

Analysis of Framed Space Structures

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THIS PAPER PRESENTS the procedure for the analysis of general framed space structures, and rules for the determination of redundancy, stability, and statical determinacy. It is shown that practically any shape of framework as would be desired to enclose a given space can be arranged and braced so that it is rigid, stable, statically determinate and subject to rational analysis. The analysis consists of the solution by computer of a set of simultaneous equations, equal in number to the number of framing members.

INTRODUCTION

Framed structures that are statically determinate are ordinarily employed in arrangements which may be analyzed by reasonably simple and straightforward methods. It does not follow, however, that the analysis of all statically determinate frameworks is simple, and this fact has perhaps impeded the development of many types of very useful structures.

Consider one familiar example, that of framed domes over stadiums, arenas, etc.; these are almost invariably made spherical, whereas the useful area below is seldom of that shape. It is unfortunate that the analytical complexity of a non-spherical shape should preclude the use of a more desirable shape, such as an oval, for example.

The principles of analysis for framed structures of all-inclusive shapes, including those with or without symmetry, have been well-established and documented.¹ The analysis of framed structures of other than very simple configurations, however, depends on the solution of a set of simultaneous equations, written as expressions of the force components of the framing members at each intersection throughout the framework. It will be found, for a structure that has been properly framed so that it is statically determinate, that the number of unknown force components in the members will be exactly equal to the number of independent equations that can be written.

1. *Timoshenko, S. P. and Young, D. H. Theory of Structures McGraw-Hill, New York, 1965*

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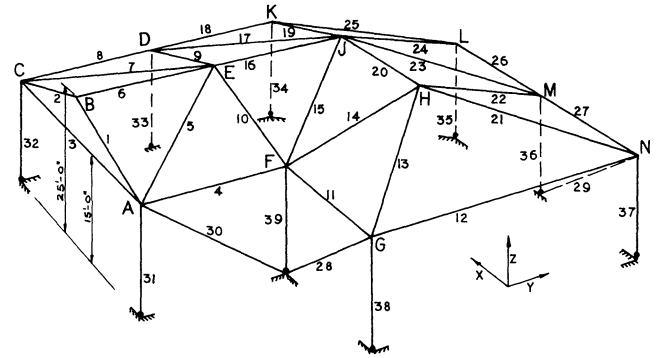
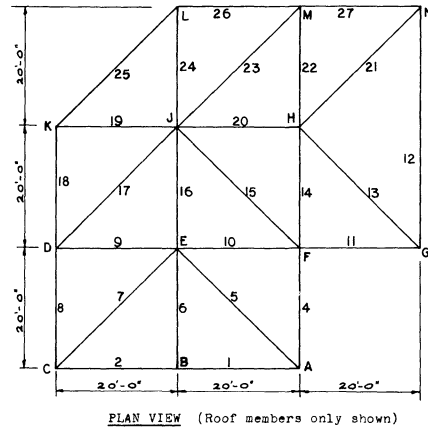


Figure 1

For a framed dome of irregular shape over an arena, the number of members and hence unknown force components will be in the hundreds, resulting in this same number of simultaneous equations; while the writing of the proper equations presents no particular difficulty except that of the possibility of mistakes, the solution of the equations has, until the advent of the computer, been insurmountable.

Even though the solution time by computer is relatively short, a considerable effort is still required for the design and analysis of a structure with hundreds of members. After the framing arrangement has been selected, the framework must be checked carefully for statical determinacy. The equations of force components then must also be carefully written and checked; since the

equations are interdependent, a mistake in just one of them can invalidate the entire solution. Checking for correctness of the solution is relatively straightforward; direct substitution into the equations is sufficient to determine the validity of the computer output. From this brief evaluation, it can be seen that the design and analysis of these structures must be carried out carefully and thoroughly, but in that it is possible with this method to treat structures susceptible of no other practical analytical approach, the effort seems well worthwhile.

The treatment that follows will be concerned with the framework indicated in Fig. 1; it is readily seen that no particular symmetry or regularity has been provided; the enclosing surfaces consist of numerous triangular shaped elements in the roof, with vertical posts at the joints of the periphery. In this example, the length of all the vertical members is the same, but no such requirement is necessary. Likewise, the ridge points need not have been all at the same elevation. None of the triangular plates need have been coplanar with the adjoining plates, and the complexity of analysis would not have been increased; in addition, the stability of the structure would have not been impaired. The only constraints imposed on the framing are that of complete internal stability of each of the plane surfaces of the roof, and that of vertical supports at the periphery of the series of adjoining plates, with proper diagonal bracing provided for the vertical members.

STATICAL DETERMINACY

The rules for statical determinacy of framework of the type illustrated in Fig. 1 are well established and are remarkably simple. The necessary and sufficient condition of determinacy, and of stability in ordinary frameworks is:

$$m = 3j$$

where m is the total number of members above the base plane, and j is the number of joints, or structural connections, above the base plane. In structures of the type illustrated, composed of numerous contiguous triangular roof elements supported at the periphery by vertical posts, it will generally be found that three and only three diagonal braces such as **28**, **29**, and **30** will be required. (These members are, of course, included in the number m .) Any fewer bracing members will result in an instability, and any more than three will result in a redundancy, immediately complicating the situation. The number of joints at the base plane is immaterial, as long as the other conditions are met; e.g., the vertical planes of the three braces must not intersect in a single line. Thus, many other configurations of bracing could have been used for the structure in Fig. 1.

The arrangement of the framing members must also follow the conventional rules for simple framed struc-

tures, especially that each flat roof panel be simply braced in a triangular pattern. It would not do, for example, to delete member **22** and to add a member **KE**. The equation $m = 3j$ would of course still be satisfied, but an instability would result in panel **HJMN**, and a redundancy would result in panel **DEJK**. The treatment at hand will be confined to the conventional framing systems described above.

STABILITY

Many different framing arrangements are possible for a given structure, especially if some of the contiguous triangular panels are coplanar; e.g. in Fig. 1 member **17** could have been deleted and replaced by a member **KE** without affecting the stability. However, member **17** could not have been deleted and replaced with a member from point **D** to the bottom of member **34**. An instability would have been created.

It is often quite difficult to visualize whether a particular framing system is or is not stable. Model studies are very helpful in the preliminary stages, but are not conclusive. The most positive method of determination of instability is in the actual solution of the system; assuming that the previously stated rules are followed, and any load pattern applied to the structure results in a unique solution for stresses in the members, the system is stable; there is no other possibility.

The question of stability, however, arises generally only in other than the conventional arrangements indicated in this presentation. In cases where the roof structure is continuously subdivided into triangular panels at a reasonable angle with the horizontal, with or without hips or valleys at the junctures of the triangular panels, and with vertical supports at each of the joints on the periphery of the series of triangular panels, no problems of stability should be encountered.

LAYOUT OF FRAMING

Consider the framing arrangement shown in Fig. 1, a fairly simple example which is not susceptible of analysis by other more conventional methods. Note that no internal framing whatever is provided; all the structural framing is contained in the enclosing surfaces. The layout of the enclosing surface is determined by other considerations, and it is seen that the roof consists of various shapes of plane surfaces, bounded by framing members. As discussed before, these plane panels may be of any shape, as long as they are stable internally and are bounded by framing members; e.g. the rhombic panel **FGHJ** is contiguous with the triangular panel **GHN** and the two trapezoidal panels **HJLN** and **FABJ**. The roof, then, is composed of stable, plane panels, connected at the edges. At the periphery of the assembly of roof panels, vertical supports are provided at each panel point. Without bracing for the vertical supports, the assembly

thus far described is of course not stable against horizontal forces; the bracing arrangement is determined by the following reasoning, based on considerations both of stability and statical determinacy:

To determine the number of bracing members that must be provided, the rules for statical determinacy, or $m = 3j$, are followed. First are counted m , the number of roof members plus vertical supports, and this number is compared with 3 times j , the number of joints. Now the number of roof members, 27, plus the number of vertical supports, 9, equals 36, or three less than 3 times 13, the number of joints. This difference of three will be the ordinary situation for structures of this type, and represents the number of bracing members that must be judiciously located.

Table Ia. Vertical Loads	Table Ib. Horizontal Loads
$Z(A) = 3.0$	$X(A) = 0.8$
$Z(B) = 6.0$	$X(B) = 0.6$
$Z(C) = 3.0$	$X(E) = 1.0$
$Z(D) = 6.0$	$X(F) = 1.2$
$Z(E) = 12.0$	$X(H) = 1.0$
$Z(F) = 9.0$	$X(G) = 2.0$
$Z(G) = 3.0$	$X(N) = 2.0$
$Z(H) = 12.0$	
$Z(J) = 12.0$	
$Z(K) = 5.0$	
$Z(L) = 5.0$	
$Z(M) = 6.0$	
$Z(N) = 3.0$	

To determine the vertical bracing locations, first consider the possible action of the structure that would result from horizontal loads in the absence of any vertical bracing whatever. Of course horizontal motion of all the joints would result, but note that the configuration of roof members would retain its shape; that is, there would be no distortion of the relative positions of any roof member to any other; the only distortion would be between the vertical members and the roof members. Now consider what would happen under horizontal loads if any one of the joints, such as **F**, is held rigidly in space, which can be done very easily by two of the bracing members, such as **28** and **30**. The entire roof structure would pivot around the braced point, but again without distortion of the roof system. It can easily be visualized that the third bracing member can then be placed to prevent the rotation merely by bracing any other joint in such a way that vertical member **39** does not fall in the vertical plane of the third brace. The bracing member **29** could have been located to brace the top of any of the other vertical members.

It can be seen, then, that the addition of the three

bracing members is both necessary and sufficient to stabilize the structure, and that the relationship $m = 3j$ is exactly satisfied. Further it is evident that the addition of any more members would create a redundancy, and the omission of any roof member, vertical member, or bracing member would create an immediate instability.

With the completion of the general statical determinacy and stability requirements, the numerical analysis may now be carried out. It may be somewhat difficult to determine that the structure is absolutely stable at this point; some structures may have a critical shape that is not evident. But if the structure is subjected to a general loading condition, and a definite solution is found for the member stresses, the structure is both stable and valid.

ANALYSIS

First consider any particular set of loads applied to the structure at the joints. In actual practice, the set may be a combination of vertical and horizontal loads; for the example at hand, one set of vertical loads and one set of horizontal loads will be considered, as indicated in Tables Ia and Ib.

The analytical reasoning behind the basic equation $m = 3j$ will now become more apparent. Note that at each of the joints there can be written three equations of loads and forces, one for each of the X , Y , and Z directions; see Fig. 1 for the directions indicated. Also, since the axial force in each member represents one unknown quantity to be determined, the number of unknowns equals the number of framing members. It follows that the number of unknowns exactly equals the number of equations available, which is just the proper condition for a solution.

To proceed with the analysis for the member forces resulting from the application of the loads in Table Ia: Three equations of loads and forces are written at each joint. In the example, member forces are taken as positive when the member is in compression; then if a negative sign appears in the solution, tension is indicated. The unknown compression forces in each member are designated by the term F_n where n is the member under consideration. The decimal coefficients for each unknown depend on the slope of the member and will become readily apparent with very little study of the configuration. The expressions $Z(i)$, $X(i)$, and $Y(i)$ represent the Z , X , and Y loads at the joint.

At Joint **A**, the following three equations can be written for the vertical load condition:

$$\text{For } Z \text{ forces and loads: } F_{31} + 0.6F_{30} - 0.333F_5 - 0.447F_1 = Z(A) = 3.0$$

$$\text{For } Y \text{ forces and loads: } -F_4 - 0.8F_{30} - 0.667F_5 = Y(A) = 0$$

$$\text{For } X \text{ forces and loads: } F_3 + 0.894F_1 + 0.667F_5 = X(A) = 0$$

At Joint **B**:

$$\text{For } Z \text{ forces and loads: } 0.447F_1 + 0.447F_2 \\ = Z(B) = 6.0$$

$$\text{For } Y \text{ forces and loads: } -F_6 = Y(B) = 0$$

$$\text{For } X \text{ forces and loads: } 0.894F_2 - 0.894F_1 \\ = X(B) = 0$$

At Joint **C**:

$$\text{For } Z \text{ forces and loads: } F_{32} - 0.447F_2 - 0.333F_7 \\ = Z(C) = 3.0$$

$$\text{For } Y \text{ forces and loads: } -F_8 - 0.667F_7 = Y(C) = 0$$

$$\text{For } X \text{ forces and loads: } -F_3 - 0.894F_2 - 0.667F_7 \\ = X(C) = 0$$

At Joint **D**:

$$\text{For } Z \text{ forces and loads: } F_{33} - 0.447F_9 - 0.333F_{17} \\ = Z(D) = 6.0$$

$$\text{For } Y \text{ forces and loads: } F_8 - F_{18} - 0.667F_{17} \\ = Y(D) = 0$$

$$\text{For } X \text{ forces and loads: } -0.894F_9 - 0.667F_{17} \\ = X(D) = 0$$

and so on for each of the 13 joints. The 39 equations resulting are then the necessary and sufficient number for a solution.

The solution of 39 simultaneous equations each with three to six or so terms may not be impossible to someone trained in this field, but it is certainly impractical for the average engineering office. The employment of a digital computer at this stage of the operation appears to be the only practicable procedure. Inasmuch as the computer equipment available will largely determine the detailed procedure for proper arrangement of the coefficients in the equations and their presentation to the machine, very little discussion will be devoted to this step in the procedure, even though it is most vital. The ordinary procedure for a typical problem will be, after the equations are written, to establish contact with a computer center and to inquire as to the method of procedure. It may be more practicable to turn the entire problem over to the computer center, or it may be more practicable to learn the necessary programming technique; it will be found that the latter course of action will result in speedier solutions and more accurate writing of the equations, which is an absolute essential. For the problem illustrated, 128 items of information are arranged in a particular order and fed to the machine, as there are 128 non-zero terms in the equations. The results from the computer are as shown in Table IIa, and are substituted in the original equations for verification of the validity of the computer solution.

Upon receipt of a unique solution to the original equations, the validity of the structure as being stable and statically determinate is assured. If both these conditions are not met, no solution would be obtained from the

equations; if the structure is not stable, the determinant would be zero, resulting in no solution, and if the structure is not statically determinate, a redundancy would prevent a solution from being obtained.

Table IIa. Member Forces from Table Ia Loads	Table IIb Member Forces from Table Ib Loads
$F_1 = 6.71$	$F_1 = -0.34$
$F_2 = 6.71$	$F_2 = 0.34$
$F_3 = -20.02$	$F_3 = -0.75$
$F_4 = -14.02$	$F_4 = -1.85$
$F_5 = 21.01$	$F_5 = 2.78$
$F_6 = 0.00$	$F_6 = 0.00$
$F_7 = 21.01$	$F_7 = 0.67$
$F_8 = -14.02$	$F_8 = -0.45$
$F_9 = -2.23$	$F_9 = -0.84$
$F_{10} = -2.23$	$F_{10} = -2.48$
$F_{11} = -14.02$	$F_{11} = -4.91$
$F_{12} = -14.02$	$F_{12} = -0.36$
$F_{13} = 21.01$	$F_{13} = 0.54$
$F_{14} = -2.23$	$F_{14} = -0.41$
$F_{15} = -18.02$	$F_{15} = -2.23$
$F_{16} = 28.00$	$F_{16} = 2.30$
$F_{17} = 3.00$	$F_{17} = 0.11$
$F_{18} = -16.01$	$F_{18} = -0.52$
$F_{19} = 17.91$	$F_{19} = 0.59$
$F_{20} = 28.03$	$F_{20} = 1.73$
$F_{21} = 21.01$	$F_{21} = 0.54$
$F_{22} = -2.23$	$F_{22} = -0.40$
$F_{23} = 3.00$	$F_{23} = 0.54$
$F_{24} = 17.91$	$F_{24} = 0.59$
$F_{25} = -22.64$	$F_{25} = -0.74$
$F_{26} = -16.01$	$F_{26} = -0.52$
$F_{27} = -14.02$	$F_{27} = -0.16$
$F_{28} = 0.00$	$F_{28} = 8.18$
$F_{29} = 0.00$	$F_{29} = 2.25$
$F_{30} = 0.00$	$F_{30} = 0.0$
$F_{31} = 13.00$	$F_{31} = 0.78$
$F_{32} = 13.00$	$F_{32} = 0.37$
$F_{33} = 6.00$	$F_{33} = -0.00$
$F_{34} = 13.00$	$F_{34} = 0.26$
$F_{35} = 13.00$	$F_{35} = 0.26$
$F_{36} = 6.00$	$F_{36} = -0.00$
$F_{37} = 10.00$	$F_{37} = -1.17$
$F_{38} = 10.00$	$F_{38} = 4.73$
$F_{39} = 1.00$	$F_{39} = -1.99$

The above procedure consisting of the writing of 39 equations is then repeated for any other set of loads to which the structure might be subjected; for example, a set of horizontal loads as indicated in Table Ib. This substitution is quite simple, in that only the right hand sides of the equations are changed, by deleting the $Z(i)$ terms, and by insertion of the $X(i)$ load terms in the appropriate equations of X forces and loads. The computer solution for the number forces resulting is as indicated in Table IIb, and is checked by substitution as before.

The results obtained and shown in Tables IIa and IIb constitute the solution desired. The members are then designed for any bending or other local conditions to which they may be subjected.

CONCLUSIONS

It has been shown that structures which may have a very complicated appearance can, if properly framed, be made stable, rigid, and statically determinate. Provided that a one-story building with vertical walls has a roof with a reasonable amount of rise, and all the panels of the roof can be made flat, stable, and connected at each

panel framing point to other contiguous panels, with vertical supports at the panel points on the periphery of the assemblage of interconnected roof panels, the roof may be of practically any shape. By the application of the well-established rules for statical determinacy, a preliminary assessment of stability is also accomplished; solution of the set of simultaneous equations describing the load and force relationship at each joint throughout the structure verifies the stability and the statical determinacy of the structure. Solution is by means of digital computer, and is limited only by the capacity of the equipment.