The Application of Flexural Methods to Torsional Analysis of Thin-walled Open Sections

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INTRODUCTION

The analysis of torsionally loaded thin-walled members of open cross-section is complicated by the presence of warping stresses. Warping normal and shear stresses arise from restraint of longitudinal warping of the cross-section due to end restraints or variations in the internal torque along the length of the member. In a typical thin-walled open member, such as a rolled or cold-formed steel section, warping normal stresses tend to be the critical torsional stress component. The usual method for analysis of warping stresses involves solution of the torsion differential equation to determine rotation of the member about the longitudinal axis. Torsional and warping stresses are functions of various order derivatives of the rotation with respect to the length. This method is described in detail in many sources, including Refs. 1 and 4-9. A recent AISC publication[1] presents non-dimensionalized solutions of the torsion differential equation for various end conditions and loading cases of a single span straight member, which can be used for determination of torsional stresses.

For steel I sections, torsional loading can be resolved into opposite lateral forces acting on flanges and an upper bound to warping stresses can be obtained by determining the resulting bending normal and shear stresses in the flanges. This flexural analogy is applied to determination of warping normal and shear stresses in Refs. 6, 8 and 9. A more general bending-warping analogy, presented in Ref. 7, can be applied to C, Z and other sections commonly used in cold-formed construction.

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A method is presented here where the more general bending-warping analogy is applied to solution of warping torsional stresses in prismatic thin-walled members of arbitrary open cross-section continuous over any number of supports. Solution of warping stresses by these methods is analogous to the more familiar methods of flexural analysis, and in many cases the methods of flexural analysis can be applied directly to solution of warping stresses.

TORSION OF THIN-WALLED OPEN CROSS-SECTIONS

In members subjected to torsional loading, two types of torsional stresses result: St. Venant torsional shear stresses result from twisting of members circular in cross-section or members subjected to uniform twisting moments with both ends free to warp. Where warping, or out-of-plane displacement of the cross-section, is restrained by end conditions or variations in the twisting moment, a pattern of normal stresses, known as warping normal stresses, results. Variations in these stresses along the member produce torsional shear stresses in addition to the St. Venant torsional shear stresses. The moment resultant of these warping sheer stresses is a torque known as warping torque. The total torque at a point on a member is a combination of warping torque and St. Venant torque. For a straight prismatic member subjected to torque loading, with $\phi(z)$ defined as the rotation about the longitudinal member axis z, the St. Venant torsional shear stresses are proportional to $\phi'(z)$, the warping normal stresses are proportional to $\phi''(z)$, and the warping shear stresses are proportional to $\phi'''(z)$. A quantity known as a "bimoment" defined as

$$B = -EC_w \phi'' \tag{1}$$

is useful in the computation of warping normal and shear stresses. Although Eq. 1 is the rigorous definition of a

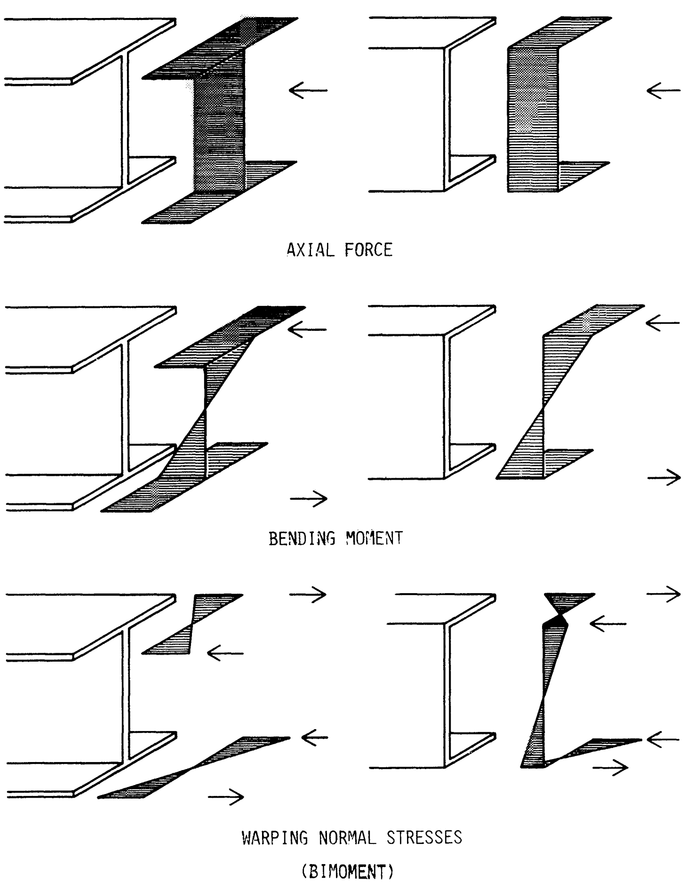

Fig. 1. Bimoment as an internal resultant

bimoment, it is helpful to visualize a bimoment as a type of internal resultant, much like axial force or bending moment. Figure 1 shows the normal stresses in an I-section and a C-section resulting from axial thrust, bending moment and restrained warping. The bending stresses result in no axial force, but in a couple of equal and oppositely directed forces separated by a certain distance. Similarly, the warping normal stresses result in no axial force and no bending moment, but may often be visualized as resulting in a "couple" of equal and oppositely directed moments. This is reflected in the units of a bimoment, force times distance.[2]

For an I-section, the bimoment effect takes the form of equal and opposite lateral bending of the flanges. This allows warping normal stresses to be computed in I-sections by analogy with lateral bending stresses in the flanges, as in Refs. 8 or 9. The more general concept of the bimoment, however, simplifies the use of the bending-warping analogy to compute warping stresses in other commonly used sections such as C- and Z-sections.

BENDING MOMENT-BIMOMENT ANALOGY

As discussed in Refs. 7 and 9, the determination of warping stresses in a beam given the angle of rotation is analogous to the determination of bending stresses given the deflection. Table 1 illustrates the computation of bimoment, warping torque and warping normal and shear stresses in a member as a function of the angle of rotation. It also shows the analogous computation of bending moment, shear and bending normal and shear stresses as a function of the deflection. In members where St. Venant torsional stiffness is negligible with respect to warping torsional stiffness, or where distribution between the St. Venant and warping torsion is known, warping stresses can be determined using techniques of flexural analysis. This is illustrated by the charts in Appendix B, Ref. 1.

Table 1. Bending—Warping Analogy

Bending		Warping	
Quantity	Definition	Quantity	Definition
Deflection	v	Rotation	ϕ
Slope	v'	Angle of twist	ϕ'
Curvature	v''	Curvature	ϕ''
Bending Moment	$M = EIv''$	Bimoment	$B = -EC_w\phi''$
Shear	$V = EIv'''$	Warping Torque	$T_w = -EC_w\phi'''$
Transverse Load	$w = -EIv^{iv}$	Torque Load*	$m_z = EC_w\phi^{iv}$
Bending Normal Stress	$f_b = \dfrac{My}{I}$	Warping Normal Stress	$f_w = \dfrac{Bw_n}{C_w}$
Bending Shear Stress	$\tau_b = \dfrac{VQ}{It}$	Warping Shear Stress	$\tau_w = \dfrac{T_w S_w}{C_w t}$

*This formula applies where St. Venant torque is negligible. Otherwise Eq. 4 applies.

For example, Case 4 illustrates a simply supported member with both ends free to warp, subjected to a uniformly distributed twisting moment m_z. As L/a becomes small, that is as J becomes small with respect to C_w, the warping torque predominates over the St. Venant torque, and the bimoment approaches the value of $m_z L^2/8$ compared with the midspan bending moment in a uniformly loaded simple span beam of $wL^2/8$. Similarly, warping torque at ends of the member approaches m_z L/2. For the simply supported member with both ends free to warp subjected to concentrated midspan torque M_z, the midspan bimoment approaches $M_z L/4$ and shear approaches $M_z/2$ as L/a becomes small.

To illustrate application of the bending moment-bimoment analogy to solution of warping normal stresses, consider the three-span continuous girder shown in Fig. 2a. Properties of the girder are $J = 17{,}500$ in.^{4}, $C_w = 2.062 \times 10^9$ in.^{6}, $G/E = 1/3$. Hence $L/a = 1.01$, a relatively small value. The girder is subjected to a uniformly distributed eccentric load w. The eccentricity of the load application is e, so the member is subjected to a uniformly distributed torque $m_z = we$. The bending moment diagram shown in Fig. 2d is obtained by using moment distribution (or any other method) and, based on load m_z, gives a bimoment diagram accurate to within 5%. The exact bimoment diagram, obtained using the method of Ref. 10, is also shown in Fig. 2d. Once the bimoment at a point is known, the warping normal stress is determined by the formula from Table 1.

$$f_w = \frac{Bw_n}{C_w} \tag{2}$$

A noteworthy feature of this method of determining warping normal stresses is that it yields an upper bound solution, since the bimoment is found by integrating total torque, which is greater than warping torque alone. In the above example, reasonable accuracy was obtained because of the relatively small value of L/a. However, as L/a becomes larger, that is, as warping torsional stiffness decreases with respect to St. Venant torsional stiffness, accuracy of the direct application of flexural methods decreases. For larger values of L/a, bimoments in continous prismatic members can be determined by a modification of the moment distribution method.

BIMOMENT DISTRIBUTION METHOD

This method relies on principles similar to the moment distribution method of flexural analysis. Using the differential equations for torsion of thin-walled members, accurate fixed-end bimoments, stiffness factors and carry-over factors can be found and applied to the moment distribution procedure to find the approximate bimoments at the supports of a continous prismatic member. The differential equation for a member subjected to a single torque M_z, as given in Eq. 4 and other sources, is

$$GJ\,\phi' - EC_w\,\phi''' = M_z \tag{3}$$

and for a member subjected to uniformly distributed torque m_z

$$EC_w\,\phi^{iv} - GJ\,\phi'' = m_z \tag{4}$$

The stiffness and carry-over factors are both found by considering a bar fixed at one end with a unit bimoment applied at the opposite end. The carry-over factor is the bimoment at the fixed end, and the stiffness factor is the ratio of the applied bimoment to warping at the point of application. Thus, the boundary conditions to Eq. 3 are

$$\text{Free End}\quad \begin{aligned} \phi\,(0) &= 0 \\ EC_w\,\phi''\,(0) &= 1 \end{aligned} \tag{5}$$

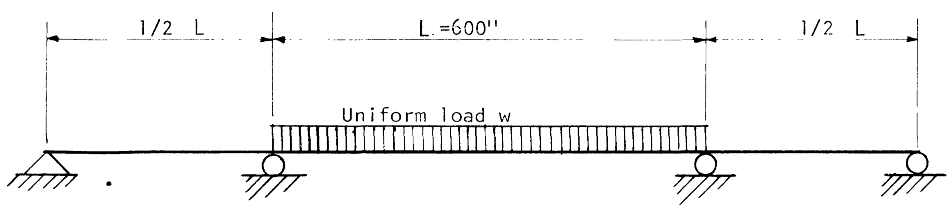

Fig. 2a. Bending support conditions

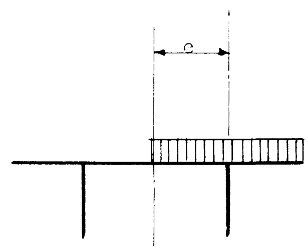

Fig. 2b. Cross-section

Fig. 2c. Torsion support conditions

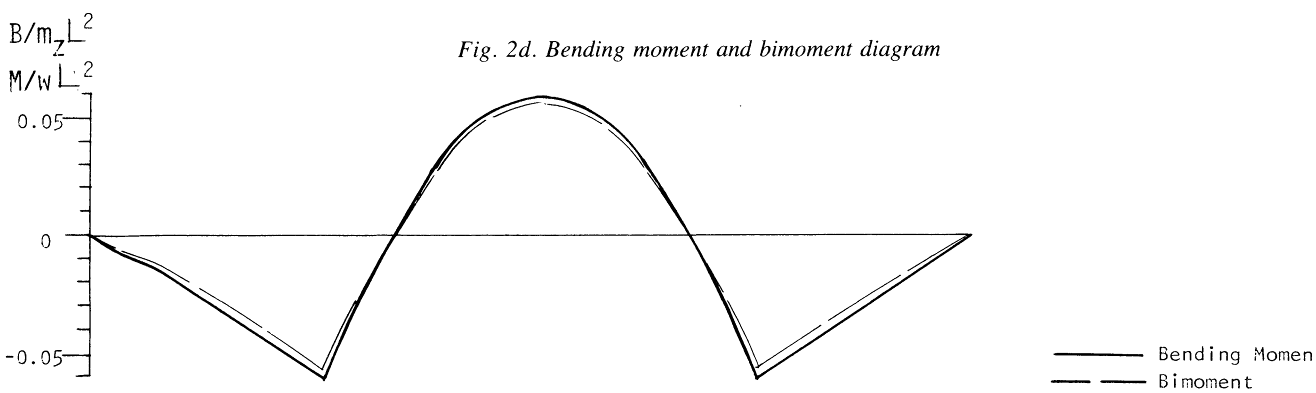

Fig. 2d. Bending moment and bimoment diagram

Fig. 2. Example problem

$$\text{Fixed End} \quad \begin{array}{l} \phi\ (L) = 0 \\ \phi'\ (L) = 0 \end{array}$$

Upon substitution of the boundary conditions, and solution for the constants of integration, the values of K and C are found

$$K = \frac{EC_w}{L} \frac{(L/a)^2 \cosh (L/a) - (L/a) \sinh (L/a)}{2(1 - \cosh (L/a)) + (L/a) \sinh (L/a)} \tag{6}$$

$$C = \frac{\sinh (L/a) - (L/a)}{(L/a) \cosh (L/a) - \sinh (L/a)} \tag{7}$$

The values of these constants are principally dependent on the parameter L/a. They are tabulated for various values of L/a in Table 2. As L/a becomes very small, C converges to ½ and K converges to $4EC_w/L$.

For a member with a warping hinge at one end, a modified stiffness factor may be used, as for a fixed-pinned beam in the moment distribution method, provided the fixed-end bimoments are modified suitably.

$$K' = K(1 - C^2) \tag{8}$$

To find the fixed end bimoments on a beam under uniformly distributed twisting moment, Eq. 4 is solved for the boundary conditions $\phi = 0$ and $\phi' = 0$ at each support. The fixed end bimoment for this condition is

$$FEB = m_z\, L^2 \left(\frac{1}{2\ (L/a) \tanh (L/2a)} - \frac{1}{(L/a)^2} \right) \tag{9}$$

Values of the fixed end bimoment are given in Table 2 for various values of L/a. It can be seen in Table 2 that, as L/a becomes very small, the fixed end bimoment approaches $M_z\, L^2/12$. Similarly, the fixed end bimoment on a beam fixed at one end with a warping hinge at the other is

$$FEB = m_z\, L^2 \left(\frac{(L/a) \sinh (L/a) + 2\ [1 - \cosh (L/a)]}{2\ (L/a)\ [(L/a) \cosh (L/a) - \sinh (L/a)]} \right) \tag{10}$$

Values of these fixed end bimoments are also tabulated in Table 2.

To determine the fixed end bimoments on a beam subjected to a single concentrated torque M_z applied at $z = \alpha L$, the beam must be divided at the point of load application into two parts with a separate Eq. 3 for each part. In addition to the six constants of integration, the reactions at the supports, M_{z1} and M_{z2} are also unknown. So eight equations are required to solve for the constants of integration and the external reactions. The boundary conditions are

$$\begin{aligned} \phi_1\ (0) &= 0 \\ \phi_1'\ (0) &= 0 \\ \phi_2\ (L) &= 0 \\ \phi_2'(L) &= 0 \end{aligned} \tag{11}$$

Continuity at the point of load application provides three additional conditions

$$\begin{aligned} \phi_1\ (\alpha L) &= \phi_2\ (\alpha L) \\ \phi_1'\ (\alpha L) &= \phi_2'\ (\alpha L) \\ \phi_1''\ (\alpha L) &= \phi_2''\ (\alpha L) \end{aligned} \tag{12}$$

And finally, equilibrium requires that

$$M_z = M_{z1} - M_{z2} \tag{13}$$

These equations may be solved for the constants of integration for various values of L/a and α. The fixed end bimoments for both the fixed-fixed and the fixed-pinned case are listed in Tables 3 and 4. The charts of Ref. 1 may also be used to compute fixed end bimoments.

The above equations for fixed-end bimoments, stiffness and carry-over factors are given without signs. Since

Table 2. Factors for Use in Bimoment Distribution Method

	Stiffness	Carry-over	Fixed-End Bimoment for Uniform Torque	
	Factor	Factor	Fixed-Fixed	Fixed-Pinned
L/a	$KL/4EC_w$	C	$FEB/m_z\, L^2$	$FEB/m_z L^2$
0.5	4.0332	0.4938	0.0829	0.1239
1.0	4.1316	0.4762	0.0819	0.1210
1.5	4.2915	0.4496	0.0803	0.1165
2.0	4.5075	0.4174	0.0782	0.1109
2.5	4.7730	0.3825	0.0757	0.1047
3.0	5.0808	0.3476	0.0730	0.0984
3.5	5.4242	0.3145	0.0701	0.0921
4.0	5.7968	0.2842	0.0671	0.0862
4.5	6.1931	0.2570	0.0642	0.0807
5.0	6.6084	0.2331	0.0613	0.0756
5.5	7.0389	0.2122	0.0585	0.0710
6.0	7.4816	0.1940	0.0559	0.0668
6.5	7.9339	0.1782	0.0534	0.0630
7.0	8.3938	0.1645	0.0511	0.0595
7.5	8.8600	0.1525	0.0489	0.0564
8.0	9.3312	0.1420	0.0469	0.0535
8.5	9.8064	0.1328	0.0450	0.0509
9.0	10.2849	0.1247	0.0432	0.0486
9.5	10.7662	0.1174	0.0415	0.0464
10.0	11.2497	0.1110	0.0400	0.0444
11.0	12.2221	0.0999	0.0371	0.0409
12.0	13.1999	0.0908	0.0347	0.0378
13.0	14.1818	0.0833	0.0325	0.0352
14.0	15.1666	0.0769	0.0306	0.0329
15.0	16.1538	0.0714	0.0288	0.0309
16.0	17.1428	0.0666	0.0273	0.0291
17.0	18.1333	0.0624	0.0259	0.0275
18.0	19.1249	0.0588	0.0246	0.0261
19.0	20.1176	0.0555	0.0235	0.0248
20.0	21.1111	0.0526	0.0225	0.0236

Table 3. Values of FEB/M_zL for Fixed-Fixed Beam Subjected to Concentrated Torque M_z

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
L/A									
0.5	0.08087	0.12763	0.14643	0.14333	0.12435	0.09547	0.06265	0.03183	0.00895
1.0	0.08050	0.12657	0.14476	0.14137	0.12245	0.09394	0.06164	0.03133	0.00882
1.5	0.07991	0.12486	0.14210	0.13825	0.11945	0.09151	0.06003	0.03055	0.00862
2.0	0.07912	0.12260	0.13860	0.13416	0.11552	0.08835	0.05796	0.02954	0.00837
2.5	0.07815	0.11988	0.13442	0.12933	0.11091	0.08466	0.05554	0.02837	0.00807
3.0	0.07705	0.11681	0.12976	0.12398	0.10585	0.08063	0.05291	0.02710	0.00775
3.5	0.07585	0.11349	0.12479	0.11834	0.10055	0.07643	0.05019	0.02579	0.00742
4.0	0.07456	0.11002	0.11965	0.11258	0.09519	0.07222	0.04748	0.02449	0.00709
4.5	0.07322	0.10647	0.11449	0.10686	0.08992	0.06810	0.04483	0.02323	0.00678
5.0	0.07185	0.10290	0.10938	0.10127	0.08482	0.06415	0.04231	0.02204	0.00648
5.5	0.07047	0.09937	0.10441	0.09591	0.07998	0.06043	0.03995	0.02092	0.00621
6.0	0.06908	0.09590	0.09962	0.09081	0.07542	0.05695	0.03775	0.01989	0.00596
6.5	0.06769	0.09252	0.09503	0.08601	0.07118	0.05373	0.03572	0.01894	0.00573
7.0	0.06632	0.08926	0.09068	0.08151	0.06724	0.05076	0.03385	0.01806	0.00552
7.5	0.06497	0.08611	0.08657	0.07732	0.06360	0.04803	0.03214	0.01726	0.00533
8.0	0.06364	0.08309	0.08269	0.07342	0.06025	0.04553	0.03058	0.01653	0.00515
8.5	0.06234	0.08020	0.07904	0.06980	0.05716	0.04324	0.02915	0.01586	0.00499
9.0	0.06107	0.07743	0.07562	0.06645	0.05433	0.04114	0.02783	0.01525	0.00485
9.5	0.05982	0.07479	0.07241	0.06334	0.05172	0.03922	0.02663	0.01468	0.00471
10.0	0.05861	0.07227	0.06941	0.06047	0.04933	0.03745	0.02552	0.01416	0.00459
11.0	0.05627	0.06759	0.06395	0.05534	0.04508	0.03433	0.02356	0.01322	0.00437
12.0	0.05405	0.06335	0.05916	0.05091	0.04146	0.03166	0.02187	0.01241	0.00417
13.0	0.05195	0.05950	0.05494	0.04709	0.03834	0.02937	0.02041	0.01170	0.00400
14.0	0.04996	0.05600	0.05122	0.04376	0.03564	0.02738	0.01913	0.01107	0.00384
15.0	0.04808	0.05283	0.04792	0.04084	0.03329	0.02564	0.01800	0.01051	0.00370
16.0	0.04630	0.04994	0.04498	0.03828	0.03122	0.02411	0.01700	0.01000	0.00357
17.0	0.04461	0.04731	0.04236	0.03600	0.02939	0.02274	0.01610	0.00954	0.00346
18.0	0.04302	0.04491	0.04001	0.03398	0.02777	0.02152	0.01529	0.00912	0.00335
19.0	0.04151	0.04271	0.03789	0.03217	0.02631	0.02043	0.01456	0.00873	0.00324
20.0	0.04007	0.04070	0.03598	0.03053	0.02499	0.01944	0.01389	0.00838	0.00315

Table 4. Values of FEB/M_zL for Fixed-Pinned Beam Subjected to Concentrated Torque M_z

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
L/A									
0.5	0.08529	0.14335	0.17737	0.19048	0.18576	0.16625	0.13496	0.09486	0.04889
1.0	0.08471	0.14150	0.17412	0.18611	0.18078	0.16127	0.13058	0.09161	0.04717
1.5	0.08379	0.13861	0.16910	0.17940	0.17317	0.15368	0.12394	0.08670	0.04456
2.0	0.08261	0.13493	0.16279	0.17104	0.16375	0.14435	0.11581	0.08072	0.04139
2.5	0.08124	0.13073	0.15567	0.16171	0.15335	0.13413	0.10697	0.07423	0.03797
3.0	0.07975	0.12623	0.14816	0.15202	0.14266	0.12373	0.09803	0.06771	0.03454
3.5	0.07818	0.12160	0.14058	0.14238	0.13219	0.11366	0.08945	0.06150	0.03128
4.0	0.07658	0.11698	0.13315	0.13311	0.12225	0.10422	0.08148	0.05576	0.02829
4.5	0.07497	0.11244	0.12601	0.12436	0.11303	0.09557	0.07426	0.05060	0.02560
5.0	0.07337	0.10804	0.11925	0.11623	0.10460	0.08777	0.06781	0.04603	0.02324
5.5	0.07179	0.10381	0.11289	0.10873	0.09695	0.08078	0.06210	0.04201	0.02117
6.0	0.07023	0.09976	0.10694	0.10186	0.09006	0.07457	0.05708	0.03850	0.01936
6.5	0.06871	0.09590	0.10140	0.09559	0.08386	0.06906	0.05266	0.03543	0.01780
7.0	0.06723	0.09223	0.09625	0.08986	0.07830	0.06417	0.04877	0.03275	0.01643
7.5	0.06578	0.08874	0.09147	0.08464	0.07330	0.05983	0.04535	0.03040	0.01524
8.0	0.06438	0.08544	0.08704	0.07989	0.06881	0.05596	0.04233	0.02834	0.01420
8.5	0.06301	0.08230	0.08292	0.07555	0.06476	0.05252	0.03965	0.02652	0.01328
9.0	0.06167	0.07933	0.07909	0.07158	0.06111	0.04943	0.03727	0.02490	0.01246
9.5	0.06038	0.07652	0.07554	0.06795	0.05780	0.04666	0.03514	0.02347	0.01174
10.0	0.05912	0.07385	0.07224	0.06463	0.05480	0.04416	0.03323	0.02218	0.01109
11.0	0.05671	0.06891	0.06631	0.05877	0.04959	0.03986	0.02995	0.01998	0.00999
12.0	0.05443	0.06448	0.06115	0.05379	0.04522	0.03629	0.02725	0.01817	0.00908
13.0	0.05228	0.06047	0.05664	0.04954	0.04154	0.03329	0.02499	0.01666	0.00833
14.0	0.05026	0.05686	0.05269	0.04586	0.03839	0.03075	0.02307	0.01538	0.00769
15.0	0.04834	0.05358	0.04920	0.04268	0.03567	0.02856	0.02142	0.01428	0.00714
16.0	0.04654	0.05061	0.04611	0.03988	0.03331	0.02666	0.01999	0.01333	0.00666
17.0	0.04483	0.04791	0.04336	0.03743	0.03123	0.02499	0.01874	0.01249	0.00624
18.0	0.04321	0.04545	0.04091	0.03525	0.02940	0.02352	0.01764	0.01176	0.00588
19.0	0.04169	0.04320	0.03870	0.03330	0.02777	0.02222	0.01666	0.01111	0.00555
20.0	0.04024	0.04114	0.03671	0.03156	0.02631	0.02105	0.01578	0.01052	0.00526

the sign of a bimoment can be difficult to define in physical terms, and since the proper assessment of warping stresses requires knowledge of the sign of w_n in a consistent sign convention, it is simplest to evaluate the signs of warping stresses by analogy with bending stresses. A torque loading of a certain sign may be designated arbitrarily as an analogous "downwards" load. Signs used in the bimoment distribution procedure will then be the same as in the analogous moment distribution method. When the support bimoments are known, the bimoment diagram can be drawn by analogy with a bending moment diagram, that is by converting the joint sign convention of the bimoment distribution procedure to a beam sign convention for the bimoment diagram. To assess the significance of the sign of the bimoment, a cantilever beam subjected to a positive, downwards torque can be visualized, in conjunction with the w_n diagram to locate zones of tension and compression due to a negative bimoment, such as would be encountered at supports of a continous beam. The example which follows will illustrate the application of these procedures.

Example Problem

The three-span continous beam shown in Fig 3a can now be considered, using the "bimoment distribution" method. The properties of the C12x30 are J = 0.864 in.^{4}, C_w = 151 in.^{6}, maximum w_n = 11.7 in.^{2}. G/E = 0.4 is used, giving $1/a$ = 0.04787 in.$^{-1}$. For members 1 and 3, L/a = 5.74 so the stiffness factor is 7.3538 EC_w/L, and the carry-over factor is 0.2030, so the stiffness factor modified for the hinged end is 6.9548 EC_w/L. For member 2, L/a = 11.48, K = 12.6992 EC_w/L, and C = 0.953. So, at joint 2, 0.532 of the unbalanced bimoment will be distributed to Member 1 and 0.477 to Member 2. The distribution at Joint 3 is similar. The fixed end bimoments on Member 2, considering the applied torque as an analogous downwards load are -0.0615 ML at Joint 2 and $+0.0227$ ML at Joint 3. The complete bimoment distribution is illustrated in Table 5. The bimoment is found to be 193.2 k in.^{2} at Joint 2 and 78.9 k in.^{2} at Joint 3. Using the method of Ref. 9 the bimoment at Joint 2 is 193.0 k in.^{2} and at Joint 3, 79.1 k in.^{2}, giving an error of less than 1%.

Given any single span, once the end bimoments are known, the forces within the span can be determined by superposition of the forces due to the applied loading on a simply supported beam with the forces due to the end bimoments, that is, the bimoments at the supports. Forces on a simple span beam due to applied torque may be computed from the charts in Ref. 1. The bimoments on a simple span beam of length L subjected to end bimoments B_1 and B_2 are computed by the following equation

$$B(z)=B_1\left[\cosh(z/a)-\frac{\sinh(z/a)}{\tanh(L/a)}\right]+\frac{B_2\sinh(z/a)}{\sinh(L/a)} \tag{14}$$

The beam sign convention used to draw moment or bimoment diagrams is used for B in the above equation. For the previous example, these forces are plotted in Fig. 3a and 3b. In general, lengthy computations are not required to check critical stresses. In most cases, the combined normal stress due to bending moment plus bimoment governs the design over the shearing stresses due to a combination of bending, St. Venant torsion and warping torsion effects. For this problem, signficance of the negative sign of the bimoment at the supports is shown in Fig. 4a, where a positive torque, or analogous downwards load is applied to a cantilever channel section. Note that the most severe stresses, at the flange tip, are of opposite sign to the bending stresses, so the worst case of combined normal stresses occurs at the intersection of the flange and the web. For the above example, the stresses at the left support, Joint 2, are combined as shown in Figs. 4b and 4c.

Table 5. Bimoment Distribution—3-span C12x30
***FEB* Joint 2 = − 0.0615 ML = −363.1**
Joint 3 = +0.0227 ML = + 134.0

	Joint 2			Joint 3	
	.523	.477	C.O. = 0.095	.477	.523
FEB		−363.1		+134.0	
Dist 1.	189.9	+173.2		−63.9	−70.1
C.O. 1		−6.1		+16.5	
Dist. 2	+2.9	+3.2		−7.9	−8.6
C.O. 2		−0.8		+0.3	
Dist. 3	+0.4	+0.4		−0.1	−0.2
	+193.2	−193.2		+78.9	−78.9

APPLICATION OF OTHER FLEXURAL ANALYSIS METHODS

Referring to Table 2, note that for very small values of L/a, the moment distribution method can be accurately applied to determination of bimoments without any modifications to usual values for fixed end forces, stiffness or carry-over factors. It would seem that reasonable accuracy ($\pm 10\%$) can be obtained from this method for values of L/a less than about 2, bearing in mind this method inherently yields a conservative upper bound solution. Once the constraint of modifying basic flexural analysis is removed, the scope of problems that can be solved increases greatly.

The example shown in Fig. 5a shows a use of this method for beams with small values of the parameter L/a. The problem is a two-span AISI standard 8-in. × 3-in. × 0.060-in. "C" purlin.[2] The properties are J = 0.00106 in.^{4}, C_w = 12.4 in.^{6}, G/E = 0.4, $1/a$ = 0.00584 in.$^{-1}$. As in many modern roof systems, there is no direct connection between roof panels and roof framing members. Resistance to rotation of the purlin between main framing members is provided by "sag angles" lo-

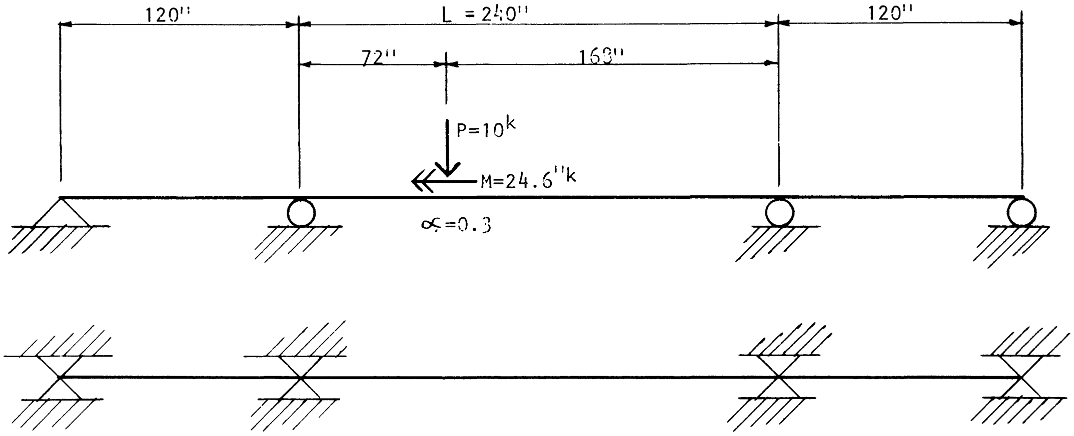

Fig. 3a. Bending and torsional support conditions

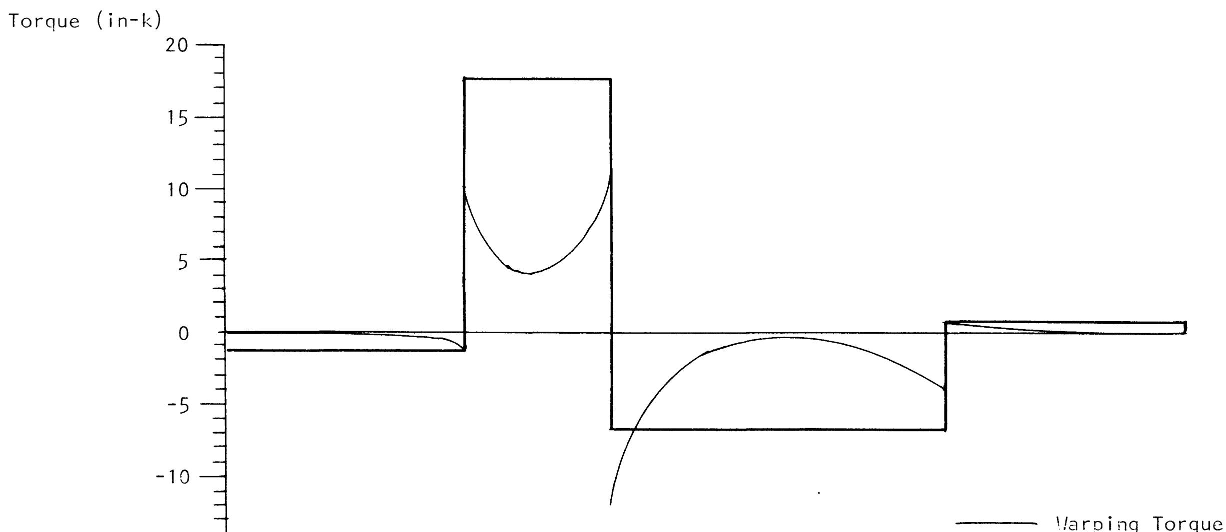

Fig. 3b. Total and warping torque

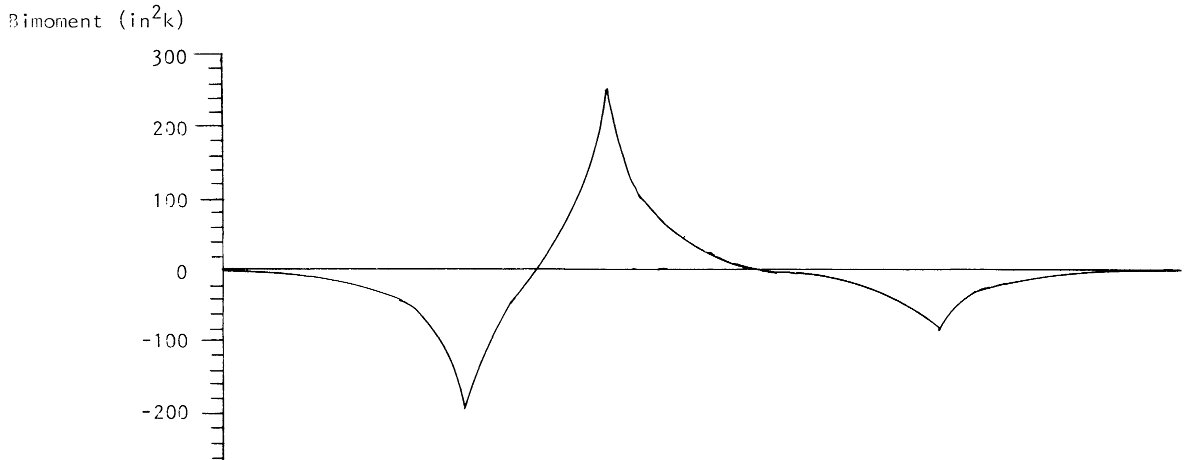

Fig. 3c. Bimoment

Fig. 3. Example problem

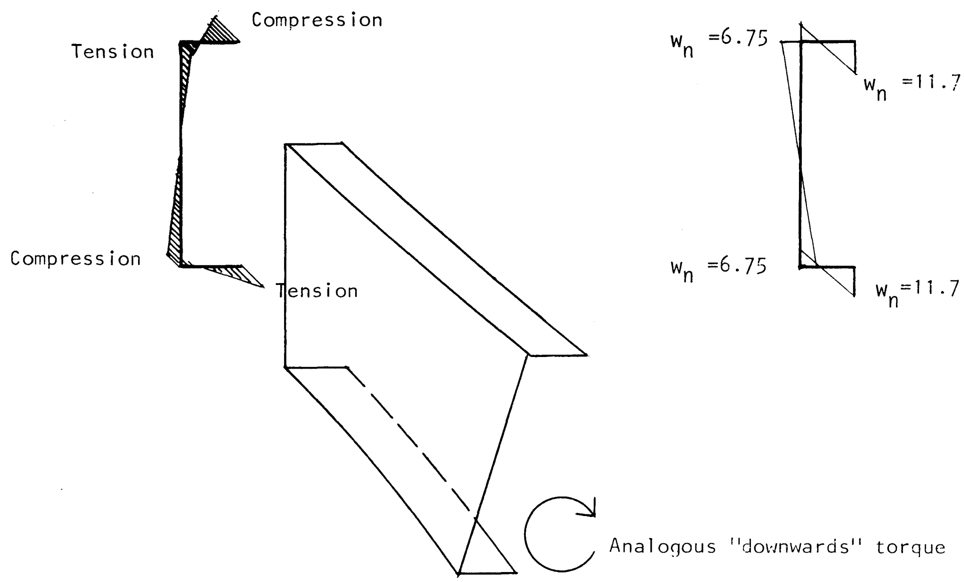

Fig. 4a. Sign of support bimoment

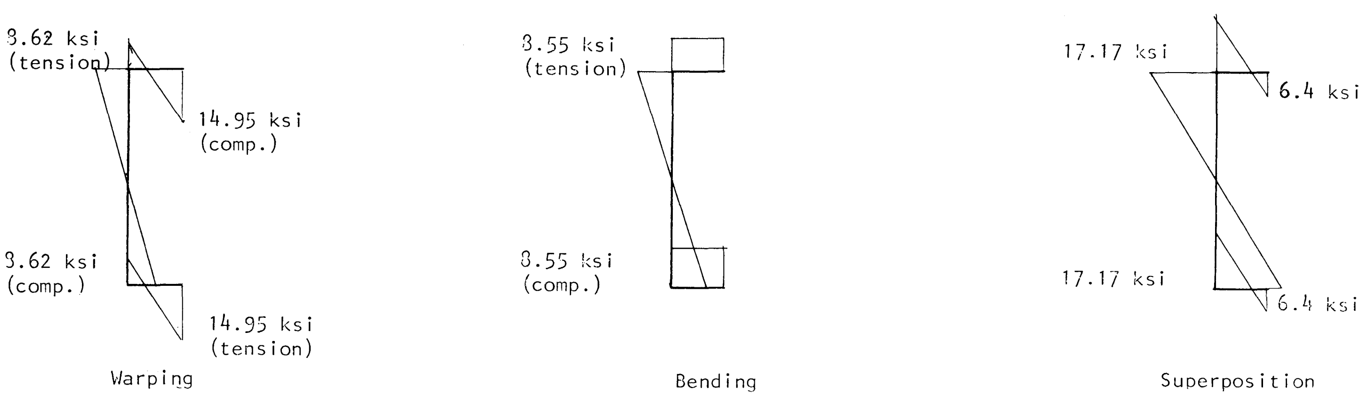

Fig. 4b. Normal stresses at Joint 2

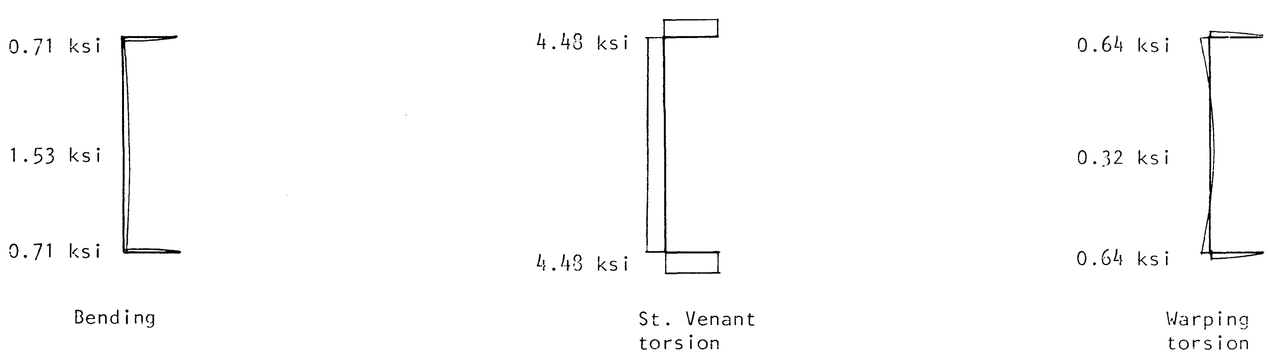

Fig. 4c. Shear stresses at Joint 2

Fig. 4. Example problem-continued

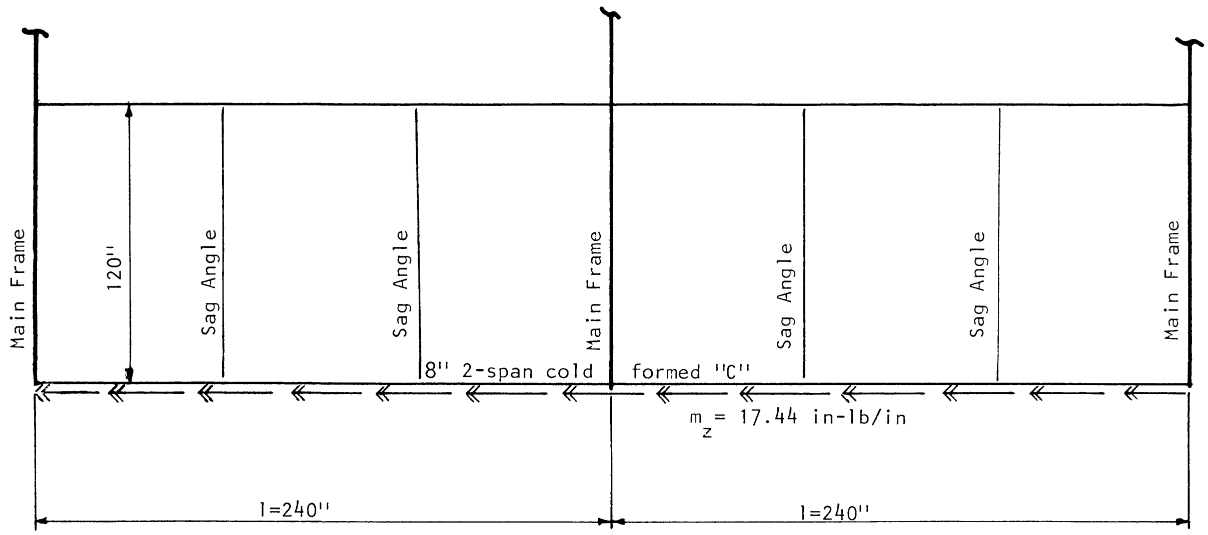

Fig. 5a. Structural layout

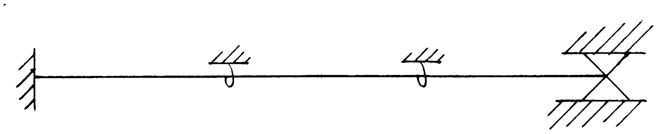

Fig. 5b. Idealized structure

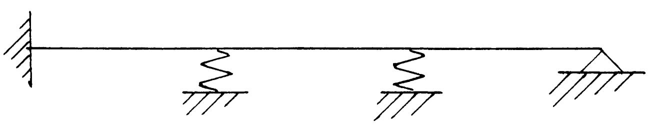

Fig. 5c. Analogous structure

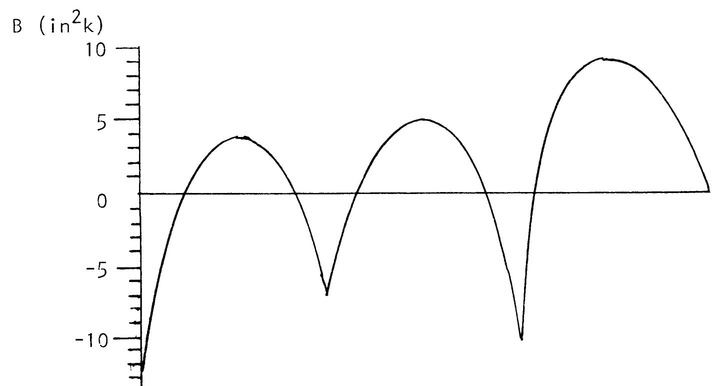

Fig. 5d. Bimoment diagram

Fig. 5. Example problem—cold-formed C-purlin

cated at third points of the span. This is not a rigid rotational restraint, but rather must be considered an elastic restraint. The sag angles are AISI standard 2-in. × 2-in. × 0.060-in. sections with I = 0.0947 in.^{4}. For a rotation of ϕ, the resisting moment developed in the sag angle bracing is $4EI\phi/L$, so an elastic twisting resistance of 94,700 in./lb. per radian is provided at the third points of the span. The load of 75 lbs./ft or 6.25 lbs./in. is applied through the top flange with an eccentricity of 2.79 in. The uniformly distributed torque load is thus equal to 17.43 in.-lbs./in. L/a in between rotational supports is equal to 0.47.

The analysis for torque loads is accomplished separately from the vertical load analysis. The structure with applied torque loading can be idealized as shown in Fig. 5b. Taking the torque load as an analogous downwards load, an analogous beam with elastic supports, as shown in Fig. 5c can be considered. This vertically loaded beam may be analyzed by any appropriate method, or continuous beam computer program. The use of the stiffness method produces the bimoment diagram shown in Fig. 5d. The warping normal stresses may then be computed using Eq. 2.

FURTHER PRACTICAL APPLICATIONS

This method permits the solution of many types of torsionally loaded, thin-walled beams, especially beams with small values of L/a. For instance, consider the differential equation for bimoment in a vertically loaded, horizontally curved beam given in Eq. 3.

$$B'' - (1/a)^2 B = \frac{M}{r} \tag{15}$$

where M is the bending moment due to vertical load and r is the radius of curvature of the beam. For small values of L/a, the second term of the left hand side vanishes. The value of B is then analogous to the bending moment on a beam with load M/r. Because of the nature of the moment diagram, B will usually have to be evaluated by a numerical procedure such as Newmark's method. Nonprismatic beams of small L/a can be analysed by the conjugate beam method, column analogy method, stiffness method or any other appropriate flexural method. Also, for beams of small L/a, the torsional stiffness is fairly simply established and related to the bending stiffness, making possible a simplified analysis of grids including warping effects.

CONCLUSION

The analogy between bending moment and bimoment provides an opportunity for structural designers to accomplish analysis of thin-walled members of open cross-section, including non-prismatic, curved or continuous members, with relative ease, often using existing computer programs. Furthermore, the analogy facilitates the understanding and visualization of problems involving warping torsion.

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NOMENCLATURE

B = bimoment (defined in Eq. 1)
C = carry-over factor
C_w = warping constant
E = modulus of elasticity
f_b = bending normal stress
f_w = warping normal stress
FEB = fixed end bimoment
G = modulus of elasticity in shear
J = torsion constant
K = stiffness factor
K' = stiffness factor modified for pinned end
L = length of a member
M_z = concentrated torque
m_z = uniformly distributed torque
r = radius of curvature
S_w = warping statical moment
t = thickness of a plate element
v = vertical displacement
z = longitudinal coordinate
α = distance from end to point of load application divided by the length of member
ϕ = angle of rotation of a member about a longitudinal axis
τ_b = bending shear stress
τ_w = warping torsional shear stress