LRFD Analysis and Design of Beams with Partially Restrained Connections

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It has been well established that connections considered to be simple, non-restraining connections have some predictable amount of moment capacity. Goverdhan[1] has collected many of the moment-rotation (M-θ) curves for these so-called simple connections and derived expressions for prediction of the M-θ characteristics. Through LRFD,[2] a mechanism for designing structures using the predictable amount of connection restraint is now available. The purpose of this paper is to show how, using the M-θ curves for single-plate framing connections[3] and LRFD design techniques, a system of beams can be designed utilizing, without any change in the simple connection, its natural end restraint to reduce the size and deflection of beams with this type of connection.

One actual design application for such a design procedure is in the design of roof purlins where purlins cannot be continuous, but must frame into or between supporting members. Heretofore, in these cases simple-span purlins or expensive moment connections were used on the ends of the purlins to develop continuity to reduce deflections and beam sizes. This paper demonstrates that using the natural restraint of the so-called simple connection can reduce the size and deflection of a simply supported purlin. Although only the design of roof purlins with single plate shear connections is illustrated in this paper, the technique is general and can be applied to any set of beams using partially restrained (PR) connections with known moment-rotation characteristics.

BACKGROUND

To review how connection restraint affects performance of a beam, a single-span member supporting a uniform load is examined. Figure 1 shows the moment diagram for the case of the pure simple connection, with a maximum bending moment of $WL^2/8$ and maximum deflection of $5WL^4/(384EI)$. This same beam with a fully restrained (FR) connection is shown in Fig. 2, with a maximum moment of $WL^2/12$ and maximum deflection of $WL^4/(384EI)$. Thus, by fixing the ends of the beam, a reduction of 33% in the bending moment and 80% in deflection is obtained. It is immediately obvious from this simple example that end fixity is desirable. Since most connections possess some amount of restraint, why not use this restraint to reduce the size and deflection of the member? Under Sect. A2 of AISC's new LRFD Specification, PR connections can be used directly in design whereas under Sect. 1.2 of the 1978 AISC allowable stress design specification[4] the subject is not addressed sufficiently for design.

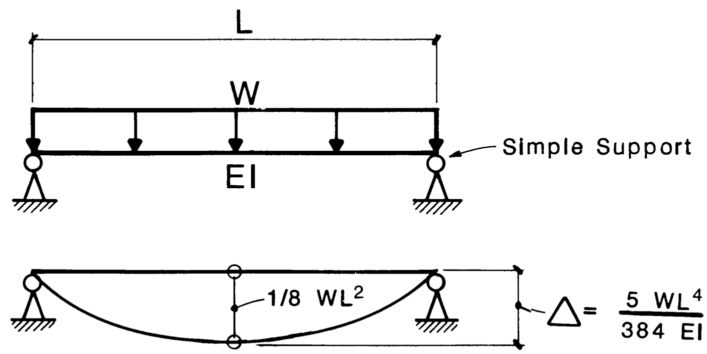

Figure 1

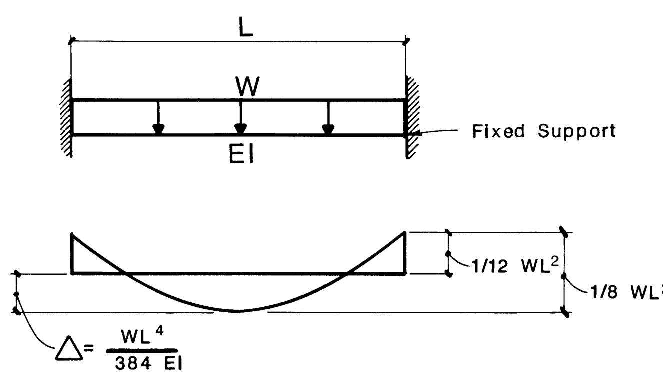

Figure 2

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The same beam, modeled with a PR connection is shown in Fig. 3. Notice the maximum moment may be either at the ends, if the connection has sufficient restraint, or at the middle with more flexible connections. The moment is some factor A times the simple moment, where A is 1 for a pure pin and ranges to 0.67 for a pure fixed condition. The deflection is still maximum at midspan and the factor B, shown in Fig. 3, is 5 for a pure pin and 1 for the pure fixed condition. Thus, the design problem for PR connections becomes one of establishing the amount of end restraint furnished by the connection at various levels of load and its effect on the maximum moment and deflection. A model for analyzing beams with PR connections is shown in Fig. 4. The PR connection is modeled as a rotational spring, characteristics of which are set by the actual moment-rotation curve of the connection assembly. It is important that, when considering moment-rotation curves for various connections, to also consider the entire assembly of members and parts comprising the connection. Extrapolation of data on PR connections to fit assemblies other than those used to establish the M-θ curves can lead to erroneous results. The following section describes the analysis technique for beams with PR connections and the use of the moment-rotation curves in the analysis.

ANALYSIS

A typical moment-rotation curve is shown in Fig. 5. The degree of non-linearity of the connection is dependent on the connection type and, to some extent, is dependent on certain parameters within the connection type. For instance, the M-θ relationship for flexible connections deviates from the straight line, which defines the initial stiffness almost immediately, while for stiff connections a linear approximation can be used for a greater range of moment values. The distribution of moments to the beam and supporting member is determined through this relationship. To arrive at consistent and predictable end moments, the moment-rotation curve for a particular connection must be known within a reasonable degree of accuracy. From the limited test data available, it appears connection behavior is fairly consistent, although dependent on a large number of parameters. In this paper, analysis of a system with semi-rigid connections by the stiffness method requires the knowledge of the connection stiffness at any moment level M. If the non-linear connection curve is available, the secant or tangent to the curve may be used as the spring stiffness, depending on the method of analysis (see Fig. 5).

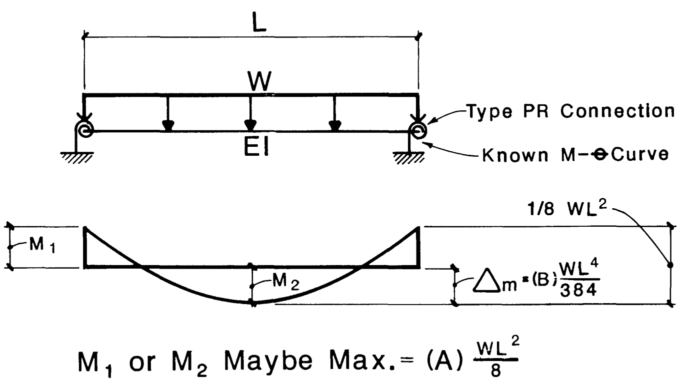

Figure 3

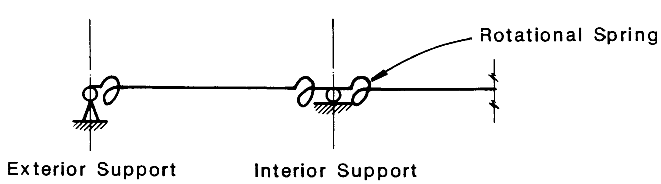

Figure 4

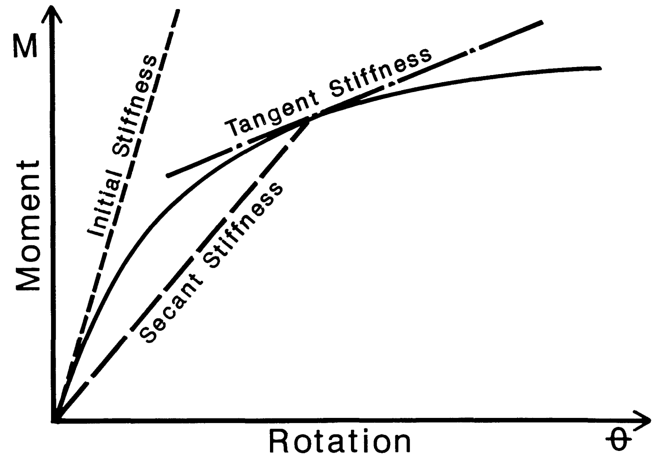

Figure 5

While the tangent stiffness method should be used following an incremental solution procedure for stability/buckling problems, the authors prefer to use the secant stiffness method to determine distribution of forces to components of the structural system. The tangent stiffness method requires knowledge of the loading history to determine tangential stiffness at each load level. The load increment has to be sufficiently small to minimize errors. The secant stiffness method cannot control errors at a local level, but captures the global behavior,[5] which the authors contend is an averaged value since it is impossible to know the exact loading history of the structure. In addition to knowing the secant stiffness at any level of moment, it is essential to know the extent of plastic rotation the connection can endure to guard against an abrupt failure.

The non-linear nature of connection behavior requires an iterative solution procedure. This can easily be undertaken on a computer using the stiffness method of analysis. The rotational stiffness of the connection is represented by a two degree-of-freedom spring element connecting the end of the beam to the supporting member. Compatibility of horizontal and vertical displacements at the joint is maintained, while relative rotation between beam and supporting member is allowed. The solution procedure begins with the assumption of a starting value for the secant stiffness of the connections.

A good initial guess assures fast convergence. Assuming the beams remain elastic, the starting value for the secant stiffness of each connection can be found by using a beam-line theory. The beam line is superimposed on the connection curve plot from 0, M_f to θ_o, 0, where M_f is the end moment of the beam, assuming fixed ends, and θ_o is the rotation of the beam end assuming a simple connection (Fig. 6). Considering various loading patterns, it can be shown that $\theta_o = 0.021$ rad. represents a fairly typical value for most beam/purlins. In any event, the intent is to arrive at a good starting value without a lot of effort. A close starting point rather than an exact guess is all that is required.

Starting the solution procedure with these initial values results in convergence in 3 or 4 iterations, while convergence was achieved only after 8 to 20 iterations when starting with the initial stiffness values. Once the iteration process is underway, a further improvement is possible by using weighing factors to determine connection stiffnesses based on stiffness values employed in previous iterations. The iterative procedure is continued until a desired tolerance level is reached.

The nonlinear behavior of the system requires separate analyses for each design load combination, because the principle of superposition is no longer valid. Connection stiffnesses are dependent on the load level and, therefore, the load-factor method of design is the best method to insure the required factor of safety. Similarly, deflection calculations require separate analyses because of the inelastic behavior of the system. The live-load deflection is found by computing the dead-load-plus-live-load deflection with unfactored loads and subtracting from that the value of dead-load deflection obtained from a separate analysis again using unfactored loads. This has been demonstrated graphically in Fig. 7.

EXAMPLE DESIGNS

With the above analysis technique available, the actual design of such a system becomes very familiar to the designer. An immediate and beneficial practical application, and one the writers use in their consulting practice, is the design of a roof purlin system. Many times space limitations do not permit purlins to run over the top of a girder, making the purlins at the same level as the top of the girders. Therefore, continuity without special, expensive moment connections is not possible. By using the inherent stiffness of a single-plate framing connection, however, the system of purlins can benefit from the available degree of fixity at the ends, thus reducing the size and deflection of the purlins. The family of M-θ curves for single-plate framing connections developed by Richard, et al.,[3] is used by the writers to analyze and design roof purlin systems. The single-plate connections are, in most cases, the connections required for the shear connections of these purlins. Therefore, using their natural restraint to reduce members and deflections is a very logical thing to do, especially since there is *no* added cost for use of this type of connection. The following design example illustrates this design procedure and shows some comparisons in steel sizes and deflections between a semi-rigid design approach and a simple-beam design approach for a system of roof purlins.

Example 1

Given:

Design a simple-span roof purlin system to carry a dead load of 18 psf, a live load of 20 psf and a snow load of 10 psf. The purlins are 8 ft o.c. and span 33 ft (5 equal spans). Assume a sloped roof with no ponding.

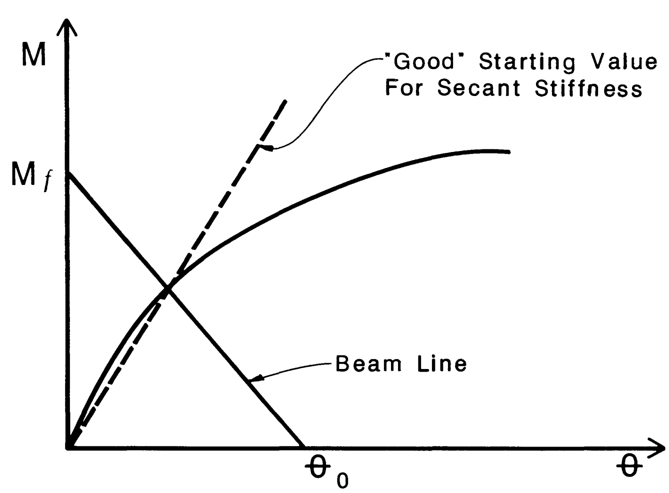

Figure 6

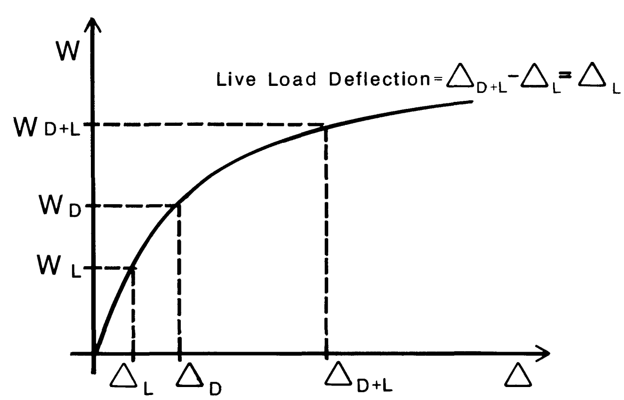

Figure 7

Solution:

Load combinations to be considered (ANSI A58.1):

Loading A: $1.4D$

Loading B: $1.2D + 1.6\,(L_r \text{ or } S \text{ or } R)$

where

D = dead load

L_r = roof live load

S = snow load

R = load due to rainwater or ice (exclusive of ponding)

Design load:

Loading A: $1.4\,(8 \times 0.018) = 0.202$ kips/ft

Loading B: $1.2\,(8 \times 0.018) + 1.6\,(8 \times 0.020) = 0.429$ kips/ft since $L_r > S$ or R

Loading B governs (See Fig. 8)

Design moment:

$$M_u = \tfrac{1}{8}\,(0.429)(33)^2 = 58.4 \text{ kip-ft}$$

Try W12 × 19 (A36 steel):

$M_p = 74$ kip-ft

Top flange is braced by deck; λ_p's and L_p **o.k.**

$\phi_b M_n = \phi_b M_p = 0.9\,(74) = 66.6 \text{ kip-ft} > 58.4$ **o.k.**

Shear **o.k.**

Deflections:

Actual W (unfactored load) = 8(0.038) = 0.304 kips/ft

$$\Delta_{D+L} = \frac{5WL^4}{384EI}$$

$$= \frac{5(0.304)(33)^4(12)^3}{384\,(29 \times 10^3)(130)} = 2.15 \text{ in.}$$

$$\Delta_L = \frac{L}{D+L}\,(2.15) = \frac{20}{38}\,(2.15) = 1.13 \text{ in.}$$

$$\frac{\text{span length}}{360} = \frac{33 \times 12}{360} = 1.10 \text{ in.} < 1.13 \quad \textbf{o.k.}$$

Use: W12 × 19, A36 steel.

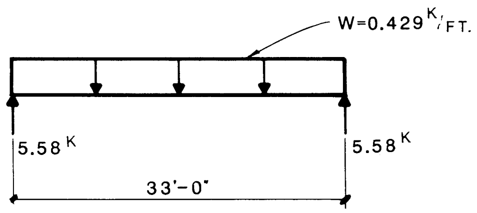

Figure 8

Example 2

Given:

Redesign purlin system for the same spans and loadings as Ex. 1 utilizing type PR connections. Use ¼-in. shear tabs and moment-rotation curves from Richards.[3] See Figs. 9, 10.

Solution:

Load combinations:

Case I:	$1.2D + 1.6\,(L_r \text{ or } S)$	$W = 0.429$ kips/ft
Case II:	$1.0D + 1.0L$	$W = 0.304$ kips/ft
Case III:	$1.0D$	$W = 0.160$ kips/ft

Case I represents the worst design case for gravity loading. Case II minus Case III gives the actual deflection of the member under live load due to the non-linearity of the system. A W12 × 16 is assumed as a trial size for the design. The final moment and deflection for spans 1 and 3 are shown in Table 1.

Span 1:

Design moment (M_u):

From Table 1, max. moment = 54.14 kip-ft

Try W12 × 16 (A36 steel):

$M_p = 60$ kip-ft; λ_r and λ_p **o.k.**

$\phi_b M_n = \phi_b M_p = 0.9\,(60) = 54.0 \text{ kip-ft} \approx 54.14$ **o.k.**

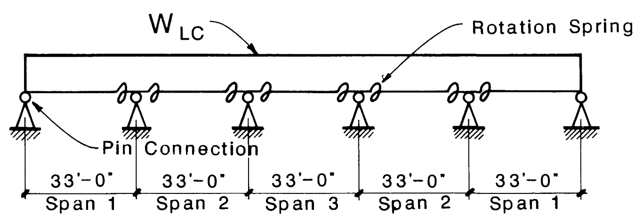

Fig. 9. Model for purlin design

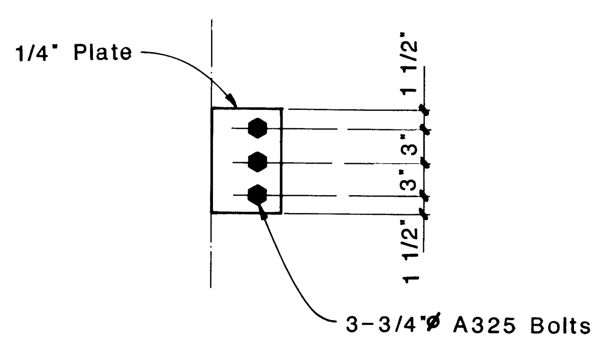

Fig. 10. Typical shear tab

Table 1

	LOADING I SPAN 1			LOADING II SPAN 1			LOADING III SPAN 1		
X	V	M	DEFL	V	M	DEFL	V	M	DEFL
0,00	6,816	0,000	0,00000	4,772	0,000	0,00000	2,441	0,000	0,00000
1,00	6,387	6,601	-0,34340	4,468	4,620	-0,23729	2,281	2,361	-0,11742
2,00	5,958	12,774	-0,68300	4,164	8,936	-0,47192	2,121	4,561	-0,23349
3,00	5,529	18,517	-1,01523	3,860	12,949	-0,70140	1,961	6,602	-0,34692
4,00	5,100	23,832	-1,33677	3,556	16,657	-0,92340	1,801	8,482	-0,45654
5,00	4,671	28,717	-1,64455	3,252	20,061	-1,13578	1,641	10,203	-0,56126
6,00	4,242	33,173	-1,93573	2,948	23,161	-1,33656	1,481	11,763	-0,66009
7,00	3,813	37,201	-2,20774	2,644	25,958	-1,52397	1,321	13,164	-0,75212
8,00	3,384	40,799	-2,45825	2,340	28,450	-1,69637	1,161	14,404	-0,83654
9,00	2,955	43,969	-2,68518	2,036	30,638	-1,85233	1,001	15,485	-0,91264
10,00	2,526	46,709	-2,88669	1,732	32,522	-1,99057	0,841	16,405	-0,97979
11,00	2,097	49,021	-3,06121	1,428	34,103	-2,11002	0,681	17,166	-1,03745
12,00	1,668	50,903	-3,20738	1,124	35,379	-2,20976	0,521	17,766	-1,08520
13,00	1,239	52,356	-3,32413	0,820	36,351	-2,28904	0,361	18,207	-1,12267
14,00	0,810	53,381	-3,41061	0,516	37,019	-2,34730	0,201	18,487	-1,14961
15,00	0,381	53,976	-3,46623	0,212	37,384	-2,38417	0,041	18,608	-1,16587
16,00	-0,048	54,143	-3,49064	-0,092	37,444	-2,39942	-0,119	18,568	-1,17138
17,00	-0,477	53,880	-3,48375	-0,396	37,200	-2,39303	-0,279	18,369	-1,16614
18,00	-0,906	53,188	-3,44572	-0,700	36,652	-2,36513	-0,439	18,009	-1,15029
19,00	-1,335	52,068	-3,37693	-1,004	35,800	-2,31605	-0,599	17,490	-1,12403
20,00	-1,764	50,518	-3,27805	-1,308	34,645	-2,24626	-0,759	16,810	-1,08766
21,00	-2,193	48,540	-3,14996	-1,612	33,185	-2,15645	-0,919	15,971	-1,04157
22,00	-2,622	46,132	-2,99381	-1,916	31,421	-2,04746	-1,079	14,971	-0,98625
23,00	-3,051	43,295	-2,81099	-2,220	29,353	-1,92030	-1,239	13,812	-0,92228
24,00	-3,480	40,030	-2,60315	-2,524	26,982	-1,77618	-1,399	12,492	-0,85032
25,00	-3,909	36,335	-2,37217	-2,828	24,306	-1,61646	-1,559	11,013	-0,77115
26,00	-4,338	32,212	-2,12019	-3,132	21,326	-1,44269	-1,719	9,373	-0,68561
27,00	-4,767	27,659	-1,84959	-3,436	18,042	-1,25661	-1,879	7,574	-0,59466
28,00	-5,196	22,677	-1,56302	-3,740	14,455	-1,06009	-2,039	5,614	-0,49933
29,00	-5,625	17,267	-1,26335	-4,044	10,563	-0,85524	-2,199	3,495	-0,40076
30,00	-6,054	11,427	-0,95371	-4,348	6,367	-0,64428	-2,359	1,215	-0,30018
31,00	-6,483	5,159	-0,63748	-4,652	1,867	-0,42966	-2,519	-1,224	-0,19890
32,00	-6,912	-1,539	-0,31828	-4,956	-2,936	-0,21397	-2,679	-3,823	-0,09834
33,00	-7,341	-8,665	0,00000	-5,260	-8,044	0,00000	-2,839	-6,583	0,00000

	LOADING I SPAN 3			LOADING II SPAN 3			LOADING III SPAN 3		
X	V	M	DEFL	V	M	DEFL	V	M	DEFL
0,00	7,078	-8,612	0,00000	5,016	-7,964	0,00000	2,640	-6,363	0,00000
1,00	6,649	-1,748	-0,29123	4,712	-3,100	-0,18915	2,480	-3,803	-0,07945
2,00	6,220	4,687	-0,58349	4,408	1,460	-0,38010	2,320	-1,403	-0,16110
3,00	5,792	10,693	-0,87306	4,104	5,716	-0,57023	2,160	0,837	-0,24358
4,00	5,362	16,270	-1,15646	3,800	9,668	-0,75706	2,000	2,917	-0,32558
5,00	4,933	21,418	-1,43048	3,496	13,316	-0,93832	1,840	4,837	-0,40589
6,00	4,504	26,137	-1,69212	3,192	16,660	-1,11188	1,680	6,597	-0,48342
7,00	4,076	30,427	-1,93866	2,888	19,700	-1,27582	1,520	8,197	-0,55714
8,00	3,646	34,288	-2,16763	2,584	22,436	-1,42838	1,360	9,637	-0,62613
9,00	3,217	37,720	-2,37677	2,280	24,868	-1,56798	1,200	10,917	-0,68955
10,00	2,788	40,723	-2,56412	1,976	26,996	-1,69320	1,040	12,037	-0,74666
11,00	2,359	43,297	-2,72793	1,672	28,820	-1,80282	0,880	12,997	-0,79682
12,00	1,931	45,442	-2,86672	1,368	30,340	-1,89579	0,720	13,797	-0,83946
13,00	1,502	47,158	-2,97923	1,064	31,556	-1,97121	0,560	14,437	-0,87413
14,00	1,072	48,445	-3,06449	0,760	32,468	-2,02840	0,400	14,917	-0,90046
15,00	0,643	49,303	-3,12174	0,456	33,076	-2,06681	0,240	15,237	-0,91817
16,00	0,214	49,732	-3,15049	0,152	33,380	-2,08611	0,080	15,397	-0,92707
17,00	-0,214	49,732	-3,15049	-0,152	33,380	-2,08611	-0,080	15,397	-0,92707
18,00	-0,643	49,303	-3,12174	-0,456	33,076	-2,06681	-0,240	15,237	-0,91817
19,00	-1,073	48,445	-3,06449	-0,760	32,468	-2,02840	-0,400	14,917	-0,90046
20,00	-1,502	47,158	-2,97923	-1,064	31,556	-1,97121	-0,560	14,437	-0,87413
21,00	-1,931	45,442	-2,86672	-1,368	30,340	-1,89579	-0,720	13,797	-0,83946
22,00	-2,359	43,297	-2,72793	-1,672	28,820	-1,80282	-0,880	12,997	-0,79682
23,00	-2,788	40,723	-2,56412	-1,976	26,996	-1,69320	-1,040	12,037	-0,74666
24,00	-3,217	37,720	-2,37677	-2,280	24,868	-1,56798	-1,200	10,917	-0,68955
25,00	-3,646	34,288	-2,16763	-2,584	22,436	-1,42838	-1,360	9,637	-0,62613
26,00	-4,075	30,427	-1,93866	-2,888	19,700	-1,27582	-1,520	8,197	-0,55714
27,00	-4,505	26,137	-1,69212	-3,192	16,660	-1,11188	-1,680	6,597	-0,48342
28,00	-4,934	21,418	-1,43048	-3,496	13,316	-0,93831	-1,840	4,837	-0,40589
29,00	-5,363	16,270	-1,15646	-3,800	9,668	-0,75706	-2,000	2,917	-0,32558
30,00	-5,792	10,693	-0,87306	-4,104	5,716	-0,57023	-2,160	0,837	-0,24358
31,00	-6,220	4,687	-0,58349	-4,408	1,460	-0,38010	-2,320	-1,403	-0,16110
32,00	-6,649	-1,748	-0,29123	-4,712	-3,100	-0,18915	-2,480	-3,803	-0,07945
33,00	-7,078	-8,612	0,00000	-5,016	-7,964	0,00000	-2,640	-6,363	0,00000

Deflections:

Δ_{D+L} (Loading II) = 2.399 in. (from Table 1)
Δ_D (Loading III) = 1.171 in. (from Table 1)
$\Delta_L = 2.399 - 1.171 = 1.228$ in.

$$\frac{\text{span length}}{360} = \frac{33 \times 12}{360} = 1.10 \text{ in.} < 1.228 \quad \textbf{o.k.}$$

Shear: **o.k.**

Use: W12 × 16 (A36 steel)

Span 3:

Design moment (M_u):
From Table 1, max. moment = 49.73 kip-in.

Try W12 × 16 (A36 steel):
$\phi_b M_n = 54.0$ kip-ft > 49.73 **o.k.** λ_r and λ_p **o.k.**

Deflections:

Δ_{D+L} (Loading II) = 2.086 in. (from Table 1)
Δ_D (Loading III) = 0.927 in. (from Table 1)
$\Delta_L = 2.086 - 0.927 = 1.159$ in.

$$\frac{\text{span length}}{360} = 1.10 \text{ in.} < 1.159 \quad \textbf{o.k.}$$

Use: W12 × 16 (A36 steel)

As can be seen from Ex. 2, the effect of a connection's natural restraint can be used to benefit the designer in reducing member sizes, thus reducing structural system costs. In Span 1, the end restraint of the connection was only used on the interior support so the exterior support did not have to supply any torsional resistance. Although not used in the example, in actual practice, many cases have the ability to furnish restraint at the exterior support, which further helps the overall performance of the system. In the case of Span 1, the purlin was reduced one size and deflections were only slightly increased (2.399 in. vs 2.15 in. and 1.228 in. vs 1.13 in.) over that of a simple-span purlin one size larger (W12 × 19 vs. W12 × 16). In the case of Span 3, where restraint was offered on each end of the purlin, it was also reduced one size. It had excess bending capacity and had about the same deflections as a simple-span purlin one size larger.

As a note, some designers might choose to use a W12 × 19 for the end span purlin size and reduce only the interior spans to W12 × 16. Other designers might even consider the same size purlin as the simple-span size for all spans and consider only the restraint offered by the connection in reducing ponding or roof deflection. In either event, using connection restraints is very beneficial and should be recognized by the designer, whether in size reduction or deflection reduction, or both. The natural restraint is there and should be made to work for the overall economy of the structure.

CONCLUSION

LRFD design equations give the designer the ability to use PR connections to reduce member sizes, which can, in turn, reduce structural costs. Since PR connection performance information is available, the use of their beneficial aspects is logical and sound. Although this paper discussed only a roof purlin system, any beam or girder that has a connection with a known moment-rotation curve can be designed by the techniques of this paper or by other similar techniques. The limit state approach to the PR connection is one that predicts the true performance of the beam or girder and its connection. The design example of this paper is a simple but very, very important one because it shows just how powerful the limit-state approach can be to designers, since inclusion of natural connection restraint reduced the member size. One can imagine, as more experience is gained using PR connections in this and many other conditions, the economies to be gained. The use of the limit state approach opens vast new areas of analysis and design that can give more economical structures.

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