+2(b/a)c}{1+b/a}\right)$$



provide the deflection of various points as function of geometric and loading parameters were developed for frames with members of uniform cross-section (typical sample charts are presented). Means of modifying these charts to include the effect of haunches are being explored. It is proposed that until such modifications are accomplished, the values calculated based on the present charts be used as conservative estimate of haunched frames deflections.

ACKNOWLEDGEMENT

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APPENDIX B

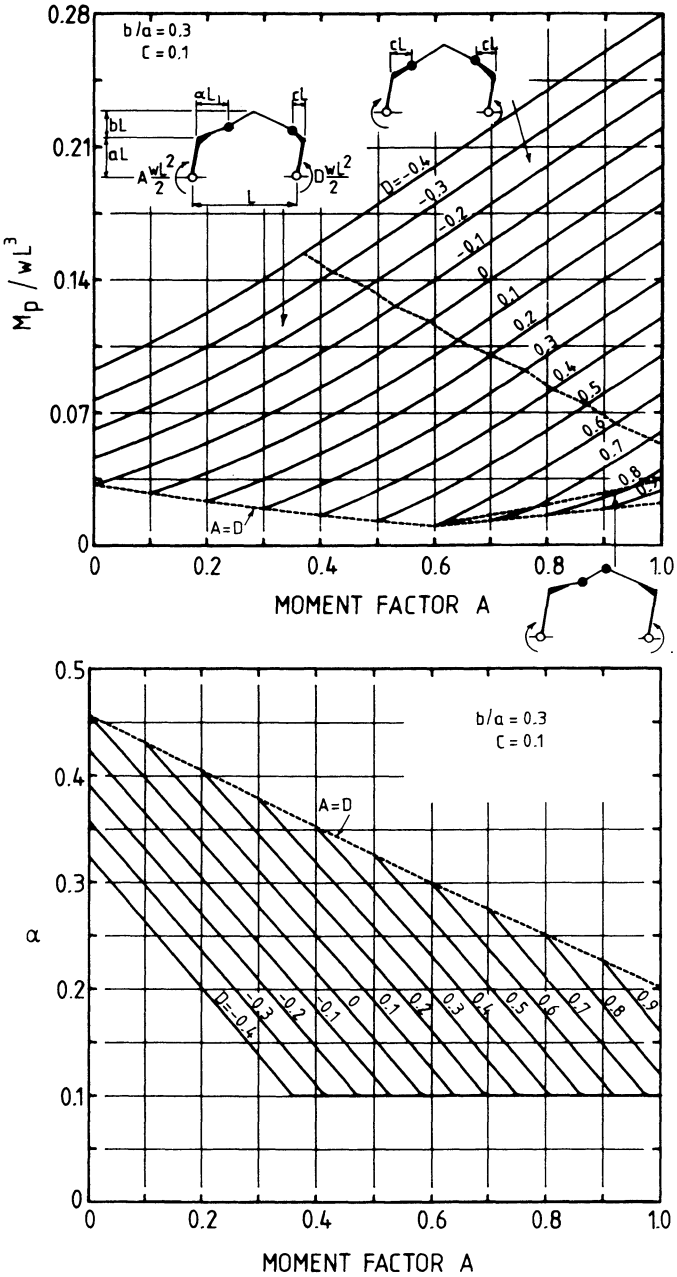

Fig. B1. Design curves
(a) Determination of required plastic moment
(b) Location of plastic hinge

APPENDIX C

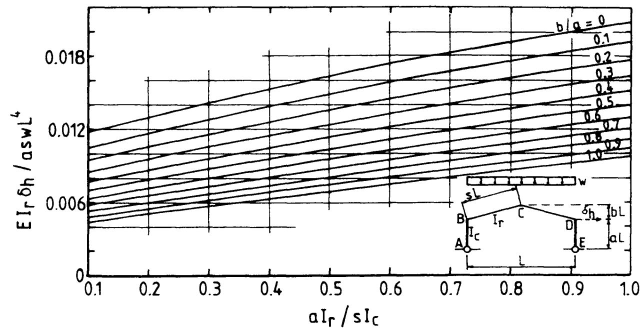

Fig. C1. Horizontal eave deflection due to vertical load w

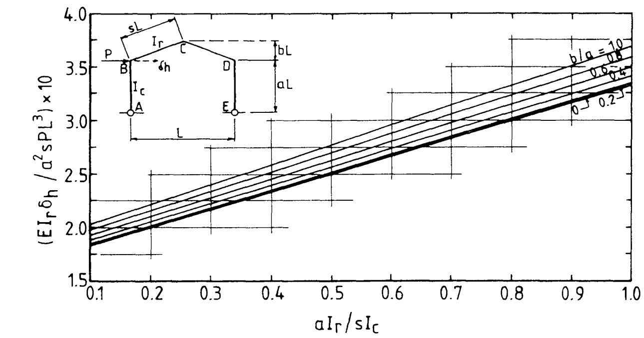

Fig. C2. Horizontal leeward eave deflection due to horizontal windward load P

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