Torsion in Closed Sections

AVINADAV SIEV

ACADEMIC DISCUSSIONS of torsion in closed sections can be found in most textbooks on elasticity; however, there is a notable lack of literature containing an engineering approach to this subject. This may be due to the fact that, until recently, pipes were the only closed sections commonly used in engineered structures. With the increased use of welding, and with developments such as "orthotropic" steel bridge decks, closed sections will undoubtedly be encountered more frequently in engineering practice.

This paper will discuss the analysis of torsion in closed sections from an engineering point of view. Emphasis is placed on the physical significance of equations, rather than exact theoretical proofs. It is hoped that this treatment will lead to a better understanding of the engineering aspects of the subject.

The advantage of a closed section under torsional loading is illustrated by a comparison of the bridge deck ribs in Fig. 1. An eccentric load P exerts a force larger than P on the right hand rib in Fig. 1a. In the case of a closed rib (Fig. 1b), the force may be replaced by load P at the centroid of the section plus a couple Pe. The load P is equally divided between the sides of the closed member, permitting savings in material. The torsion must be resisted by the rib but, as will be shown later, it is absorbed without the introduction of additional longitudinal stresses and hardly interferes with the moment-resistant capacity of the rib.

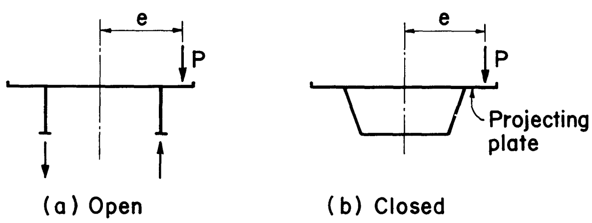

Fig. 1. Bridge sections

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TORSION IN OPEN SECTIONS OF NARROW RECTANGLES

Most structural steel open sections can be considered to be made up of combinations of narrow rectangular "building blocks".

The torsional characteristics of a rectangular element are normally given by the following equations:

$$K = \frac{bt^3}{3} \tag{1}$$

$$\tau = \frac{3M_p}{bt^2} = \frac{M_p t}{K} \tag{2}$$

$$\theta = \frac{M_p}{GK} \tag{3}$$

where

K = torsional resistance constant (torsional rigidity)
τ = torsional shear stress
M_p = primary torsional resisting moment (St. Venant)
θ = angle of twist per unit length
G = modulus of elasticity in shear

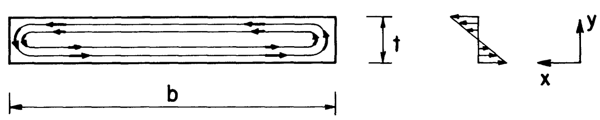

Fig. 2. Shear distribution in a narrow rectangular section

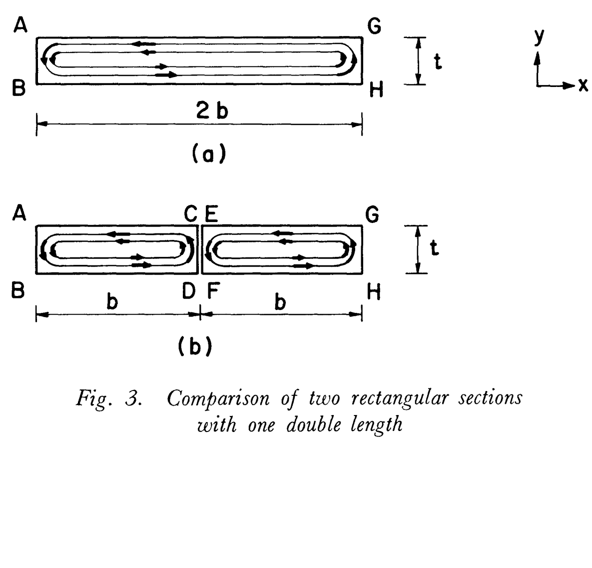

Fig. 3. Comparison of two rectangular sections with one double length

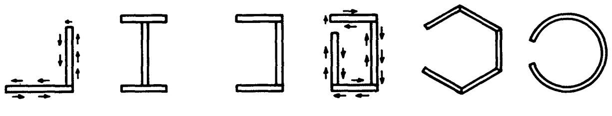

Fig. 4. Examples of sections consisting of narrow rectangles

Since torsional rigidity of a rectangular element is a linear function of its width, it is evident that the rectangular section $2b \times t$ in Fig. 3a can be considered to have the same rigidity as the assumed combination of two rectangles $b \times t$ in Fig. 3b.

The distribution of torsional shear in a single rectangular section is compared with the distribution of stresses in two rectangular elements of equivalent total width in Figs. 3a and 3b. The central region of each rectangle is stress-free in the y-direction, but contains stresses in the x-direction. There is negligible difference between the sum of the internal force couples, resulting from shear stresses near the edges of the rectangles, whether the section is a single element or is made up of component parts. The shear forces acting in the y-direction at the adjacent faces of the two-part section are self-neutralizing. The unit shear stresses acting in the x-direction as a couple in the two-part section will not be effective in the corners of the rectangles; thus, there will be a very slight reduction in torsional moment capacity from that provided by the single element section. The larger the b/t ratio, the smaller the difference in torsional moment capacity between sections represented by Figs. 3a and 3b.

In considering most standard rolled steel sections as combinations of rectangular elements, the b/t ratio is sufficiently large to make valid the assumption of equal moment capacity.

As for twist, the edges **CD** and **EF** in Fig. 3b remain straight under pure torsional loading, and the two rectangles in combination can be assumed to deform the same as the single, wider rectangle in Fig. 3a. The angle θ will be the same, since K will be the same in each case.

The same method of combining narrow rectangular elements is applicable to other shapes, such as angles, WF-beams, channels, etc. (see Fig. 4). The torsional rigidity of the shape can be considered to be approximately the sum of the torsional rigidities of the individual components. If the thickness of all the rectangles in a shape is the same, the total rigidity is that of a rectangle of total developed length. The same concept applies to curved sections, such as the slotted ring of Fig. 4f, but the shape does not provide any greater torsional strength than shapes such as Figs. 4a through 4e, assuming equal thickness and equal developed length.

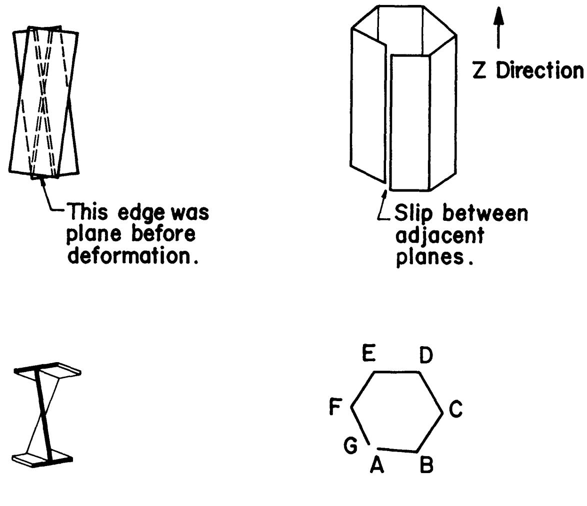

Fig. 5. Warping of open sections

WARPING

Analysis of sections such as the WF, I, channel and open polygon shows that twisting is accompanied by displacements in the z-direction, and the section does not remain plane under torsional loading unless restrained (see Fig. 5). This kind of deformation is called warping.

In the case of the open polygon (Fig. 5b), warping causes relative movement between edges **A** and **G**. This movement is prevented in the closed section (Fig. 6a) by shear induced between **A** and **G**, which is transmitted along the whole perimeter. By isolating one rectangular element of the polygon (Fig. 6b), it is seen that the vertical shear is accompanied, as is always the case, by an equal horizontal counterpart. The longitudinal shear stresses are in equilibrium, but the transverse shear stresses are additive along the perimeter of the polygon, and combine to counteract part of the external torque.

Figure 7 shows that the resistance of a closed section to external torque may be assumed to consist of two parts. For example, the deformation of plate **AB** in Figs. 7c and 7d can be arbitrarily considered as two effects: 1) twisting of the element about the longitudinal axis, and 2) shear deformation in the plane of the plate. The twisting effect is an angular displacement of line **A′B′** with respect to line **AB**, and results in shear stresses as shown in Fig. 7e. The shear deformation in Fig. 7d is a deformation of the plate in its own plane with the points **A** and **B** shifting to the right along **A′B′**, thereby transforming the rectangle into a parallelogram. This deformation is caused by pure shear in the plates as was shown in Fig. 6. The stresses due to pure shear in the closed section are shown in Fig. 7f. The actual stress distribution shown in Fig. 7g is then the sum of the two components: twisting and pure shear stress. Essentially

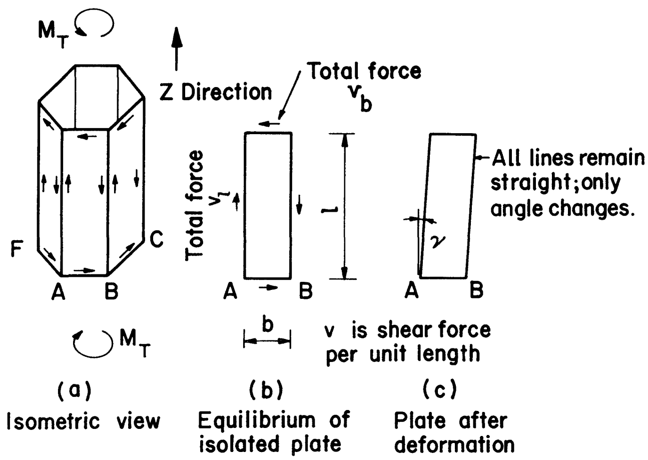

Fig. 6. Shear and deformation of closed sections

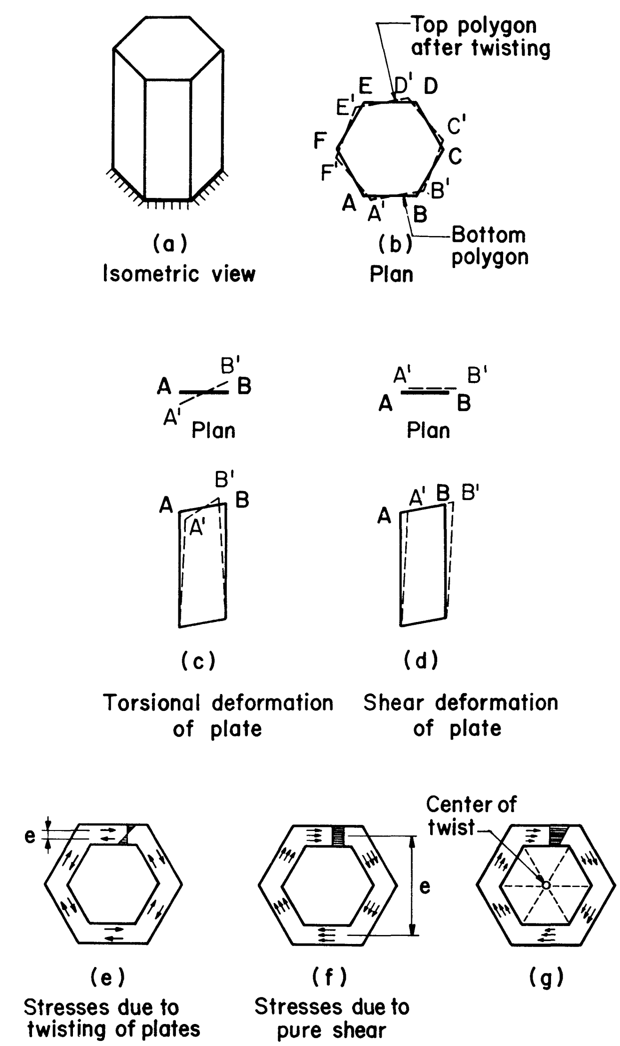

Fig. 7. Deformation and stresses of closed section

only the shear stresses due to twisting of each element, the smaller of the two components, resist pure shear in open sections.

The ratio of the two components is influenced by the geometry of the section. Even without mathematical analysis, it is obvious that the thinner the plate (compared with its other dimensions), the smaller will be the stresses due to twisting compared to the stressses due to restrained warping. Moreover, the distance, e, between the stress resultants in Fig. 7e is much smaller than in Fig. 7f.

The role of pure torsional or twisting stresses in resisting torque can be considered to be negligible in the case of closed sections. The advantage of closed sections over open sections, in which twisting is a major factor, is clear. For simplicity, the analysis in this paper will be based on the assumption that the torque is absorbed by shear forces uniformly distributed over the thickness of the plate as in Fig. 6.

CHARACTERISTICS OF CLOSED SECTIONS UNDER TORSION

From the foregoing, it may be concluded that:

a. Pure torsion M_p induces only shear stresses, unaccompanied by tension or compression stresses.
b. Shear forces acting on edges of component elements are equal. (This follows from elementary conditions of equilibrium.)
c. Shear force per unit length is uniform throughout the polygon. (Note the reference to force per unit length rather than unit stress. This follows from equilibrium conditions with regard to interaction of each plate with adjacent elements (Fig. 6b)).
d. Unit shear force is not applicable to plates projecting from a closed section (such as the cantilevers in Fig. 1b).
e. The resultant of the shear forces acting on the perimeter of a closed cross-section is zero in the x- and y-directions, so that only a moment remains.

Analysis—The capacity of a closed section can now be calculated by considering any point 0 as a reference point (Fig. 8). For the nth plate element with length a_n, the shear force is v_{an}, and the moment about 0 is

$$M_{n,0} = va_ne_n \tag{4}$$

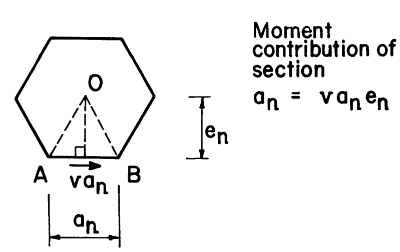

Fig. 8. Torque capacity of closed section polygon is sum of its sections

Now, $a_n e_n$ is double the area of triangle **ABO**. Thus the contribution of each section is v, multiplied by the double area of the corresponding triangle between the section edges and point 0. The overall capacity of the section is

$$M_t = 2tA_{\text{polygon}} \tag{5}$$

where A_{polygon} is the area of the polygon.

Specifically, the center lines of the plates should be taken as the sides of the polygon. This is correct for a stress distribution as in Fig. 7f. According to the actual distribution (Fig. 7g), the stress resultant is slightly off center, but the deviation is negligible.

As the plates are not necessarily of the same thickness, the stresses will also vary between them. The stress τ_n in the nth plate will be

$$\tau_n = \frac{v_n}{t_n} \tag{6}$$

and substitution in Equation (5) yields

$$M_t = 2\tau t A_{\text{polygon}} \tag{7}$$

which is a convenient form for comparison with a narrow rectangular section. Substituting K from Equation (1) in Equation (2), and bearing in mind that

$$A = bt \tag{8}$$

the torque for a narrow rectangular section can be formulated by analogy to Equation (7):

$$M_t = \tfrac{1}{3}\,\tau t A_{\text{polygon}} \tag{9}$$

The difference between Equations (7) and (9) lies not so much in the coefficients (2 against $\frac{1}{3}$) as in the character of A. For the hollow section, A is the area bounded by the plate, whereas for the narrow rectangular section, it is that of the material. The former exceeds the latter many times, hence the vast superiority of the closed section to resist torsion is evident.

Warping of Closed Sections—It has already been shown that shear stresses develop in closed sections and that there can be no relative displacement of the edges of the theoretical plate elements. Warping is reduced considerably as a result, and is often completely eliminated as will be explained later. The stresses in a closed section under torsion are, to a certain degree, similar to those involved in warping torsion. In both cases the external forces are resisted by shear forces uniformly distributed over the thickness of the plate, with larger lever arms between stress resultants. The presence of warping torsion in open sections, however, depends on compatibility between load distribution and end conditions (Fig. 9). In closed sections normal stresses are not

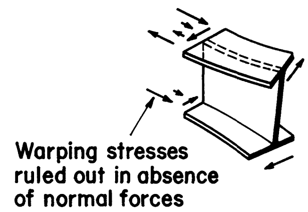

Fig. 9. Warping stresses of a cantilevered open section

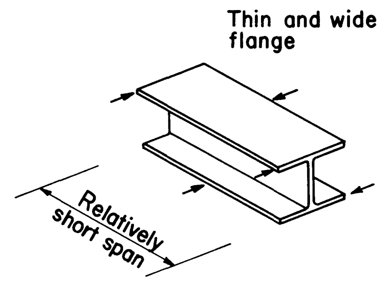

Fig. 10. When flexural stiffness of flanges is high compared with torsional stiffness, flanges act as horizontal beams

essential since the transverse shear is balanced by a longitudinal internal counterpart (Fig. 6b).

Continuing this analogy, the ratio of pure torsion stresses to warping stresses in open sections can be determined. Evidently, the higher the ratio EC_w/GK, the greater the role of the warping stresses as against those of pure torsion. If the flexural stiffness of the flanges in the planes of the flanges is very high, the lateral beam deflection is very small, and twist is limited, even though the torsional stiffness is very low (Fig. 10). This is analogous with accepted practice in fully continuous structures where the strong elements take a larger share of the acting forces. The rigidity of a plate in a closed-section is extremely high, since the pure shear forces cause no bending.

Relative slip at edges between adjacent plate elements is prevented, but the original planar condition of the edges is not necessarily preserved. The edges between adjacent plates remain straight as shown in Fig. 6c in cross-sections of uniform thickness. Now, let us assume that the end-polygon remains planar i.e., no warping, and consider the necessary conditions (Fig. 11). For small angles of twist

$$u_n = e_n \tan \phi = e_n \phi \tag{10}$$

where

u_n = displacement of the end of the nth plate in its plane
e_n = distance from center of twist
ϕ = total angle of twist

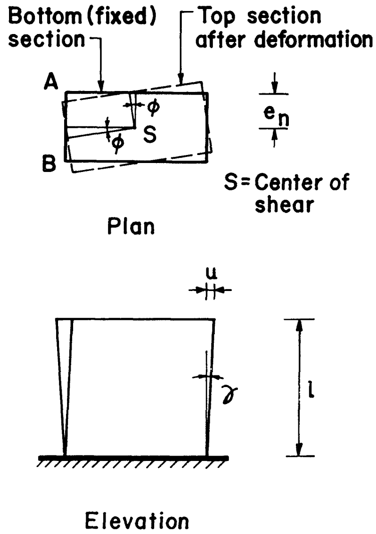

Fig. 11. Geometric conditions for warp-free torsion of closed section

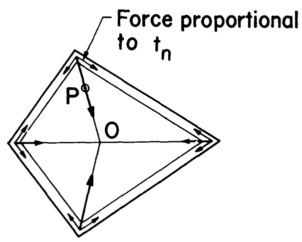

Fig. 12. Graphical solution for center of twist

As the plate is further removed from the center of twist, the larger will be the displacement at its end. On the other hand

$$u_n = l \tan \gamma = l\gamma_n \tag{11}$$

where γ is the shearing strain, actually an angle. Equating the right-hand side of Equation (10) to that of Equation (11), we have

$$e_n\phi = l\gamma_n \tag{12}$$

$$\frac{\gamma_n}{e_n} = \frac{\phi}{l} = \theta = \text{constant for all plates} \tag{13}$$

This implies that the ratio of shear strain to distance from twist center is equal for all plates, and those nearer the twist center should have a smaller shear strain. The latter is given by

$$\gamma_n = \frac{\tau_n}{G} = \frac{v}{t_nG} \tag{14}$$

Substituting in Equation 13 and rearranging:

$$e_nt_n = \frac{lv}{\phi G} = \text{constant} \tag{15}$$

If this relationship does not exist, then line **AB**, which was in a plane perpendicular to the axis, will be warped out of the plane after the deformation.

It may be concluded that to avoid warping the plates nearer the center of twist should be thicker, and those further removed should be thinner. In this case, the center of twist can be found graphically (Fig. 12).

Imaginary forces proportional to plate thickness and directed towards the center are assumed to act at the edge of each plate. A line in the direction of the resultant is drawn through each corner. The sum of the moments of the two forces t_{n-1} and t_n about any point **P** on the line of the resultant is zero

$$e_{n-1}t_{n-1} + e_nt_n = 0 \tag{16}$$

and

$$|e_{n-1}t_{n-1}| = |e_nt_n| \tag{17}$$

Equation (15) is thus satisfied for any point on the line of the resultant with respect to the adjoining plates, and if all resultants meet in a single point **O**, this point is the center of twist and Equation (15) is satisfied for all plates. In this case there will be no twist. However, in all other cases, the plates must undergo a certain rotation, so that the overall displacement of one end relative to the other, is in accordance with the geometrical conditions specified before. This rotation causes the warping W.

Analysis of the warping is based on the condition of no slip between plates (Fig. 13). If the displacement of point **A** in the z-direction is denoted by W_A, that of point **B** is

$$W_B = W_A + (\beta_1 - \gamma_1)a_1 \tag{18}$$

and that of point **C** is that above plus the contribution due to the section **BC**:

$$W_C = W_B + (\beta_2 - \gamma_2)a_2 = W_A \pm (\beta_1 - \gamma_1)a_1 + (\beta_2 - \gamma_2)a_2, \text{ etc.} \tag{19}$$

In short, the longitudinal displacement of each joint is the sum of the terms of Equation (18) from point **A**

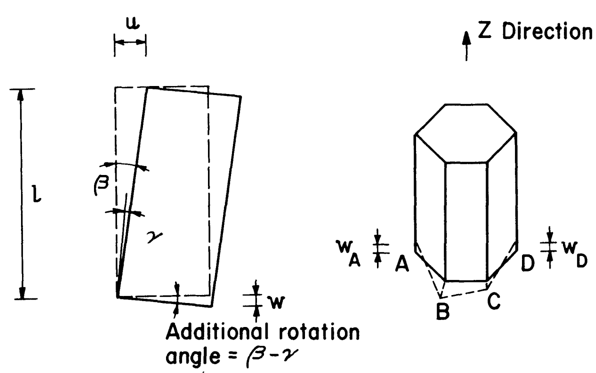

Fig. 13. Geometric relations when plate undergoes shear strain and rotation

to that in question. Each term may be positive (for $\beta > \gamma$) or negative ($\beta < \gamma$), and thus the longitudinal displacement of each consecutive point may increase or decrease. The sequence concludes again with point **A**, in accordance with the non-slip condition:

$$W_A + \Sigma(\beta_n - \gamma_n)a_n = W_A$$

or

$$\sum(\beta_n - \gamma_n)a_n = 0 \tag{20}$$

where the summation symbol covers all plates forming the closed section. The value of β_n is obtainable from the geometrical condition, using Equation (10) and rewriting Equation (11) as

$$u_n = l \tan \beta_n = l\beta_n \tag{21}$$

Substituting u_n from Equation (10) and transforming, β_n is obtained as:

$$\beta_n = e_n \, \phi/l = e_n\theta \tag{22}$$

where θ is the angle of twist per unit length, unknown for the time being. Substitution of β_n and γ_n in Equation (20) yields an equation for the unknown θ:

$$\sum \left(e_n\theta - \frac{v}{t_nG}\right) a_n = 0 \tag{23}$$

Rearranging,

$$\theta \sum e_n a_n = \frac{v}{G} \sum \frac{a_n}{t_n} \tag{24}$$

and since

$$\Sigma \, e_n a_n = 2A_{\text{polygon}} \tag{25}$$

θ is obtained as

$$\theta = \frac{v}{2GA_{\text{polygon}}} \sum \frac{a_n}{t_n} \tag{26}$$

θ may now be substituted in Equation (22) to yield β_n, and further substitution of β_n in Equations (18) and (19) yields the warping.

Closed Sections with Partitions—Consideration of bending in the plates, or buckling, often necessitates recourse to partitioned closed sections (Fig. 14). The latter are statically indeterminate, since any of those shown in Fig. 15 in solid lines are essentially capable of absorbing torsion, and the overall torsional capacity is the sum of the component capacities. The only condition for equilibrium is that the shear force v per unit length be the same for any junction of two plates (Fig. 16a), and that for any junction of three plates (Fig. 16b):

$$v_1 = v_2 + v_3 \tag{27}$$

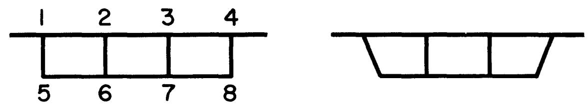

Fig. 14. Partitioned closed sections

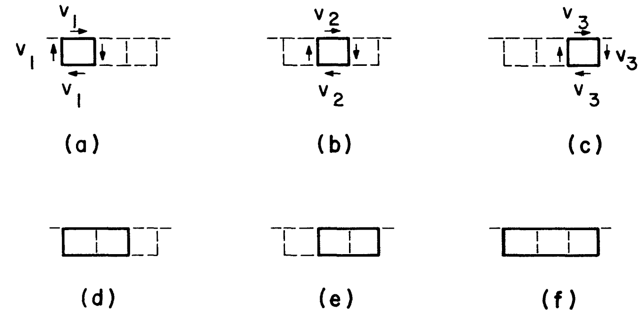

Fig. 15. Parts of section capable of absorbing torsion

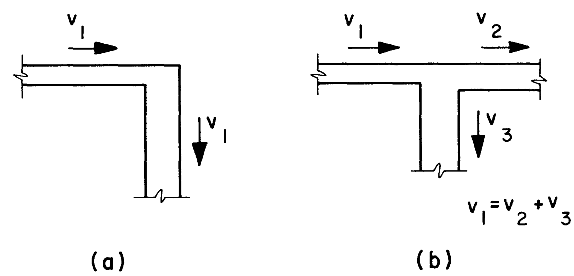

Fig. 16. Condition of equilibrium at junctions of plates

The direction of the arrow is considered positive. Only in plastic design should these conditions be satisfied, and as many sections as possible should be assumed to yield.

In elastic analysis, the deformations should be taken into account, and equations based on non-slip in the rectangles of Fig. 15 should be formulated. These equations are similar to Equation (23), except that v may vary along the perimeter according to Equation (26), whereby

$$\sum \left(e_n\theta - \frac{v_n}{t_nG}\right) a_n = 0 \tag{28}$$

The equations need be written only for part of the rectangles, the rest being satisfied automatically. Judicious choice of the rectangles, however, may reduce the number of unknowns (and equations) and simplify computations.

Need for Stiffeners—In the preceding discussion, only shear stresses were mentioned. Stiffeners should, however, be provided in two cases. First, when the shear stresses may cause buckling of the plate, where the design should then be the same as that of webs in pure shear. Second, when the manner of application of the external torque is such that transverse shear force is not applied to each plate element, and the section is in-

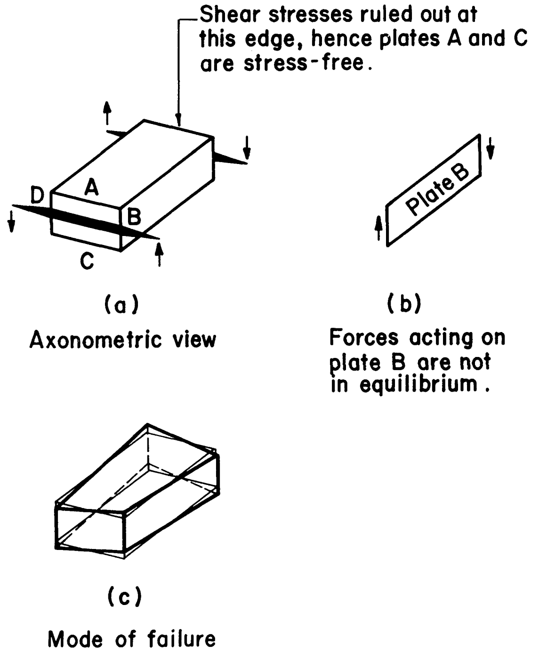

Fig. 17. Sections without stiffeners are unstable in pure torsion

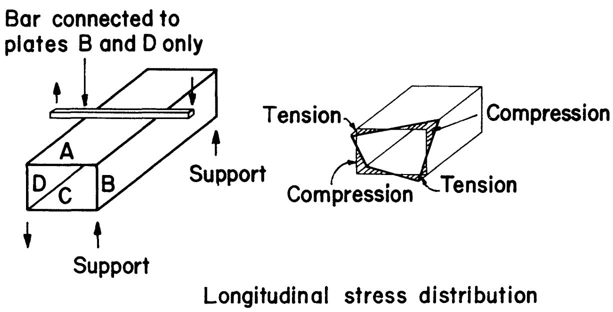

Fig. 18. Sections without stiffeners absorb torsion by bending

capable of producing this distribution as, for example, when no interior diaphragms are present. Two examples of this category are shown in Figs. 17 and 18. If the edge between plates **A** and **B** is assumed to be a hinge, only capable of transmitting shear between the plates and forces are applied through simple supports only, then pure torsion is ruled out. For equilibrium of the system shown in Fig. 17, the shear must be distributed between the plates by some kind of stiffness at the edges. If external forces act only at the edges, no intermediate stiffeners are needed. Obviously, if the plates are stiff enough, the shear is transmitted to plates **A** and **C** by transverse bending in the vicinity of the ends. The mode of failure shown in Fig. 17c is ruled out in this case.

The system shown in Fig. 18 is slightly different. Under the same conditions torsion will result in a couple, inducing moments in plates **B** and **D**. Each of the latter act as a simply-supported beam, while **A** and **C** act as flanges reducing the stresses in **B** and **D**. Shears and stresses as shown in Fig. 18, are as in hipped-plate structures. Equilibrium without stiffeners is possible in this case, but shear stresses are higher and longitudinal stresses are also present; the latter combined with longitudinal stresses due to other loads reduces the capacity of the system. If possible, diaphragms should be inserted at both ends, and at the section where torsion is applied. Whenever external torsion is distributed throughout the span, or likely to arise at any point (as in bridges), a set of stiffeners should be specified. The resulting stress distribution for torsion applied between stiffeners is as shown in Fig. 18. These stresses may be considered as acting on a secondary system, while the main system acts in pure shear as shown before.

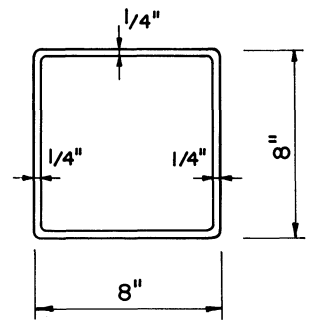

Fig 19. Hollow square section for Example A

NUMERICAL EXAMPLES

Example A

Given: Torque of 50 kip-ft acting on a hollow square structural section, 8 x 8 in., wall thickness $\frac{1}{4}$ in., $G = 12 \times 10^6$ psi. See Fig. 19.

Required: Shear stress τ, and angle of twist θ.

Solution: For simplicity, the section is assumed perfectly rectangular, disregarding the rounded corners.

$$A_{\text{polygon}} = 7.75 \times 7.75 = 60 \text{ in.}^2$$

Equation (7) rearranged:

$$\tau = \frac{M_t}{2tA_{\text{polygon}}} = \frac{50 \times 12}{2 \times \frac{1}{4} \times 60} = 20 \text{ ksi}$$

Since the section is symmetrical, the center of twist can be found according to the procedure shown in Fig. 12 and there is no warping. The angle of twist is obtainable with the aid of Equations (13) and (14):

$$\theta = \frac{\gamma}{e} = \frac{\tau}{Ge} = \frac{20{,}000}{12 \times 10^6 \times 3\frac{7}{8}} = 0.43 \times 10^3 \text{ in.}^{-1}$$

Example B

Given: Same torque and section as in Example A, except that the section is slotted (open, as in Fig. 5b).

Required: Shear stress.

Solution: The section perimeter is

$$b = 4 \times 7.75 = 31 \text{ in.}$$

By Equations (1) and (2)

$$\tau = \frac{3M}{bt^2} = \frac{3 \times 50 \times 12}{31 \times 0.25^2} = 930 \text{ ksi}$$

The stresses are far above the yield point. If a closed section (without a slot) is used, instead of a slotted section, the bar is 46.5 times stronger.

Example C

Given: Slotted section, as above.

Required: Moment that will produce the angle of twist $\theta = 0.43 \times 10^{-3}$ in.$^{-1}$ found in Example A, and the corresponding stresses.

Solution:
By Equation (1)

$$K = \frac{bt^3}{3} = \frac{31 \times 0.25^3}{3} = 0.162 \text{ in.}^3$$

From Equation (3), transposed,

$$M_p = \theta GK = 0.43 \times 10^{-3} \times 12 \times 10^6 \times 0.162 = 835 \text{ lb-in.} = 0.063 \text{ kip-ft}$$

$$\tau = \frac{3M}{bt^2} = \frac{3 \times 0.835}{31 \times 0.25^2} = 1.3 \text{ ksi}$$

This, in conjunction with Example A, yields the exact stresses for unslotted sections instead of the approximate ones. For $\theta = 0.43 \times 10^{-3}$ in.^{3}, the actual moment is the sum of the moments in the two examples

$$M_{\text{actual}} = 50 + 0.069 = 50.069 \text{ kip-ft}$$

The shear distribution is as in Fig. 7g. For the external face:

$$\tau = 20 + 1.3 = 21.3 \text{ ksi}$$

For the internal face:

$$\tau = 20 - 1.3 = 18.7 \text{ ksi}$$

The calculated increase in moment capacity is negligible. The variation in stress is greater with a thicker plate for sections of the same overall dimensions. In plastic design, this variation may be disregarded, since the whole thickness is under yield stress at failure.

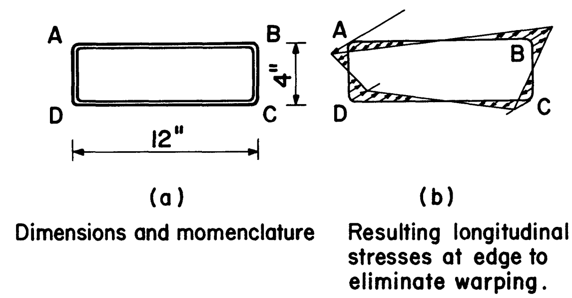

Fig. 20. Hollow rectangular section for Example D

Example D

Given: A torque of 50 kip-ft acts on a hollow rectangular section 12 × 4 with ¼ in. wall thickness (Fig. 20). Warping is ruled out because of end condition.

Required: Shear stresses τ, angle of twist θ, and warping at the corners.

Solution: $A_{\text{polygon}} = 11.75 \times 3.75 = 44.2 \text{ in.}^2$

$$\tau = \frac{50 \times 12}{2 \times \frac{1}{4} \times 44.2} = 27.2 \text{ ksi}$$

Note that the material is the same as in Example A, but the stresses are considerably higher.

The angle of twist is obtainable with the aid of Equation (25). As the plate is of constant thickness, Equation (25) reduces to

$$\theta = \frac{\tau}{2GA_{\text{polygon}}} \sum a_n$$

$$\theta = \frac{27.2}{2 \times 12 \times 10^3 \times 44.2} \times (2 \times 11.75 + 3.75) = 0.795 \times 10^{-3} \text{ in.}^{-1}$$

The twist is almost double that of a square section of the same area and moment. Substituting Equation (22) in Equations (18) and (19),

$$W_B = W_A + \left(e_n\theta - \frac{\tau}{G}\right) a_n, \text{ and so on for } \mathbf{C} \text{ and } \mathbf{D}$$

Assuming tentatively that $W_A = 0$,

$$W_B = \left(1.875 \times 0.795 \times 10^{-3} - \frac{27.2}{12 \times 10^3}\right) \times 11.75 = -8.95 \times 10^{-3} \text{ in.}$$

$$W_C = -8.95 \times 10^{-3} + \left(5.875 \times 0.795 \times 10^3 - \frac{27.2}{12 \times 10^3}\right) \times 3.75 = 0$$

and

$$W_D = W_B = -8.95 \times 10^{-3} \text{ in.}$$

The resulting longitudinal stresses are shown schematically in Fig. 20b. Coplanarity of all four corners indicates tension at **A** and **C**, and compression in **B** and **D**. These stresses are in equilibrium and their distribution is only affected in the vicinity of the edge. Calculation of these stresses is outside the scope of this paper.

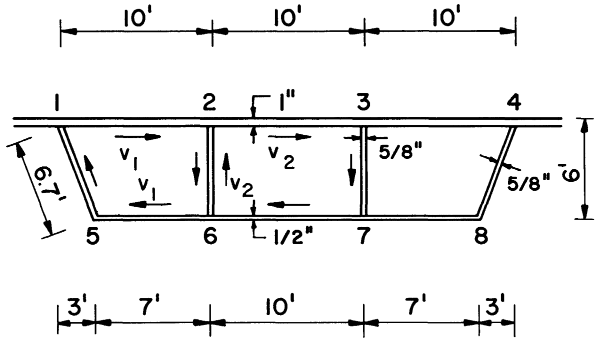

Fig. 21. Section for Example E

Example E

Given: A torque of 500 kip-ft acting on a multiple closed bridge section, dimensions and geometry as in Fig. 21.

Required: Shear stresses in plates, and angle of twist (neglecting the contribution of the top cantilevers and ribs).

Solution: The unknowns are v_1, v_2 and θ. For section **1-2-6-5** the shear in line **2-6** is $v_1 - v_2$, and for section **2-3-7-6** it is $v_2 - v_1$. This takes care of the condition for equilibrium expressed in Equation (26).

The joint action of the three quadrilaterals is given by:

$$M_t = 2v_1 \times 2A_{\text{polygon } 1-2-5-6} + v_2 \times 2A_{\text{polygon } 2-3-6-7}$$

where

$$A_{\text{polygon } 1-2-5-6} = \frac{10 + 7}{2} \times 6 = 51 \text{ ft}^2$$

$$A_{\text{polygon } 2-3-6-7} = 10 \times 6 = 60 \text{ ft}^2$$

so that the first equation is:

$$500 = 204v_1 + 120v_2 \text{ kip-ft}$$

and their mutual compatibility (non-slip) is given by:

$$2\theta G = \frac{1}{A_{\text{polygon}}} \sum \frac{v_n a_n}{t_n}$$

The second equation is:

$$2\theta G = \frac{12}{51}\left[v_1\left(\frac{10}{1} + \frac{6 \times 8}{5} + \frac{7 \times 2}{1} + \frac{6.7 \times 8}{5}\right) - v_2 \frac{6 \times 8}{5}\right]$$

and the third equation is:

$$2\theta G = \frac{12}{60}\left[v_2\left(\frac{10}{1} + 2\frac{6 \times 8}{5} + \frac{10 \times 2}{1}\right) - 2v_2 \frac{6 \times 5}{8}\right]$$

respectively.

$v_1 = 1.56$ kip/ft
$v_2 = 1.53$ kip/ft
$\theta = 3.08 \times 10^{-7}$ in.$^{-1}$

$$\tau_{1-2} = \frac{v_1}{1 \times 12} = 0.13 \text{ ksi}$$

$$\tau_{2-0} = \frac{v_1 - v_2}{\frac{5}{8} \times 12} \sim 0$$

$$\tau_{5-6} = \frac{v_1}{\frac{1}{2} \times 12} = 0.26 \text{ ksi}$$

If loading is increased beyond the elastic limit, plate **5-6** will yield first, and then plate **6-7**, and v_1 will equal v_2.

NOMENCLATURE

a	Width of a rectangular element
a_n	Width of rectangular element for nth plate
b	Length of a rectangular element
e	Eccentricity; distance between stress resultants
e_n	Distance from center of twist
l	Length of a rectangular element
l_n	Length of rectangular element for nth plate
n	Subscript denoting nth plate element
t	Thickness of a rectangular element
t_n	Thickness of rectangular element for nth plate
u_n	Displacement of end of nth plate in its plane
v	Shear force per unit length
v_n	Shear force per unit length for nth plate
A_{polygon}	Area of a polygon
C_w	Warping resistance constant
E	Modulus of elasticity
G	Modulus of elasticity in shear
K	Torsional resistance constant
M_p	Primary or pure torsional resisting moment
M_t	Total torsional resisting moment
P	Concentrated load
W	Warping
w_A, w_B, w_C	Displacement of points A, B and C
EC_w/GK	Ratio of warping rigidity to torsional rigidity
β	Angle of rotation
β_n	Angle of rotation of nth plate
γ	Shearing strain; angle of rotation
θ	Angle of twist per unit length
τ	Torsional shear stress
ϕ	Total angle of twist