Limit Analysis and Plastic Design of Grid Systems

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Limit analysis and plastic design are techniques developed to represent more realistic behavior of indeterminate steel structures. They take advantage of the fact these structures have a greater load-carrying capacity than indicated by the elastic analysis and the allowable stress design concept.

Furthermore, the theoretical and experimental research of the last 40 years or so has led to the realization that the assumption of perfectly elastic behavior of structures is by far too simple. The actual performance of structures often departs considerably from pure theory of elasticity. In many instances, the calculated local stresses will be exceeded due to residual stresses from rolling, welding and cold-forming. Additional stresses can also be a product of erection and differential settlement of foundation. When highly stressed sections of a continuous structure yield, they merely transfer additional moments to less stressed areas and readjust themselves to carry the load more efficiently. In fact, there is not a valid reason to insist the calculated service stresses in a steel structure should be below yield stress, as long as there is no danger of low cycle-fatigue or brittle failure.

This paper presents the upper bound approach to the analysis of orthogonal grid systems applying the kinematic or mechanism method. The corresponding virtual work equations are formally written and solved for the value of the ultimate load. The load obtained in this way is the correct limit load only if the corresponding bending moments nowhere exceed the maximum plastic moments (the plasticity condition).

It is assumed the grid members all lie in the same plane, and that all loads act perpendicularly to this plane. Furthermore, bending moments having axes perpendicular to this plane and axial forces in the plane are assumed to be zero. The beams of a grid system rigidly connected, in general, will transmit both bending and torsional moments. In this study, torsional moments are ignored, assuming the beams have no resistance to torque. Thus, any obtained solutions will be somewhat on the conservative side. J. Heyman[7] has shown however, that for typical I-beams the error has no practical significance since it was found to be less than 0.1% for the grids considered. The further assumption is that grid deformations are small compared with the grid size, so the equilibrium conditions may be satisfied by the undeformed rather than the deformed configuration.

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Restrictions on deflection at working as well as ultimate loads may also control the choice of structural members, and the plastic design method must allow for this. Also, the deformability of the structural system should be checked to verify that it can actually go through the required deformations and plastic hinge rotations up to the formations of the last plastic hinge (the actual collapse of the structure).

The principles of limit analysis may be applied directly to the large majority of low-rise frames, continuous beams and grid systems, without modification, as far as failure load is concerned. It has been used this way in most countries of the world[1,5,8,12] and the AISC Specification Part II has approved plastic design since 1963.[1]

Now, the principle of virtual work will be used to determine the ultimate capacity of a grid system. Moving through the collapse mechanism, the grid segments will rotate along yield lines while maintaining deflection compatibility. The principle of virtual work states that the total work done by a force system in equilibrium going through a virtual rigid body displacement is zero. By means of stated principle, one can equate the total work of exterior forces to that done by interior plastic moments, or

$$W_e = W_i \tag{1}$$

The total work done by the uniform load is equal to the volume which is outlined by the structure moving during virtual displacement multiplied by the uniform load, i.e.

$$W_e = \iiint F(x,y,z)p(x, y)dxdydz \tag{2}$$

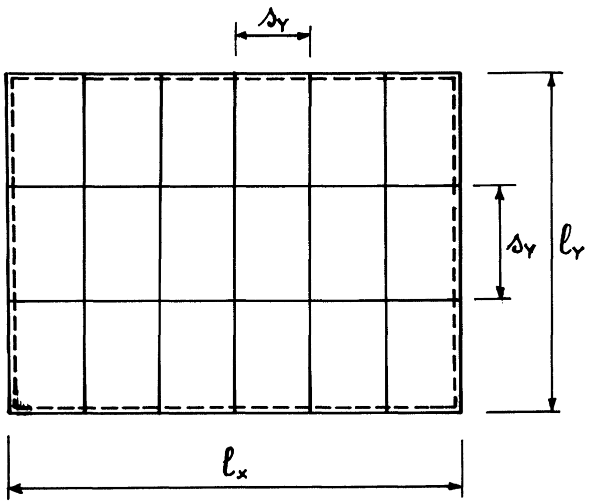


Fig. 1. Rectangular grid system

The work of interior forces on the same mechanism is equal to the sum of the plastic moment at each plastic hinge times the hinge rotation, or

$$w_i = \sum M_{pi} \rho_i \tag{3}$$

The material is assumed to be ideally elastic-perfectly plastic.

GRID ANALYSIS

Assume a rectangular steel grid system with length ℓ_x in x-direction and ℓ_y in y-direction, as shown in Fig. 1. The task is to find the grid beam sizes in order to support a load p uniformly distributed over the grid area.

The elementary beam spacings will be denoted by s_x and s_y in x- and y-direction respectively, so that:

$$(n_x + 1)s_x = \ell_x \tag{4}$$

and

$$(n_y + 1)s_y = \ell_y \tag{5}$$

For the sake of simplicity, the load will be taken as divided into equal concentrated loads of value P acting at the joints of the grid system. The replaced concentrated load is equal to:

$$P = s_x s_y p \tag{6}$$

Suppose the grid is formed of I-beams of equal plastic moments M_p placed in both directions. It is obvious that by increasing the loading on the grid, the first plastic hinges theoretically will be formed at the mid-span of the centrally placed beam where the bending moment is the largest. In practice, two hinges will not be formed in the same beam, one on either side of the joint, but only one would form on one side of the joint. The work given by Eq. 1 however remains unaffected. With further increasing of the load a chain of the plastic hinges will spread towards mid-points of the grid support sides. Because of the absence of any significant torsional rigidity of the system, the grid collapse pattern differs from that diagonally developed, usually found in the slab-type collapse.

The grid members between the plastic hinges or the plastic hinge and the support will remain straight. The dark dots represent the positions of each plastic hinge in Fig. 2. Therefore, the grid system undergoing virtual displacement will outline a hyperbolic paraboloid deflected surface, as in Fig. 3.

It is easy to prove that centrally placed collapse pattern yields the lower bound solution to the diagonal one, which usually appears in a slab-type collapse. Using the solution for a diagonal collapse pattern[4] and the one given in Table 1, it can be shown that a square 5 by 5 grid yields a 25% lower load and a 10 by 10 grid yields 36.4% less. With increasing number of beams, the difference will increase to the maximum limit of 50%.

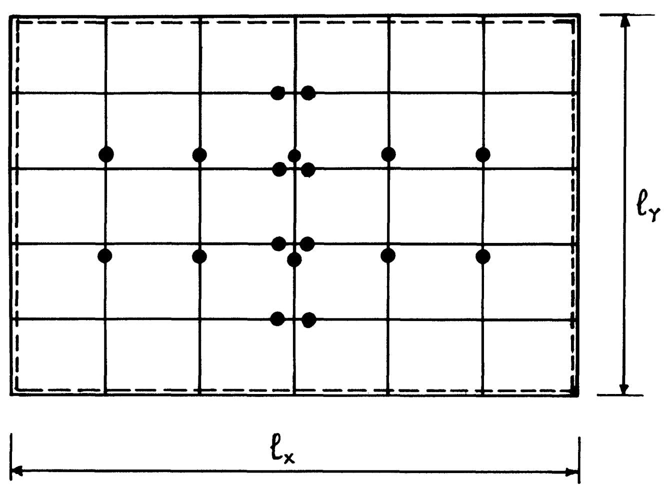


Fig. 2. Hinge formation pattern

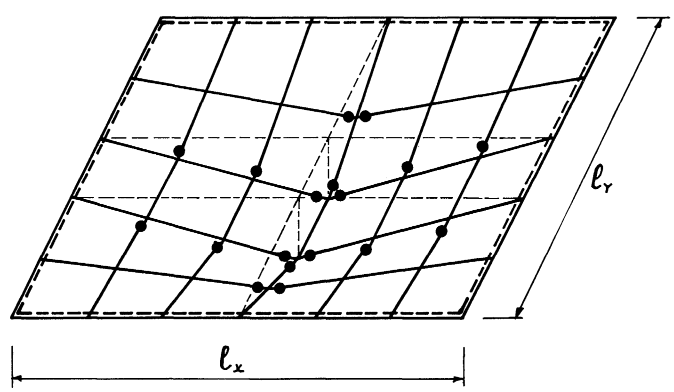
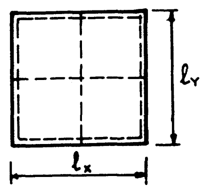
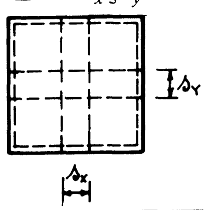
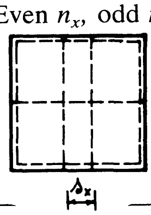
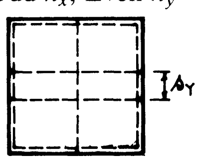


Fig. 3. Hyperbolic paraboloid collapse shape

Table 1. Work Equations for Different Beam Layout*

CASE	GRID LAYOUT	WORK DONE BY EXTERIOR LOAD W_e	WORK DONE BY INTERIOR MOMENTS $W_{i,x}$	$W_{i,y}$
1	Odd $n_x \xi n_y$ 	$\frac{1}{4}\,\ell_x \ell_y p_u$	$2\frac{M_{px}}{\ell_x}(n_y+1)$	$2\frac{M_{py}}{\ell_y}(n_x+1)$
2	Square Grid $\ell_x=\ell_y=\ell$ $n_x=n_y=n$	$\frac{1}{4}(\ell+s)^2 p_u$	$2(M_{px}+M_{py})\frac{n+2}{\ell-s}$	
3	Even $n_x \xi n_y$ 	$[\frac{1}{4}(\ell_x-s_x)(\ell_y-s_y)+ (\ell_x-s_x)\frac{s_y}{2}+ (\ell_y-s_y)\frac{s_x}{2}+ s_x s_y]p_u$	$\frac{4M_{px}}{\ell_x-s_x}\left(\frac{n_x}{2}+1\right)$	$\frac{4M_{py}}{\ell_y-s_y}\left(\frac{n_y}{2}+1\right.$
4	Even n_x, odd n_y 	$\left[\frac{1}{4}\ell_y(\ell_x-s_x)+ \frac{1}{2}\ell_y s_x\right]p_u$	$\frac{2M_{px}}{\ell_x-s_x}(n_x+1)$	$\frac{4M_{py}}{\ell_y}\left(\frac{n_y}{2}+1\right)$
5	Odd n_x, Even n_y 	$\left[\frac{1}{4}\ell_x\left(\ell_y-s_y\right)+ \frac{1}{2}\ell_x s_y\right]p_u$	$\frac{4M_{px}}{\ell_x}\left(\frac{n_x}{2}+1\right)$	$\frac{2M_{py}}{\ell_y-s_y}(n_y+1)$

*Assuming a unit for the maximum virtual displacement

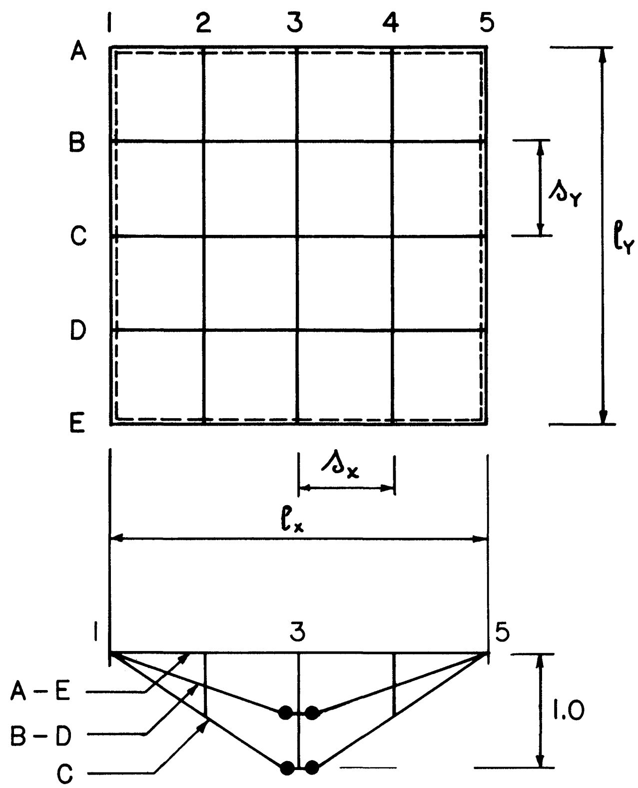


Fig. 4. Three-square grid system

Let us now appraise the collapse load of a simple supported 3 by 3 grid system, shown in Fig. 4. Assume all beams are equally spaced and are the same size. The load p is uniformly distributed over the structure area.

a. Grid Plan

$\ell_x = \ell_y = \ell$
$s_x = s_y = s$
$n_x = n_y = n$

b. Collapse Pattern

The maximum deflected point during the collapse is the center of the grid (3,c). The structure will move through the four doubly symmetrical hyperbolic paraboloid sections during virtual displacement of the work. The exterior load is equal to the described volume times the uniform load, i.e.

$$W_e = \tfrac{1}{4}\ \ell_x \ell_y p_u = \tfrac{1}{4}\ \ell^2 p_u \tag{7}$$

The work of interior plastic moments is equal to

$$W_i = 4\frac{M_p}{\ell}(n+1) \tag{8}$$

Equating the work of exterior forces to the value of the work done by interior moments:

$$\tfrac{1}{4}\ \ell^2 P_u = 4M_p(3+1)/\ell, \text{ or}$$

$$P_u = 64M_p/\ell^3 \tag{9}$$

or expressed by the concentrated load at joint:

$$P_u = 4M_p/\ell \tag{10}$$

This is the same result as found by J. Heyman[7] conducting a step-by-step search for the true collapse grid pattern.

The work done by the exterior load and the interior moments is found for different combinations of odd and even numbers of grid members. The results obtained are compiled in Table 1.

Number of beams in x-direction is:

$$n_x = \frac{\ell_y}{s_y} - 1 \tag{11}$$

similarly in y-direction:

$$n_y = \frac{\ell_x}{s_x} - 1 \tag{12}$$

The use of the table is illustrated in the following numerical examples:

EXAMPLE 1

Given

For a rectangular steel grid 56 ft by 48 ft, shown in Fig. 5. It is required to design beams made of A36 steel to support total working loads of 175 psf. Use a load factor of 1.7, see Refs. 1,5,12.

Solution

Using Table 1, and assuming maximum virtual displacement of one ft, the total work of exterior load is equal to (Case 3):

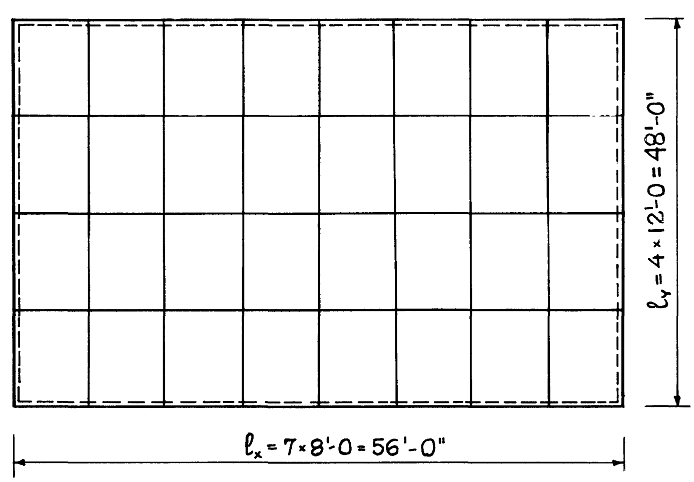


Fig. 5. Grid layout, Ex. 1

$$W_e = [\tfrac{1}{4}\ell_y(\ell_x - s_x) + \tfrac{1}{2}\ell_y s_x]p_u$$
$$= [\tfrac{1}{4}(48)(56-8) + \tfrac{1}{2}(48)(8)](1.7)(0.175)$$
$$= 228.48 \text{ kip-ft.} \tag{13}$$

In a similar way, the work done by the plastic moments is:

$$W_i = 2\frac{M_{px}}{\ell_x - s_x}(n_x+1) + 4\frac{M_{py}}{\ell_y}(\frac{n_y}{2}+1)$$

$$= 2\frac{M_{px}}{56-8}(3+1) + 4\frac{M_{py}}{48}\left(\frac{6}{2}+1\right)$$

$$= \frac{1}{6}M_{px} + \frac{1}{3}M_{py} \tag{14}$$

Typically, the relationship between M_{px} and M_{py} would be known, or could be assumed. Assuming all beams are of the same size $M_{px} = M_{py} = M_p$ and equating the value of Eqs. 13 and 14, Eq. 1 becomes,

$$228.50 = 0.5\ M_p, \text{ or}$$
$$M_p = 457.0 \text{ kip-ft}$$

This would require W24×62 beam (M_p = 459 kip-ft).
If a W24×68 with M_p = 531 kip-ft were used in the y-direction, then Eq. 14 becomes,

$$228.50 = \tfrac{1}{6}M_{px} + \tfrac{1}{3}\,531, \text{ or}$$
$$M_{pu} = 309.0 \text{ kip-ft}$$

for which an adequate beam size would be W21×50 (M_p = 330 kip-ft).

Usually in the optimum design process it is required to produce safe but economical structures minimizing the weight of steel used. The weight of the steel beams in this example using uniform beam sizes W24×62 would be 28,272 lbs., and in the case when using W21×50 in x- and W24×68 in the y-direction, the total weight of beams would be 27,984 lbs. (saving about 1%).

In this kind of analysis, the discrete available number of beam sizes is replaced by a continuous range of sections in which available beam sections represent discrete points within a continuous function. A power curve gives a good approximation of a non-linear relationship between the mass per unit length, w, of a member and its plastic moment, M_p, by

$$w = a \cdot M_p^{\gamma} \tag{15}$$

Where a and γ are constants depending on the geometry of the cross section and material of the beam. For the average value of the plastic moments of hot rolled beams made of A36 steel, the constants are:

$$a = 1.021$$
$$\gamma = 0.68$$

The a and γ constants are based on the most effective ratio of plastic moment/beam weight range of beams from M10×9 to W36×300 (total 39 beams). The calculated coefficient of determination was r^2 = 0.997, indicating the high quality of the curve fit. For example, for a plastic moment of 310 kip-ft, the expected weight of A36 steel beam would be:

$$w = (1.021)(310^{0.68})$$
$$w = 50.50 \text{ lbs/ft}$$

Corresponding to a W21×50 beam with a plastic moment of 330 kip-ft.

It is of interest to note that heavier beams are more economical to use. So, for the ratio of plastic moments 1:2:3 the corresponding beam unit mass ratio is 1:1.5:2.11.

EXAMPLE 2

It is required to find the ultimate carrying capacity of a square grid system of a 17.5-ft span shown in Fig. 6. The grid consists of six 8H2 bar joists orthogonally threaded through six 12H2 bar joists. The plastic moment capacity of a bar joist is approximately given by:

$$M_p = A_c d F_y \tag{16}$$

where

A_c = area of smaller chord member

d = effective joist depth (measured to centroids of chords)

F_y = steel yield stress

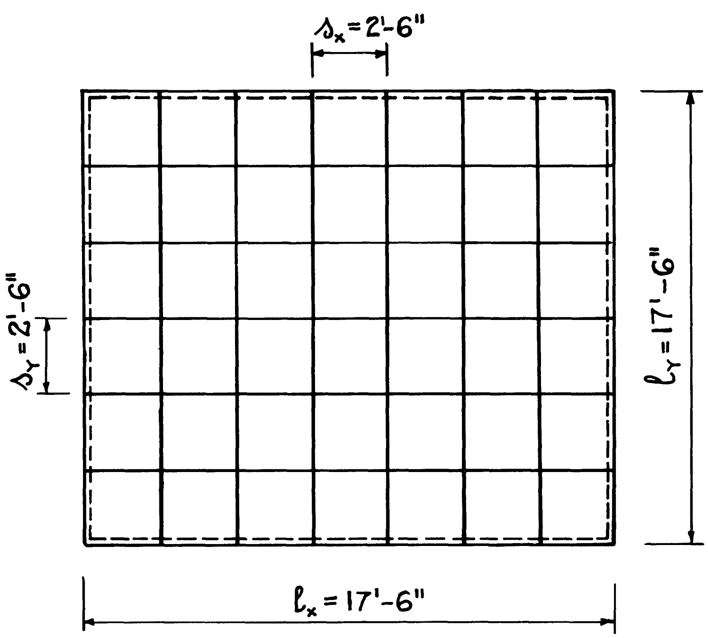


Fig. 6. Square grid 17.5 ft × 17.5 ft.

The plastic moment capacities of bar joists, calculated from Eq. 16 are:

for 8H2: $M_{px} = (0.378 \text{ in}^2)(0.5 \text{ ft})(50 \text{ ksi}) = 9.450 \text{ kip-ft}$

for 12H2: $M_{py} = (0.378 \text{in}^2)(0.85 \text{ft})(50 \text{ksi}) = 16.065 \text{kip-ft}$

Applying formulas from Table 1, the virtual work done by the uniform load is (Case 2a):

$$W_e = \tfrac{1}{4}(\ell + s)^2 w_u$$

$$= \tfrac{1}{4}(17.5 + 2.5)2 w_u = 100 w_u$$

The work done by the plastic moments is (for a maximum virtual displacement of one ft):

$$W_i = 2(M_{px} + M_{py})\frac{n+2}{\ell - s}$$

$$= 2(9.450 + 16.065)\frac{6+2}{17.5 - 2.5} = 27.216 \text{ kip-ft}$$

Equating the two expressions for work and solving for w_u,

$$100 \text{ w}_\text{u} = 27.216, \text{ or}$$
$$w_u = 272 \text{ psf}$$

J. Cannon[4] has carried out tests on the grid system considered in this example.

As indicated by the load-deflection curve in Fig. 7, the test was terminated at the load of 260 psf that is slightly lesser than the calculated ultimate load of 272 psf. However, it would appear from examination of the load-deflection curve of Fig. 7, that the grid could sustain the predicted failure load before reaching collapse.

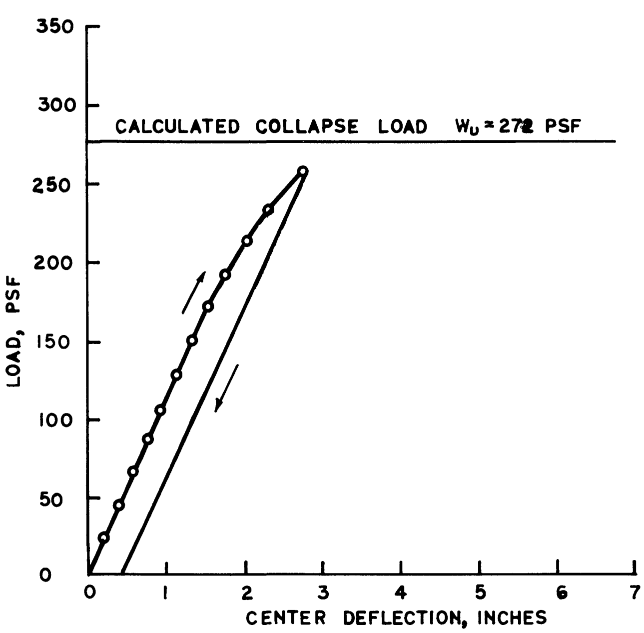


Fig. 7. Load deflection for bar-joist grid

CONCLUSION

Summarizing the results of the investigations presented in this paper it may be said it extends the theoretical work previously developed into a general multiple beam grid system. The full-scale bar joist grid and bar grids previously tested by others[4,7] seem to correspond well with the theoretical calculated collapse load. The work thus confirms the general behavior of this type of structure, in particular, the hyperbolic paraboloid modes of failure.

From the practical standpoint, the design procedure presented can be applied within given guidelines of the AISC Specification, Part II[1] to produce an economical and safe design without modifications.

ACKNOWLEDGEMENTS

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NOMENCLATURE

The following notations are used in this paper, unless otherwise stated:

A_c = area of smaller joist chord member, in^2
a = 1.021, a constant
d = effective joint depth
F_y = steel yield stress
$F(x,y,z)$ = virtual displacement function
ℓ = grid beam span
ℓ_x = grid beam span in *x-direction*
ℓ_y = grid beam span in y-direction
n = number of beams
n_x = number of beams in x-direction
n_y = number of beams in y-direction
M_p = plastic moment
M_{pi} = plastic moment at section i
M_{px} = plastic moment of a beam in x-direction
M_{py} = plastic moment of a beam in y-direction
$p(x,y)$ = load function
$p_u = w_u$ = uniformly distributed ultimate load
P_u = ultimate concentrated load
s = beam spacing
s_x = beam spacing in x-direction
s_y = beam spacing in y-direction
w = unit mass of steel beam
W_e = work done by exterior load
W_i = work done by plastic moments
γ = 0.68, a constant
ρ_i = plastic hinge rotation at section i

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