Allowable Bending Stresses for Overhanging Monorails

N. STEPHEN TANNER

In the design of monorails for trolley hoisting devices it is often necessary to extend the monorail out beyond the last available support (as in Fig. 1), so the lifting point is clear of obstructions. Such a monorail supported vertically at two locations can be analyzed as a simple beam with overhang (as in Fig. 2), so the maximum bending moment is easily obtained. However, the question of the allowable bending stress immediately presents itself. The intent of this paper is to address the problem and present a simple design solution.

DESIGN ASSUMPTIONS

LTB Brace Points

For the overhanging monorail of Fig. 1, the bottom flange is unbraced and in compression, so lateral-torsional buckling (LTB) of the section is possible. Such buckling must be prevented by proper bracing at appropriate points, here called "LTB brace points." Although the following discussion reveals the only practical brace point on the monorail is at the interior support (where "interior" is as shown in Fig. 1, generally the more "interior" part of a building), the subsequent mathematical analysis shows such bracing is adequate provided the allowable bending stress is correctly chosen.

To qualify as a brace point, the AISC Specification (Ref. 1, p. 5-22) requires the cross-section be "braced against twist or lateral displacement of the compression flange." At the interior support of Fig. 1, restraint against twist is clearly provided by the supporting beam, so the monorail is adequately braced at this point. However, at the point of maximum moment, the exterior support, the section is neither braced against twist nor against lateral displacement of the compression flange. At this location it is impossible to restrain the bottom flange laterally and still allow passage of the trolley. Likewise, restraint against twist would require relying on the low out-of-plane flexural stiffness of the monorail web; such support would be difficult to model and is not recognized

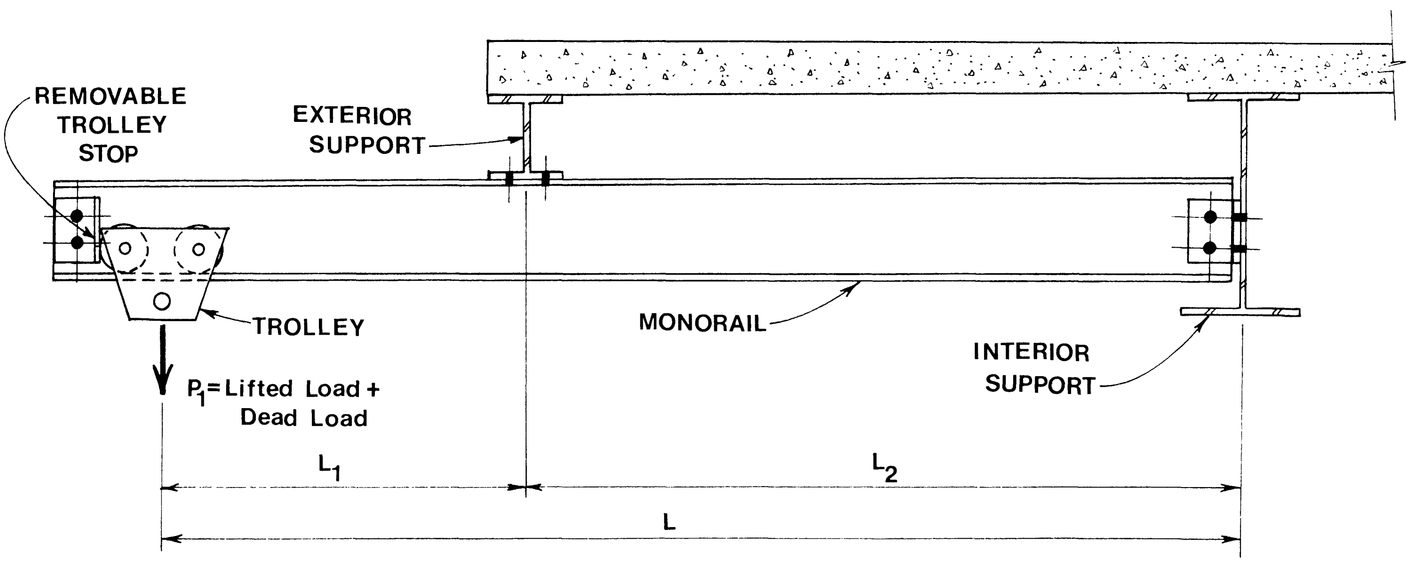

Fig. 1. Typical overhanging monorail

N. Stephen Tanner, P. E., is Civil/Structural/Architectural Department Head, Midwest Technical, Inc., Kingsport, Tennessee.

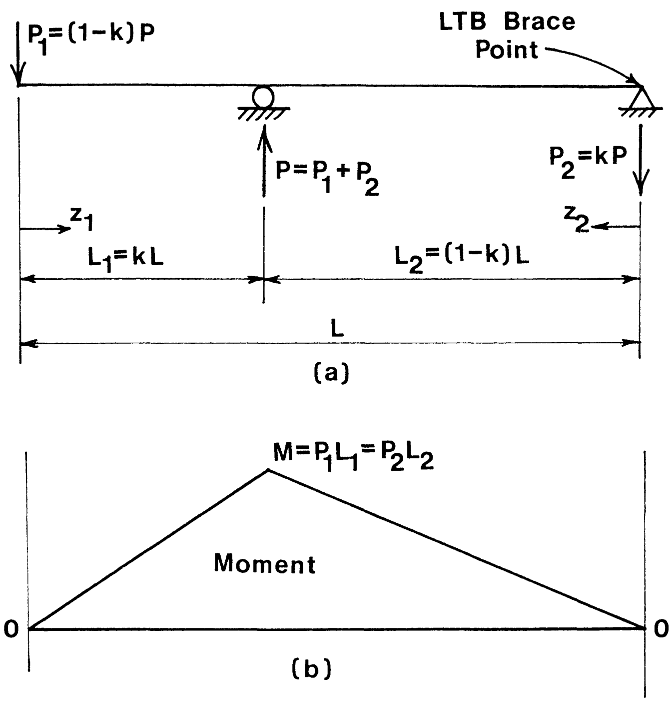

Fig. 2. Monorail load and moment diagrams

by the AISC Specification as effective lateral support for any type of beam. Finally, bracing the end of the overhang is impractical if it is desirable to have an easily removable trolley. Therefore, the overhanging monorail of Fig. 1 is regarded as braced at the interior support only.

General Mathematical Procedure

The classical mathematical approach for LTB problems is as follows:

1. Derive the equilibrium equations of the beam in its buckled configuration, assuming small displacements.
2. Simplify these equations to a single linear differential equation.
3. Obtain the general solution of the differential equation.
4. Apply the beam boundary conditions to this solution to obtain a system of linear algebraic equations.
5. Set the determinant of the coefficient matrix of the system equal to zero, yielding a transcendental equation which is a function of the unknown magnitude of the maximum moment.
6. Solve the transcendental equation for the lowest nonzero characteristic value of the moment. This value is the critical moment, defined as the moment at which bifurcation of the equilibrium position theoretically occurs.

Common assumptions made in LTB analysis are as follows:

1. The material is linearly elastic at the onset of buckling.
2. Transverse loads are applied through the shear center of the cross-section.
3. The moment diagram due to beam weight may be approximated as geometrically similar to the moment diagram due to the other, more significant, applied loads.
4. The cross-section is a prismatic doubly-symmetric I-shape.
5. The warping stiffness of the beam is negligible in comparison with the St. Venant torsional stiffness.

Using these assumptions, the following equation for the critical moment for LTB in a beam of arbitrary loading and support geometry is derived (see Ref. 2, p. 268):

$$M = \frac{\pi C_b}{l}\sqrt{EI_yGJ} \qquad (1)$$

This equation may be rewritten in terms of allowable bending stress to yield AISC Formula 1.5–7:

$$F_b = \frac{12{,}000\ C_b}{ld/A_f} \qquad (2)$$

AISC Formula 1.5–6b is derived using the above Assumptions 1 through 4 and the inverse of Assumption 5, that the St. Venant torsional stiffness of the beam is negligible in comparison to the warping stiffness. AISC Formula 1.5–6a is derived similarly to 1.5–6b, except that inelastic behavior is substituted for Assumption 1. Hoisting trolleys are normally designed to ride on narrow-flanged American standard shapes (S shapes), where the contribution of warping torsion is in fact negligible. Hence, the use of Assumption 5 and Eqs. 1 or 2 for overhanging monorails is justified.

All the variables on the right-hand side of Eqs. 1 and 2 except for C_b and l are properties of the particular beam shape and the steel used, and are therefore easily obtained. The values to be used for C_b and l are not so obvious. Immediately below Formula 1.5–7 is the following definition:

> "l = distance between cross sections braced against twist or lateral displacement of the compression flange, inches. For cantilevers braced against twist only at the support, l may conservatively be taken as the actual length."

To be consistent with this definition of l, it is convenient to take l for an overhanging monorail in an analogous manner to that for a cantilever beam. To avoid confusion in this paper, the notation L is used for the length of an overhanging monorail, while l is reserved for the general case of a beam of arbitrary support geometry. Using this notation, Eqs. 1 and 2 for an overhanging monorail become:

$$M = \frac{\pi C_b}{L} \sqrt{EI_y GJ} \tag{3}$$

$$F_b = \frac{12{,}000\, C_b}{Ld/A_f} \tag{4}$$

where L = actual length of an overhanging beam braced against twist only at the interior support. The result of this is that C_b is now the only unknown on the right-hand side of Eqs. 3 and 4. It must be noted that the assumption $l = L$ is somewhat arbitrary, so that it does not necessarily follow that $C_b = 1$ is a conservative assumption.

The approach used in this paper is to derive a formula for the critical moment of an overhanging monorail under concentrated end load and to compare it to Eq. 3. In this way a table of C_b values is readily obtained. These values can then be used in Eq. 4 for actual monorail design.

QUALITATIVE STUDY

Nature of Study

Before proceeding with the mathematical analysis, it is desirable to make a qualitative study of the problem. Referring to Fig. 2, this study involves visualizing the effect of changing the length ratio $k = L_1/L$, while holding the section properties, total length L, and maximum moment M constant. In this way, the only variable in Eq. 3 is C_b. It must be noted that holding M constant implies varying the lifted load P_1 for different values of k.

Case $k = 1$

Consider the case where k approaches unity, as shown in Fig. 3. Here the beam becomes a pure cantilever of length L and concentrated end load P_1, with the fixed end braced against twist and the free end unbraced. The AISC Specification states $C_b = 1$ may be used for this case and the allowable stress then obtained using Eq. 3. However, this is somewhat conservative, since the actual value is $C_b = 1.28$ (see Ref. 2, p. 269).

Case $k = 0$

Consider now the case where k approaches zero, as shown in Fig. 4a. Here the beam approaches the case of a simply-supported beam of length L with applied end moment M. The end moment is applied at the unbraced end, the other end being braced against twist. Owing to symmetry, the beam of Fig. 4a is identical with the beam of Fig. 4c as regards the strong-axis bending moment; this is apparent from the common moment diagram of Fig. 4b. Although it is not as apparent, these two beams are

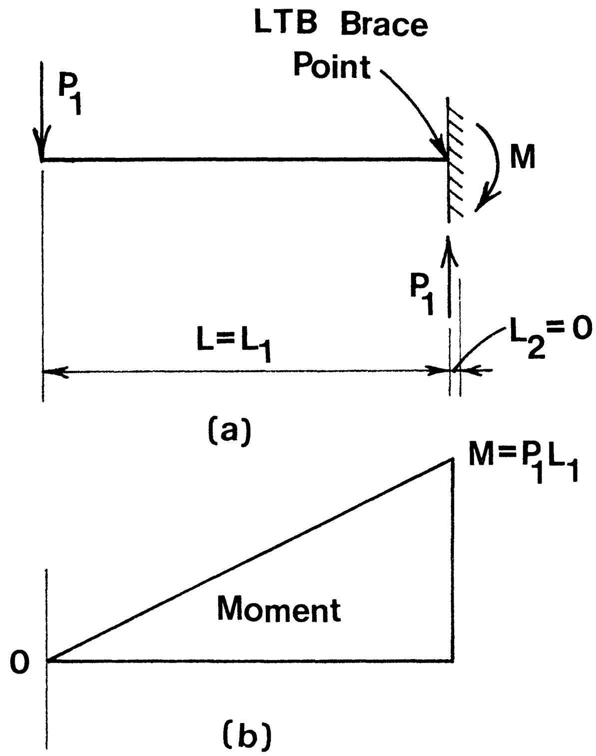

Fig. 3. Limiting case k = 1

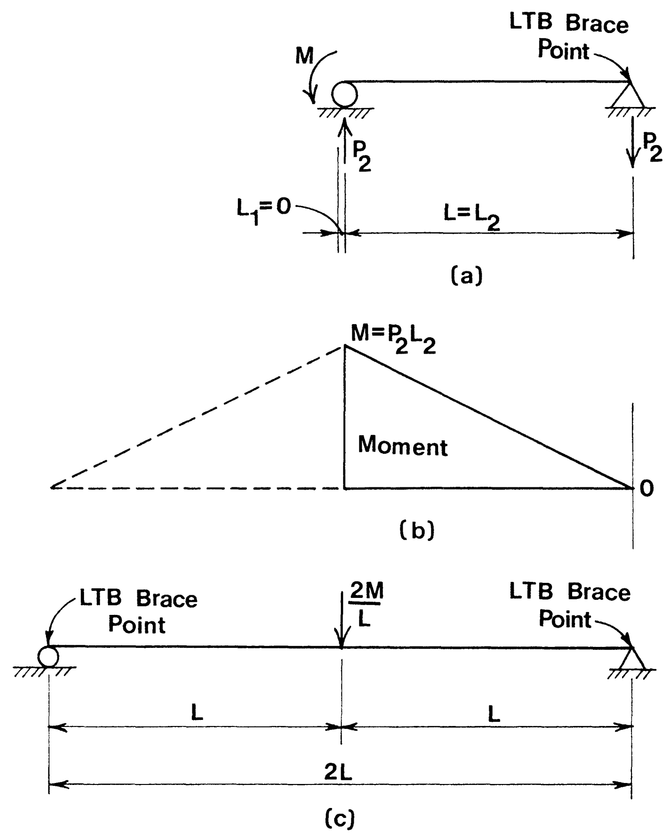

Fig. 4. Limiting case k = 0

also identical as regards LTB behavior.

To show this, first consider the left end of Fig. 4a. Here, the beam is not braced against twist and not subject to external torques; hence, torque at this point is zero. But, for a beam of negligible warping stiffness, this requires the first derivative of the torsional rotation be zero (see Ref. 3, Eq. 2.58). However, at midspan of Fig. 4c, symmetry requires the first derivative of rotation be zero here also. Thus, both beams have the same moment diagram and LTB boundary conditions, so they must also be identical with respect to LTB behavior.

Now, the critical moment for the beam of Fig. 4c is obtained from Eq. 1 using $l = 2L$ and $C_b = 1.35$ (see Ref. 2, p. 269). This is equivalent to using Eq. 3 and a value $C_b = .675$. Therefore, the limiting case $k = 0$ shown in Fig. 4a results in a value $C_b = .675$.

Conclusion

This qualitative study indicates the buckling parameter C_b is somewhat sensitive to the length ratio, varying from $C_b = .675$ for $k = 0$ to $C_b = 1.28$ for $k = 1$. Of course, this result shows clearly the assumption $C_b = 1$ is unconservative for many overhanging monorails. That lower allowable stresses occur for monorails with shorter relative overhangs is somewhat paradoxical. However, this can be explained by referring to the moment diagrams of Figs. 2b, 3b and 4b. LTB theory recognizes that as the beam twists and deflects laterally during buckling, disturbing weak-axis bending and torsional moments are introduced as components of the strong-axis bending moment. At any location along the beam, these disturbing moments are proportional to both the strong-axis moment and the twist or lateral deflection. Thus, a beam which has high strong-axis moment acting at the same location as high twist or lateral deflection will have high disturbing moments.

Referring to the moment diagrams, Fig. 4b has maximum moment acting at the left end, where the torsional rotation is the highest, so that high disturbing moments occur in this beam. In Fig. 3b, maximum moment acts at the right end, where the rotation is zero, so the disturbing moments in this beam are much smaller. Figure 2b is clearly intermediate between these two cases. Now, other things being equal, the higher the disturbing moments are, the greater the tendency for LTB. Therefore, beams with a more beneficial moment gradient, as in Fig. 3b, have higher allowable strong-axis moments than comparable beams with a more unstable moment gradient, as in Fig. 4b. This conclusion is confirmed by the numerical results in this paper.

MATHEMATICAL ANALYSIS

Differential Equation of Beam Elements

The overhanging monorail of Fig. 2 can be conveniently broken into two beam elements of length L_1 and L_2. The longitudinal axes z_1 and z_2 corresponding to elements L_1 and L_2 can then be assigned the origins and directions shown in Fig. 2a. Thus, each beam element of length L_i ($i = 1,2$) is subjected to a concentrated load P_i at the end $z_i = 0$; the strong-axis bending moment equals zero at $z_i = 0$, equals $M = P_iL_i$ at $z_i = L_i$, and varies linearly between these points. The differential equation for LTB for a beam element under such a moment gradient is derived as Eq. 2.92 of Ref. 3. Rewriting this equation to conform to the notation of this paper gives the following equation for the torsional rotation ϕ_i along each element:

$$\phi_i'' + \pi^2\eta_i^2 z_i^2 \phi_i = 0 \qquad (5)$$

Here, primes denote differentiation with respect to z_i, and η_i is defined as

$$\eta_i^2 = \frac{P_i^2}{\pi^2 EI_y GJ} \qquad (6)$$

Reference 3 gives the solution of Eq. 5, and its derivative, as

$$\begin{aligned} \phi_i &= z_i^{1/2}\left\{A_{i1}J_{-1/4}\left(\frac{\pi}{2}\eta_i z_i^2\right) + A_{i2}J_{1/4}\left(\frac{\pi}{2}\eta_i z_i^2\right)\right\} \\ \phi_i' &= \pi\eta_i z_i^{3/2}\left\{-A_{i1}J_{3/4}\left(\frac{\pi}{2}\eta_i z_i^2\right) + A_{i2}J_{-3/4}\left(\frac{\pi}{2}\eta_i z_i^2\right)\right\} \end{aligned} \qquad (7)$$

Defining the series

$$\begin{aligned} S_n(\xi) &= \left[\Gamma(n+1)\right]\left(\frac{\pi}{4}\xi\right)^{-n} J_n\left(\frac{\pi}{2}\xi\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+1)_k}\left(\frac{\pi}{4}\xi\right)^{2k} \end{aligned} \qquad (8)$$

and redefining the constants as

$$C_{i1} = \frac{1}{3\Gamma\left(\frac{3}{4}\right)}\left(\frac{\pi\eta_i}{4}\right)^{-1/4} A_{i1}$$

$$C_{i2} = \frac{1}{\Gamma\left(\frac{5}{4}\right)}\left(\frac{\pi\eta_i}{4}\right)^{1/4} A_{i2}$$

simplifies Eq. 7 to

$$\begin{aligned} \phi_i &= 3C_{i1}S_{-1/4}(\eta_i z_i^2) + C_{i2}z_i S_{1/4}(\eta_i z_i^2) \\ \phi_i' &= -C_{i1}\pi^2\eta_i^2 z_i^3 S_{3/4}(\eta_i z_i^2) + C_{i2}S_{-3/4}(\eta_i z_i^2) \end{aligned} \qquad (9)$$

Boundary Conditions

The torsional boundary conditions at the ends $z_i = 0$ are given by

$$\phi_1'(z_1 = 0) = \phi_1'(0) = 0$$
$$\phi_2(z_2 = 0) = \phi_2(0) = 0 \qquad (10)$$

The first of these expresses the condition that there is no torque at the end of the overhang (see Ref. 3, Eq. 2.58). The second condition denotes torsional restraint at the interior support. The boundary conditions at the interface between the two elements are:

$$\phi_1(z_1 = L_1) + \phi_2(z_2 = L_2) = \phi_1(L_1) + \phi_2(L_2) = 0$$
$$-\phi_1'(z_1 = L_1) + \phi_2'(z_2 = L_2) = -\phi_1'(L_1) + \phi_2'(L_2) = 0 \qquad (11)$$

These equations denote compatibility between the rotations and torques, respectively, at the ends of the elements, with due regard to the different sign conventions for the two elements.

Substituting 9 into 10 yields requirement $C_{12} = C_{21} = 0$. Conditions (see 11) then give the following system:

$$\begin{bmatrix} 3S_{-1/4}(\eta_1 L_1^2) & L_2 S_{1/4}(\eta_2 L_2^2) \\ \pi^2 \eta_1^2 L_1^3 S_{3/4}(\eta_1 L_1^2) & S_{-3/4}(\eta_2 L_2^2) \end{bmatrix} \begin{Bmatrix} C_{11} \\ C_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (12)$$

Characteristic Equation

Defining the exterior support reaction as P and noting the forces P_i and lengths L_i written in terms of P, L and k in Fig. 2a, the moment M may be written as

$$M = Pk(1 - k)L \qquad (13)$$

Equating right-hand sides of Eqs. 3 and 13, squaring, and rearranging yields the following expression for C_b:

$$C_b^2 = \frac{P^2 k^2 (1 - k)^2 L^4}{\pi^2 E I_y G J} \qquad (14)$$

Now, using the forces and lengths in Fig. 2a in conjunction with Eqs. 6 and 14, the η_i terms of Eq. 12 may be written in terms of C_b and k:

$$\eta_1 L_1^2 = k C_b$$
$$\eta_2 L_2^2 = (1 - k) C_b \qquad (15)$$
$$\eta_1^2 L_1^3 L_2 = k(1 - k) C_b^2$$

The characteristic equation, representing the solutions to the system (see 12), is obtained by setting the determinant of the coefficient matrix equal to zero. Using expressions (see 15), the characteristic equation may be written in the form

$$3\{S_{-1/4}(kC_b)\}\{S_{-3/4}[(1 - k)C_b]\}$$
$$- k(1 - k)\pi^2 C_b^2 \{S_{3/4}(kC_b)\} \qquad (16)$$
$$\times \{S_{1/4}[(1 - k)C_b]\} = 0$$

Numerical Results

For a given value of the length ratio k, the corresponding value of C_b is obtained using a computer program capable of both evaluating the series (see 8) and finding the lowest positive root of the transcendental Eq. 16. This has been done for values ranging from $k = 0$ to $k = 1$ in increments of $\Delta k = 0.1$; the results are presented in Table 1. It is apparent the assumption $C_b = 1$ is unconservative for $k < .75$, which constitutes almost all practical overhanging monorails; therefore, it is recommended the values of C_b shown in Table 1 be used in the design of such members. The values are accurate to two decimal places, which is consistent with C_b values and equations given in Ref. 1 for other types of beams. Linear interpolation for intermediate values of k is sufficiently accurate.

Table 1. Computed Values

$k = L_1/L$	C_b
0	.67
.1	.70
.2	.73
.3	.76
.4	.80
.5	.84
.6	.90
.7	.96
.8	1.05
.9	1.15
1.0	1.28

DESIGN PROCEDURE

General Method

For a given monorail layout, as shown in Fig. 1, the length ratio k is computed; then, the value of C_b is obtained from Table 1. A trial section is selected, and Eq. 4 is applied in the same manner as for any other type of beam. The most economical such trial section is then chosen.

Numerical Example

Given: $L_1 = 7'$, $L = 20'$
P_1 = Lifted Load + Impact + Trolley + Hoist = 5 kips
Solution: $k = L_1/L = 7/20 = .35$
From Table 1, $C_b = .78$
Estimate 40 lb./ft monorail weight:

$$M = 5(7) + 1/2(.04)(7)^2 = 36.0 \text{ kip-ft}$$

Try S12×40.8: $S_x = 45.4$, $d/A_f = 3.46$
From Eq. 4:

$$F_b = \frac{12{,}000C_b}{Ld/A_f} = \frac{12{,}000(.78)}{20(12)(3.46)} = 11.3 \text{ ksi}$$

$$f_b = \frac{M}{S_x} = \frac{36.0(12)}{45.4} = 9.5 \text{ ksi} < 11.3 \text{ ksi} \quad \textbf{o.k.}$$

S15×42.9 also is **o.k.**; S12×35 is **n.g.**
∴ *Use S12×40.8*

NOMENCLATURE

A_f = area of compression flange (sq. in.)
A_{ij}, C_{ij} = arbitrary constants in the solution for ϕ_i
C_b = bending coefficient dependent upon moment gradient
d = depth of monorail (in.)
E = modulus of elasticity
F_b = allowable bending stress (ksi)
f_b = computed bending stress
G = shear modulus of elasticity
I_y = moment of inertia about the Y-Y (weak) axis
i = index denoting beam element (i = 1,2)
J = torsional constant
$J_n(\xi)$ = Bessel function of first kind* of order n
j = index denoting term in the solution for ϕ_i (j = 1,2)
k = length ratio, $k = L_1/L$
L = length of overhanging beam (in.)
L_i = length of beam element
l = length of beam between LTB brace points (in.)
M = maximum moment in beam
n = index denoting order of $J_n(\xi)$, $S_n(\xi)$, ($n = \pm 1/4, \pm 3/4$)
$(n + 1)_k$ = generalized factorial function* defined as $(n + 1)_k = (n + 1)(n + 2) \ldots (n + k)$, with $(n + 1)_0 = 1$ and $(1)_k = k!$
P = exterior support reaction
P_i = end reaction or load on beam element
$S_n(\xi)$ = solution series function of order n, defined by Eq. 8
S_x = section modulus about the X-X (strong) axis
z_i = distance along beam element (see Fig. 2)
$\Gamma(n + 1)$ = gamma function*
η_i = load term defined by Eq. 6
ϕ_i = torsional rotation of beam element
ξ = general argument of J_n, S_n

*See Ref. 4, p. 168, 333 and 337.

REFERENCES

1. *American Institute of Steel Construction, Inc.* Specification for the Design and Erection of Structural Steel for Buildings *Chicago, Ill., November 1978.*
2. *Gaylord, E. H. and C. N. Gaylord* Design of Steel Structures *2nd Ed., McGraw-Hill, 1972.*
3. *Brush, D. O. and B. O. Almroth* Buckling of Bars, Plates and Shells *McGraw-Hill, 1975.*
4. *Rainville, E. D. and P. E. Bedient* Elementary Differential Equations *4th Ed., MacMillan, 1969.*