

Simplified Approach to the Analysis and Design of Columns with Imperfections

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INTRODUCTION

The design of columns makes an extensive use of the column curves. A column curve is a plot of the strength of the column (in stress or load form) vs. its slenderness ratio. The column curve for a perfectly straight column in the elastic or inelastic range can easily be derived using the eigenvalue approach. However, columns in actual building frames are rarely straight. The derivation of the column curve for an initially crooked column is very complex, with recourse to laborious numerical technique often necessary. The intent of this paper is to present a simple method to assess the strength of columns with initial out-of-straightness. A beam-column approach is used in the method. And the validity of the approach is determined by comparing it with existing column design formulas and with more elaborate computer models. A computer program is also written to help the designer use the proposed approach.

In this paper, the fundamental theories of perfectly straight and initially crooked columns will first be reviewed. The derivation of the proposed column equation will then follow. The discussion will be concluded with design examples using three different formats:

- Allowable Stress Design (AISC, Part I)
- Plastic Design (AISC, Part II)
- Load and Resistance Factor Design (AISC, Part III, proposed)

FUNDAMENTAL COLUMN THEORIES

The derivation of the present column design formulas can be related to one of the following two methods:

1. Bifurcation (Eigenvalue) Approach
2. Stability (Load-Deflection Approach)

Bifurcation Approach (for perfectly straight columns)

In the bifurcation approach, the load corresponding to the state where equilibrium is possible both in the straight and slightly bent positions is calculated using the linear

column theory. A linear differential equation was written for the column in a slightly bent configuration and the eigenvalue of the characteristic equation to this differential equation is

$$P_E = \frac{\pi^2 EA}{\left(\frac{KL}{r}\right)^2} \quad (1)$$

where

- P_E = Euler load
- E = modulus of elasticity
- A = cross-sectional area
- KL = effective length of the column
- r = radius of gyration of the cross section

The Euler Load P_E makes the point of bifurcation of equilibrium for an elastic column and is depicted schematically in Fig. 1 (curve i). As the applied load reaches P_E , the lateral deflection δ will increase without bound and the column will fail by elastic instability.

Equation 1 is valid only if the member is perfectly

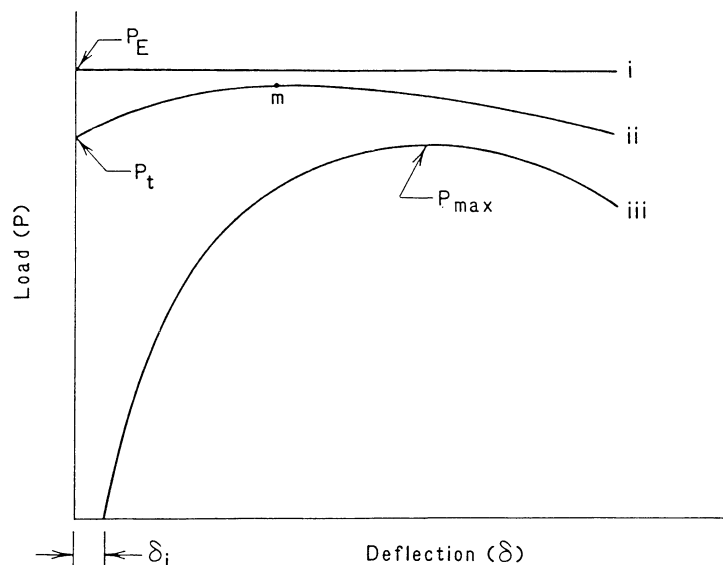


Fig. 1. Load-deflection (P - δ) curves for pin-ended columns

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straight and fully elastic. It represents the behavior of slender columns as long as the axial stresses in the member remain below the proportional limit. When the axial stress reaches the proportional limit, yielding will occur, and thus, for short columns, the yield load P_y rather than the Euler load P_E will control.

Figure 2 shows a plot of the normalized Euler load P_E / P_y with a yielding cutoff. The solid line shows the load that controls for a given non-dimensional slender-

$$\text{ness } \lambda \left(\lambda = \frac{1}{\pi} \sqrt{\frac{F_y}{E} \frac{KL}{r}} \right).$$

Test results of a number of steel columns indicate the buckling load for columns of intermediate slenderness falls significantly below those given by the solid curve in Fig. 2. In the early investigations, it was concluded this reduction in strength was due primarily to the presence of residual stresses incurred during the manufacturing process.

Figure 3 shows the stress-strain ($\sigma - \epsilon$) relationship for mild structural steel obtained from a coupon test (dashed line) and a stub column test (solid line). The $\sigma - \epsilon$ curve for the coupon test of a mild structural steel can be represented schematically as elastic-perfectly plastic. Since a coupon is free of residual stress, all of

its fibers remain elastic until the applied stress reaches the yield strength after which plastic flow will occur. In contrast, the $\sigma - \epsilon$ curve for the stub column curve shows a gradual yielding. This is because the fibers that have compressive residual stress will yield first, followed by yielding of the fibers with tensile residual stress as the applied stress is increased. Thus, yielding is a gradual, but not a simultaneous, process as in the case of a tensile coupon.

The slope of stub column stress-strain curve is called the tangent modulus E_t . To account for residual stress, Eq. 1 can be modified by replacing E by E_t ,³ i.e.

$$P_t = \frac{\pi^2 E_t A}{\left(\frac{KL}{r}\right)^2} \quad (2)$$

in which P_t is the tangent modulus load.

Since P_t is a function of E_t , and E_t reflects the influence of residual stress, use of the tangent modulus load takes into account the effect of material non-linearity.

In lieu of a stub-column test, the tangent modulus load can be determined analytically by assuming a residual stress pattern. According to the concept¹⁴ that (1) the axial load and bending moment increase simultaneously when the critical load has just been passed, and (2) no strain-reversal occurs as the column buckles, only the

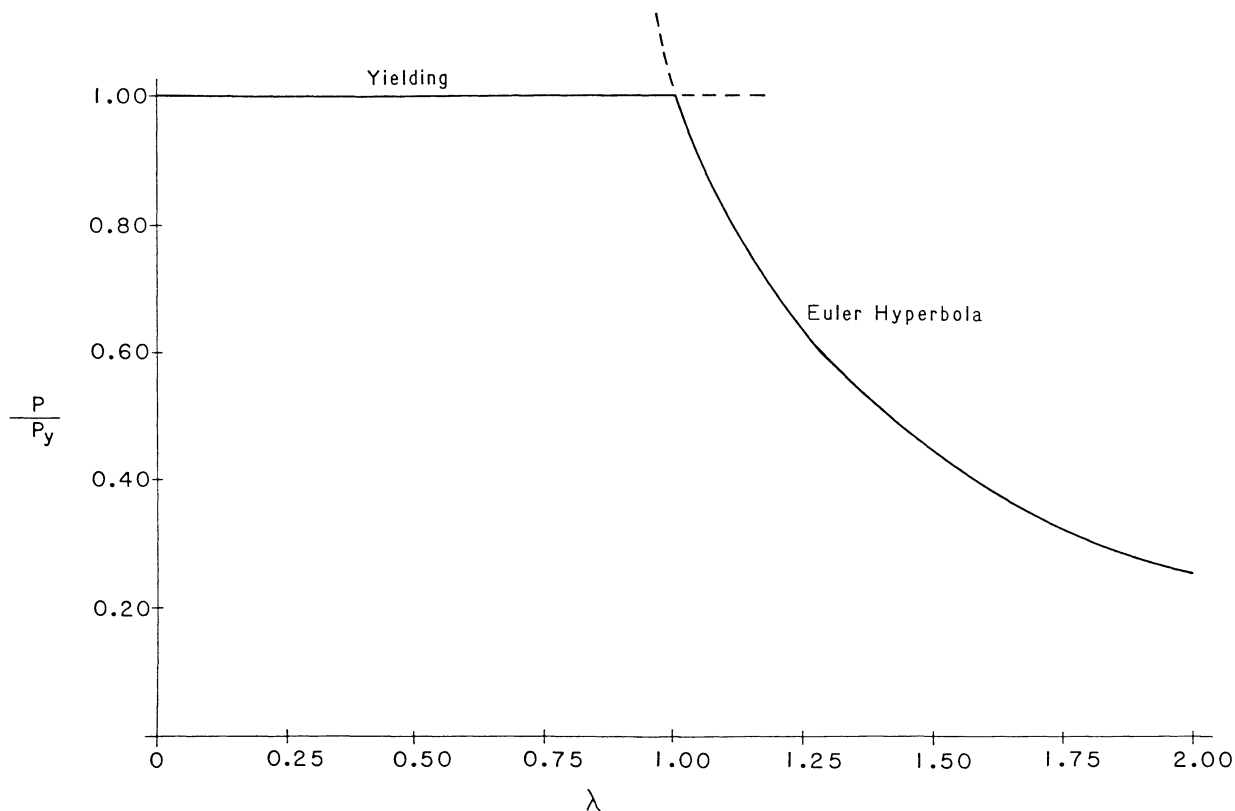


Fig. 2. Euler curve with yielding cutoff

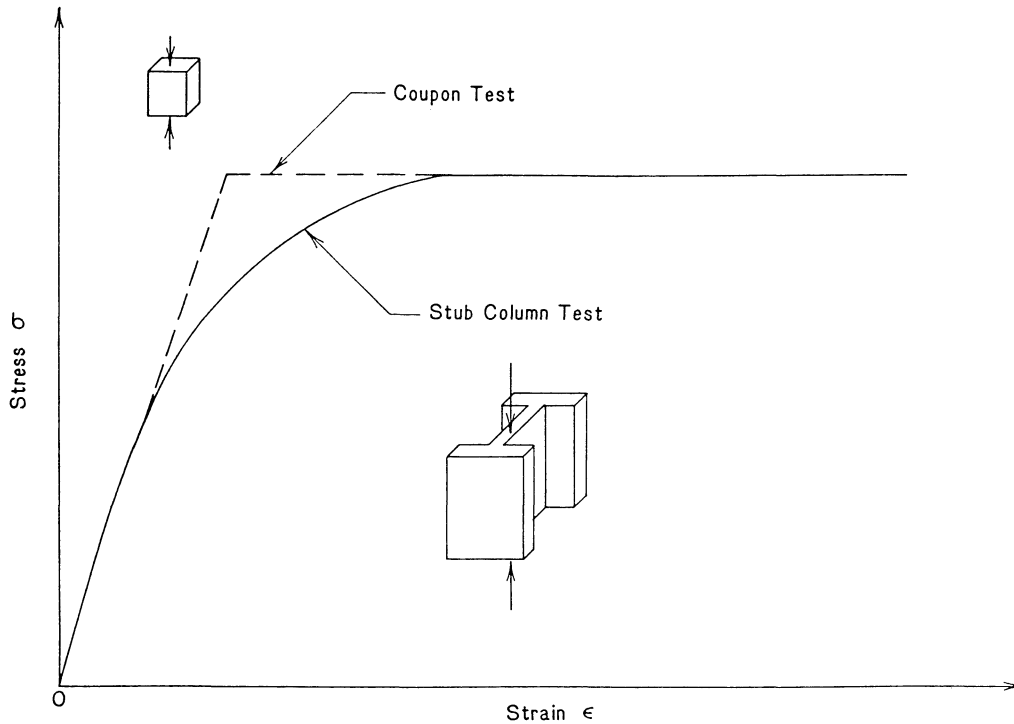


Fig. 3. Stress-strain relationship for steel

elastic core of the cross section will be effective in carrying the applied force. Thus, the tangent modulus load can be written as⁶

$$P_t = \frac{\pi^2 EA \left(\frac{I_e}{I} \right)}{\left(\frac{KL}{r} \right)^2} \quad (3)$$

in which I_e is the moment of inertia of the elastic part of the cross section. The ratio I_e/I depends on the residual stress distribution, the cross-sectional shape and the axis of buckling of the column.

The tangent modulus load P_t marks the point at which lateral deformation begins for perfectly straight inelastic column. This point is shown schematically in Fig. 1 (curve ii). It can be shown that after the load has reached P_t , an increase in lateral deflection δ is accompanied by a corresponding slight increase in load P until point m is reached. After that, the load decreases as the lateral deformation increases. Finally, the column will fail by inelastic instability as the lateral deformation becomes excessive. Since the maximum load (the load that corresponds to point m) is higher than P_t , the tangent modulus load represents a lower bound solution to the perfectly straight inelastic column problem. Nevertheless, the Column Research Council (CRC) recommended the tangent

modulus load as the ultimate load of columns failed in the inelastic range.

By using Eq. 3 and an assumed residual stress distribution, tangent modulus curves can be generated easily¹³ for idealized I-sections (I-section with the contribution of the web-to-column strength neglected).

Based on these curves, as well as on the test results of a number of small and medium-sized hot-rolled wide-flange shapes of mild structural steel with $F_y = 33$ ksi (228 MN/m^2), the Column Research Council (CRC) recommended in the second edition of the Guide⁷ a parabola proposed by Bleich³ to approximate column strength in the inelastic range.

$$F_{cr} = F_y - B \left(\frac{KL}{r} \right)^2 \quad (4)$$

B was chosen equal to $\frac{F_y^2}{4 \pi^2 E}$ so that the parabola merges with the Euler curve at $F_{cr} = 0.5 F_y$. The slenderness ratio corresponding to $F_{cr} = 0.5 F_y$ is designated as C_c where:

$$C_c = \sqrt{\frac{2\pi^2 E}{F_y}} \quad (5)$$

Thus, for columns with slenderness ratios less than, or equal to, C_c , the CRC curve assumes the shape of a

parabola (Eq. 4) and for slenderness ratio exceeding C_c , the CRC curve is the Euler hyperbola, i.e.

$$F_{cr} = F_y \left[1 - \frac{(KL/r)^2}{2C_c^2} \right] \text{ for } \frac{KL}{r} \leq C_c \quad (6)$$

$$F_{cr} = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} \text{ for } \frac{KL}{r} > C_c$$

Equation 6 expresses the CRC curve in stress form. The corresponding load form in terms of the non-dimensional slenderness ratio λ is

$$\begin{aligned} P/P_y &= 1 - 0.25 \lambda^2 \text{ for } \lambda \leq \sqrt{2} \\ P/P &= \lambda^{-2} \text{ for } \lambda > \sqrt{2} \end{aligned} \quad (7)$$

The CRC curve, divided by a variable factor of safety of

$$\left[\frac{5}{3} + \frac{3}{8} \frac{(KL/r)}{C_c} - \frac{1}{8} \left(\frac{KL/r}{C_c} \right)^3 \right]$$

in the inelastic range ($F_{cr} \geq 0.5F_y$) and a constant factor of safety of 23/12 in the elastic range ($F_{cr} < 0.5F_y$), gives the AISC curve contained in the present Specification¹¹ for the design of steel columns (Fig. 4).

Stability Approach (for initially crooked columns)

In the bifurcation approach (1) the column is assumed to be perfectly straight and (2) only the critical load corresponding to the point of bifurcation of equilibrium is obtained. In contrast, the stability approach takes into consideration that (1) the column can have initial geometric imperfections; and (2) the complete load-deflection curve starting from initial loading to final failure is traced by numerical techniques. The maximum load-carrying capacity of the column is obtained from the peak point of this load-deflection curve.

One such load-deflection curve is shown in Fig. 1 (curve iii). Unlike a perfectly straight column, an initially crooked column starts to bend as soon as the load is applied. To obtain the column curve, a number of these load-deflection curves have to be traced. Each load-deflection curve is evaluated for a specific slenderness ratio (L/r). The maximum load obtained from each curve establishes a point on the column curve. When enough points have been established, the column curve can be drawn by curve fitting. This process is shown schematically in Fig. 5. It should be noted the procedure is very laborious and time-consuming, since the use of elaborate numerical techniques is often necessary to trace the load-deflection curves. Furthermore, since only discrete points can be established on the column curve, curve-fitting

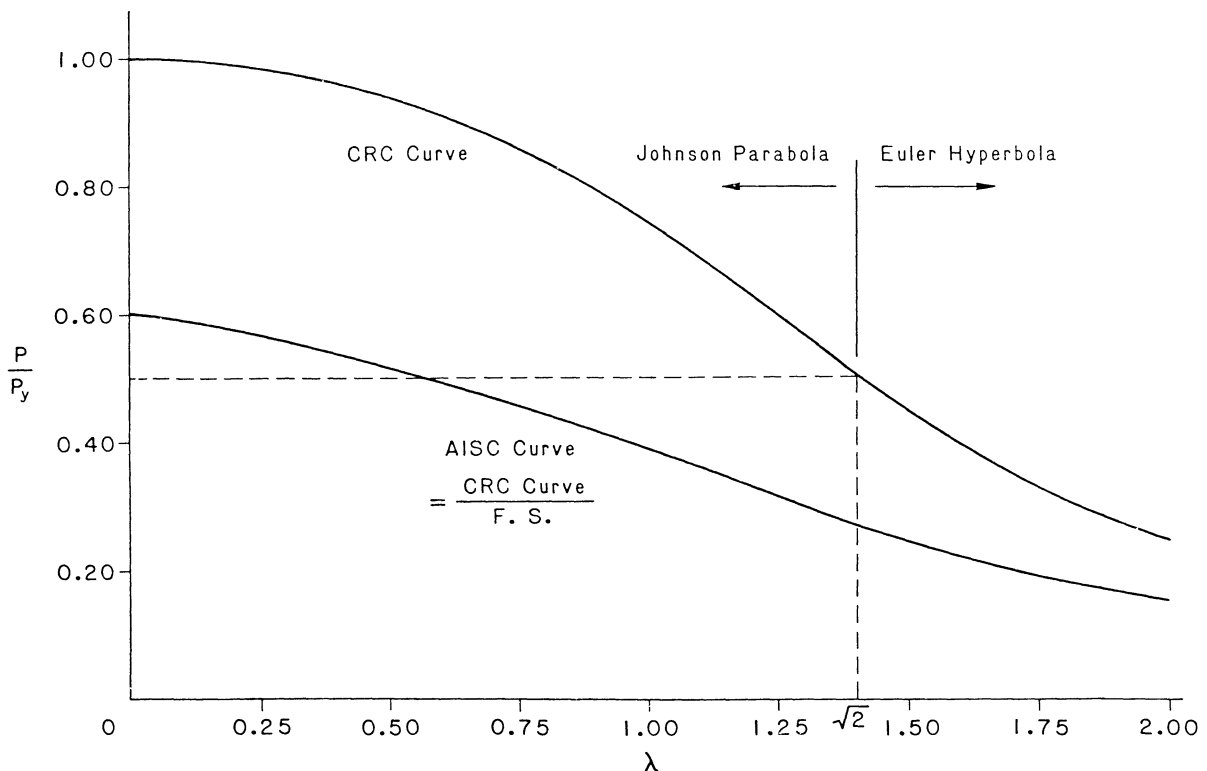


Fig. 4. CRC and AISC curves

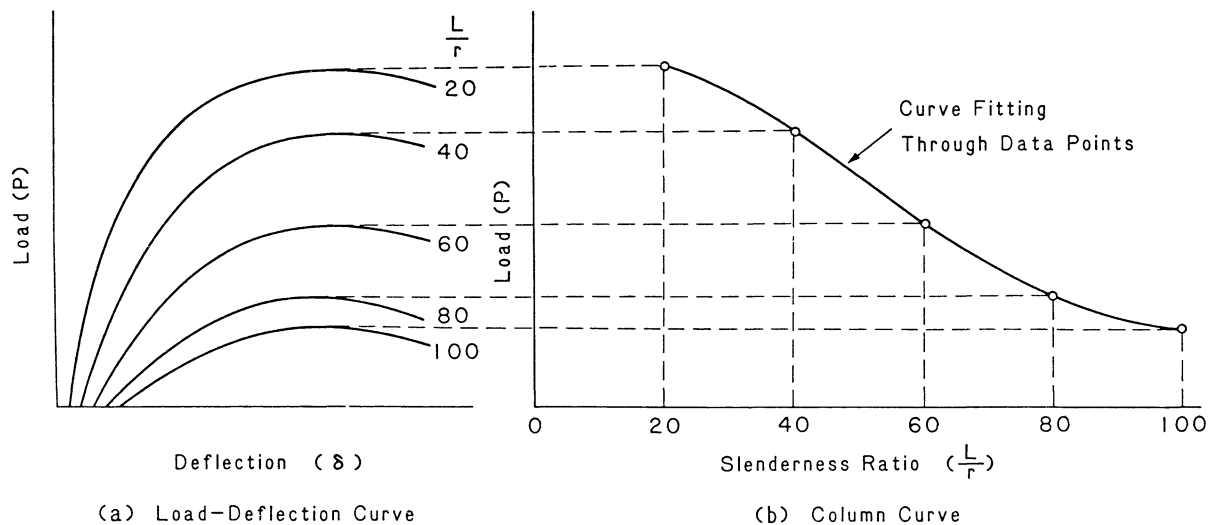


Fig. 5. Determination of column strength curve from load - deflection curves for an initially crooked column

through these points is necessary to describe the relationship between the load P and the slenderness ratio (L/r). This is in contrast to the bifurcation approach in which the load can be obtained directly as a function of the slenderness ratio upon solving the governing differential equation (Eqs. 1, 2 and 3).

The development of column curves for initially crooked columns was achieved by Bjorhovde² based on a computer model developed for a geometrically imperfect column of initial out-of-straightness at mid-height equal to $0.001L$ and for actual measured values of residual stresses. A set of three multiple column strength curves were developed from a total of 112 columns investigated. Each of these curves represents the strength of a related category of columns. Included in the categories covered by these column curves are hot-rolled and cold-straightened members, wide-flange and box shapes, as well as round bars and members composed of welded plates.

The Structural Stability Research Council (SSRC) in its third edition of the Guide, 1976⁸ presents these three curves along with the former one (CRC curve).

The general expression for these curves is

$$\frac{P}{P_y} = C_1 + C_2 \lambda^{k_1} + C_3 \lambda^{k_2} \quad (8)$$

in which

C_1, C_2, C_3, k_1, k_2 are all curve-fitting constants. The numerical values of these constants are given elsewhere.⁸ To describe each curve, nine to twelve constants are necessary to fit a parabola or hyperbola to the designated characteristic column curves which are the arithmetic mean curves of the three divided categories. This is somewhat cumbersome. Moreover, the lack of smooth

transition from one curve to another means there will be a jump in strength for some columns whose strength may be very close to one another. This is not desirable.

A plot of these three SSRC Multiple Column Curves is shown as solid lines in Fig. 6.

The use of multiple column curves is recommended in the Guide⁸ because both theoretical and experimental studies have shown that the strengths of different types of steel columns due to different manufacturing and fabrication processes, different sizes and steel grades and different axis bending may vary considerably so that more than a single design curve may be desirable.

Note that the SSRC curves were developed using the stability approach and the maximum or ultimate strength concept, rather than the bifurcation concept as discussed previously. These curves are more representative than the CRC curve because they not only consider geometric imperfections explicitly in their development but also the vast differentiation of column strength due to the variability in manufacturing processes and rather different intention of usage renders those multiple column curves more appropriate for use in a limit state approach to design.

Although the SSRC curves are more accurate than the CRC curve in representing column length, it should be kept in mind that all these multiple column curves were developed assuming an initial out-of-straightness at mid-height equal to $0.001L$. A common question posed is: will these multiple column curves be applicable if the initial crookedness is not $0.001L$? Furthermore, although both residual stresses and initial crookedness were taken into consideration in the development of these multiple column curves, these factors are not clearly identified in the column equations. Therefore, it is the intent of this paper to present a column equation to identify explicitly

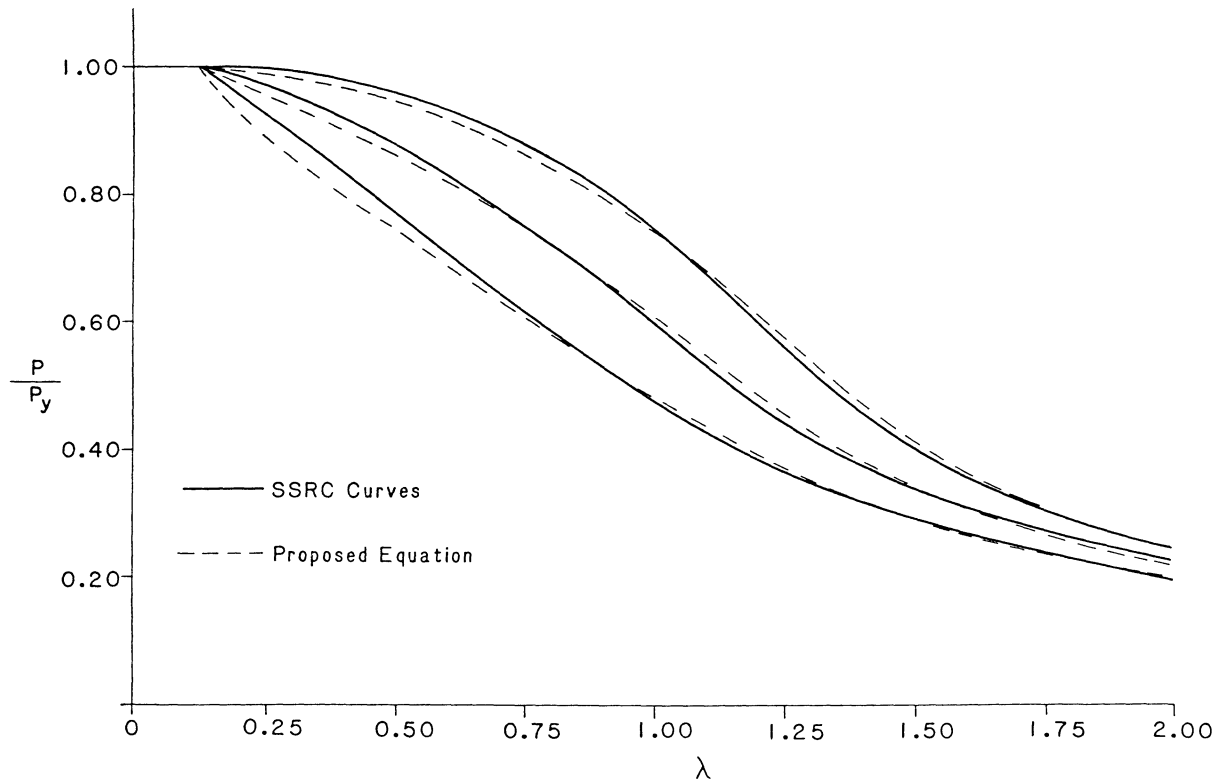


Fig. 6. Comparison of SSRC curves with proposed equation

the factors that influence column strength so the analyst or designer can have a better insight into the column problem.

DERIVATION OF COLUMN EQUATION

The derivation will follow a beam-column approach. The essential characteristic of a beam-column is the secondary moment that arises from the $P-\delta$ effect. The physical model used here is shown in Fig. 7. The imperfect column has an initial out-of-straightness of δ_i at mid-height. Upon application of an axial force P , a secondary moment M will be induced at mid-height of the column. As a column problem, the ultimate strength is reached when the axial force P equals the tangent modulus load P_t . As a beam problem, the ultimate strength is reached when the secondary moment equals the average flow moment M_{mc} . The average flow moment is depicted in Fig. 8. This moment lies between M_{yc} (the yield moment adjusted for the presence of axial force) and M_{pc} (the plastic moment adjusted for the presence of axial force). It was first used by Chen and Atsuta^{4,5} in approximating the ultimate strength of beam-columns.

Thus, the ultimate strength of this beam-column model is reached when

$$\frac{P}{P_t} + \frac{M}{M_{mc}} = 1 \quad (9)$$

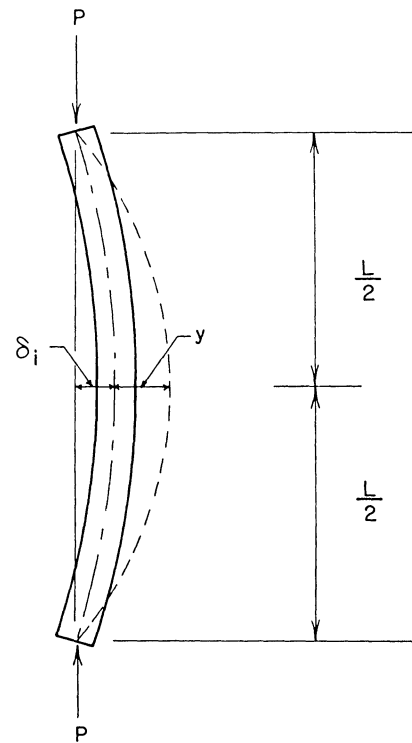


Fig. 7. Physical model for a column with initial crookedness δ_i at mid-height

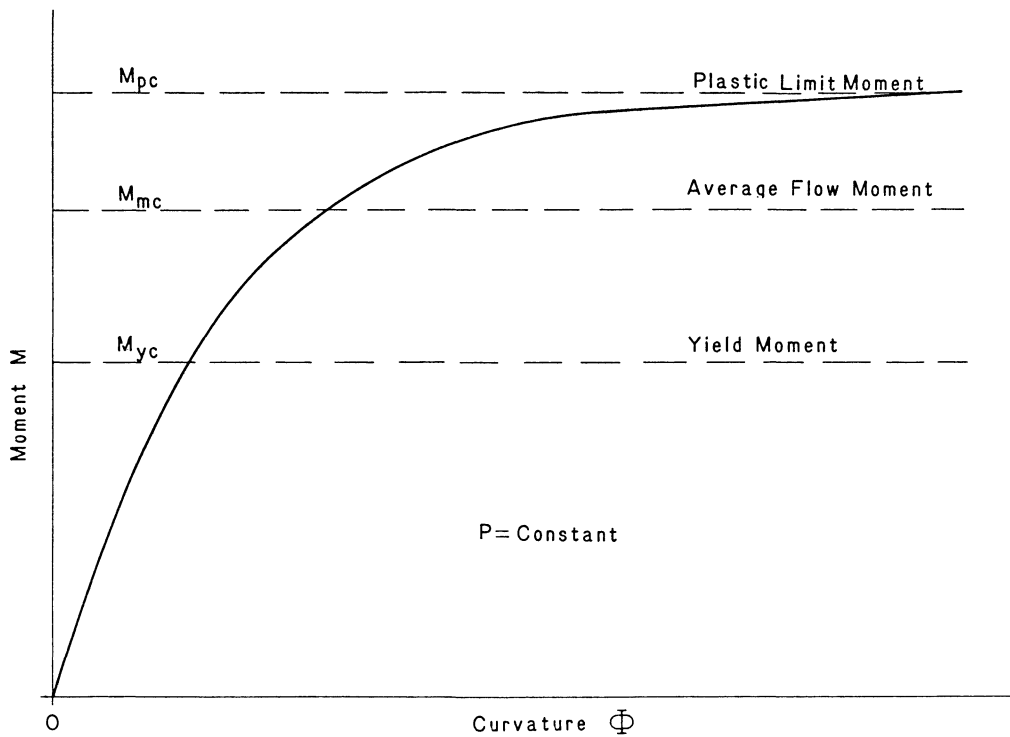


Fig. 8. Schematic representation of the average flow moment M_{mc}

Since M , the secondary moment, can be expressed as

$$M = P (\delta_i + y) = \frac{P\delta_i}{1 - \frac{P}{P_E}} \quad (10)$$

where

δ_i = initial crookedness at mid-height

y = additional mid-height deflection due to the application of P

$1/(1 - P/P_E)$ = magnification factor based on an elastic analysis

Substitution of Eq. 10 to Eq. 9 yields

$$\frac{P}{P_t} + \frac{P\delta_i}{\left(1 - \frac{P}{P_E}\right) M_{mc}} = 1 \quad (11)$$

The flow moment can be expressed as the product of the plastic section modulus Z and an average flow stress $\bar{\sigma}_y$

$$M_{mc} = Z \bar{\sigma}_y \quad (12)$$

The flow stress is such that $\sigma_y / f \leq \bar{\sigma}_y \leq \sigma_y$, in which f is the shape factor of the cross section.

Thus, Eq. 11 can be rewritten as

$$\frac{P}{P_t} + \frac{P\delta_i}{\left(1 - \frac{P}{P_E}\right) Z \bar{\sigma}_y} = 1 \quad (13)$$

or

$$\left(\frac{P}{P_y}\right) \left(\frac{P_y}{P_E}\right) \left(\frac{P_E}{P_t}\right) + \frac{P \delta_i}{\left(1 - \frac{P}{P_y} \frac{P_y}{P_E}\right) \left(\frac{Z}{S}\right) S \left(\frac{\bar{\sigma}_y}{\sigma_y}\right) \sigma_y} = 1 \quad (14)$$

where S = elastic section modulus

Defining

$$\hat{E} = \frac{E}{E_t} = \frac{P_E}{P_t} \quad (15)$$

$$f = \frac{Z}{S} \quad (16)$$

$$\hat{\sigma}_y = \frac{\bar{\sigma}_y}{\sigma_y} \quad (17)$$

and realizing that

$$\frac{P_y}{P_E} = \lambda^2 \quad (18)$$

where

$$\lambda = \frac{1}{\pi} \sqrt{\frac{F_y}{E}} \left(\frac{KL}{r} \right)$$

Eq. 14 can be written as

$$\left(\frac{P}{P_y} \right) \lambda^2 \hat{E} + \frac{P \delta_i}{\left(1 - \frac{P}{P_y} \lambda^2 \right) f \hat{\sigma}_y S \sigma_y} = 1 \quad (19)$$

or

$$\left(\frac{P}{P_y} \right) \lambda^2 \hat{E} + \frac{\left(\frac{P}{P_y} \right)}{\left(1 - \frac{P}{P_y} \lambda^2 \right)} \hat{\eta} = 1 \quad (20)$$

in which

$$\hat{\eta} = \frac{\delta_i A}{f \hat{\sigma}_y S} \quad (21)$$

is the imperfection parameter.

Solving Eq. 20 for P/P_y gives

$$\frac{P}{P_y} = \frac{\hat{\eta} + (1 + \hat{E}) \lambda^2 - \sqrt{[\hat{\eta} + (1 + \hat{E}) \lambda^2]^2 - 4\hat{E} \lambda^4}}{2\hat{E} \lambda^4} \quad (22)$$

which is the proposed column design equation.

The maximum load a column can carry is a function of $\hat{\eta}$, \hat{E} and λ . Any column curve can be generated using this equation provided that the parameters $\hat{\eta}$, \hat{E} and λ are known.

Now, expressing the initial crookedness δ_i of a column as a fraction of the column length L

$$\delta_i = \rho L \quad (23)$$

in which ρ = coefficient of initial out-of-straightness and realizing that

$$S = \frac{I}{c} = \frac{Ar^2}{c} \quad (24)$$

where

c = distance from neutral axis to extreme fiber
 r = radius of gyration

Equation (21) can be written as

$$\hat{\eta} = \frac{c\rho L}{f \hat{\sigma}_y r^2} \quad (25)$$

or in terms of λ (with $K = 1$ for pin-ended column)

$$\hat{\eta} = \pi \sqrt{\frac{E}{F_y}} (\rho) \left(\frac{1}{f} \right) \left(\frac{c}{r} \right) \left(\frac{\lambda}{\hat{\sigma}_y} \right) \quad (26)$$

Note that this imperfection parameter reflects the effects of yield stress (F_y), geometric imperfection (ρ), axis of bending (f), cross-sectional shape (c, r) and slenderness ratio (λ) on the load-carrying capacity of columns.

The average flow stress $\bar{\sigma}_y$ depends on the degree of plastification of the cross section and is a function of the load level and column type. Since, for a column, the degree of plastification depends on the load level, and the load level is a function of the slenderness ratio of the column, this flow stress can be thought of as a function of the slenderness ratio λ . If λ is very large, and P/P_y is very small, the problem resembles a beam problem, and the plastic limit moment M_{pc} will govern the ultimate state, and $\bar{\sigma}_y$ will approach σ_y . On the other hand, if λ is very small, P/P_y will approach unity, and the problem resembles an axially loaded short column problem. The yield moment M_{yc} will govern the ultimate state and $\bar{\sigma}_y$ will approach $\bar{\sigma}_y/f$ where f is the shape factor.

Based on this agreement, the following expression for the flow stress is proposed

$$\bar{\sigma}_y = \left(\frac{1}{\xi \lambda^2 + f} \right) \sigma_y \quad (27)$$

where ξ = plasticity parameter

The constant ξ can be determined from experiments or from calibration against existing column curves.

Substitution of Eq. 27 into Eq. 26 (with $\sigma_y = F_y$ and $\hat{\sigma}_y = \bar{\sigma}_y/\sigma_y$) gives

$$\hat{\eta} = \left[\pi \sqrt{\frac{E}{F_y}} (\rho) \left(\frac{1}{f} \right) \left(\frac{c}{r} \right) \right] (\xi \lambda^3 + f \lambda) \quad (28)$$

It can be seen clearly that the imperfection parameter decreases as λ , ρ and c/r decrease and as F_y and f increase. This observation agrees with the following general experimental facts:

- a. The load-carrying capacity of the column increases as the slenderness decreases.

- b. The column curves for light and medium columns are relatively higher than those for large and heavy shapes.
- c. The column has greater strength if manufactured of higher steel grade.
- d. The column strength increases as the initial out-of-straightness decreases.
- e. The column is stronger upon bending about its strong axis ($f = 1.12$ for hot-rolled wide-flange shapes) than about its weak axis ($f = 1.55$).

For a given column, the terms inside the brackets of Eq. 28 are known, so it can be written in the general form as

$$\hat{\eta} = \hat{a} \lambda^3 + \hat{b} \lambda \quad (29)$$

where

$$\hat{a} = \pi \sqrt{\frac{E}{F_y}} (\rho) \left(\frac{1}{f}\right) \left(\frac{c}{r}\right) \xi$$

$$\hat{b} = \pi \sqrt{\frac{E}{F_y}} (\rho) \left(\frac{c}{r}\right)$$

The modulus ratio \hat{E} can be evaluated if the tangent modulus E_t is known either from an experimental or a theoretical approach. Recall that the tangent modulus is the slope of the non-linear stress-strain curve. This non-linearity is due to the presence of residual stresses for steel columns. Thus, this modulus ratio will reflect material non-linearity and imperfections.

For simplicity, this modulus ratio is taken to be the ratio of the Euler curve to the CRC curve

$$\hat{E} = \frac{E}{E_t} = \frac{P_E}{P_{CRC}} = \frac{4}{(4 - \lambda^2) \lambda^2} \text{ for } \lambda \leq \sqrt{2} \quad (30)$$

$$= 1 \quad \text{for } \lambda > \sqrt{2}$$

In summary, the proposed column equation is expressed as

$$\frac{P}{P_y} = \frac{\hat{\eta} + (1 + \hat{E}) \lambda^2 - \sqrt{[\hat{\eta} + (1 + \hat{E}) \lambda^2]^2 - 4 \hat{E} \lambda^4}}{2 \hat{E} \lambda^4} \quad (22)$$

where

$$\hat{\eta} = \hat{a} \lambda^3 + \hat{b} \lambda \quad (29)$$

$$\hat{E} = \frac{4}{(4 - \lambda^2) \lambda^2} \text{ for } \lambda \leq \sqrt{2} \quad (30)$$

$$= 1 \quad \text{for } \lambda > \sqrt{2}$$

COMPARISON OF PROPOSED EQUATION WITH EXISTING DESIGN FORMULAS

This section will show the proposed column equation can be used to represent the CRC curve and to approximate the SSRC Multiple Column Curves.

CRC Curve

The CRC curve was developed for a perfectly straight column, therefore, $\delta_i = 0$. In other words, ρ will be zero (Eq. 23) and so $\hat{\eta}$ will be zero (Eq. 28). By setting $\hat{\eta} = 0$ in Eq. 22, Eq. 30 results in

$$\frac{P}{P_y} = 1 - 0.25 \lambda^2 \text{ for } \lambda \leq \sqrt{2}$$

$$= \lambda^{-2} \text{ for } \lambda > \sqrt{2}$$

which is the expression for the CRC curve (Eq. 7).

Thus, by setting $\hat{\eta} = 0$, the proposed equation reduces to the CRC formula.

SSRC Curves

The proposed formula can also be used to represent the SSRC multiple column curves through calibration. By using values of \hat{a} and \hat{b} shown in Table 1, for example, the SSRC curves can be closely approximated.

Table 1

Curve	\hat{a}	\hat{b}
SSRC 1	0.002	-0.001
SSRC 2	-0.036	0.159
SSRC 3	-0.092	0.453

These approximations are shown in Fig. 6 (dashed curves). Good correlation between the actual curves and the proposed equation is generally observed.

COMPARISON OF PROPOSED EQUATION WITH COMPUTER MODELS

The versatility and validity of the proposed column equation will be demonstrated by comparing it with three different computer models.

Upon calibration against SSRC Curve 2, the plasticity parameter ξ was found¹⁰ to be -0.378 for strong-axis bending and -0.308 for weak-axis bending. Since this plasticity parameter is independent of the magnitude of initial crookedness, the same values of ξ can be used for any value of ρ . However, since the expressions for the modulus ratio \hat{E} (Eq. 30) are, strictly speaking, valid only for light and median rolled H-sections (standard W, S, M, shapes), the use of the equation for other sizes

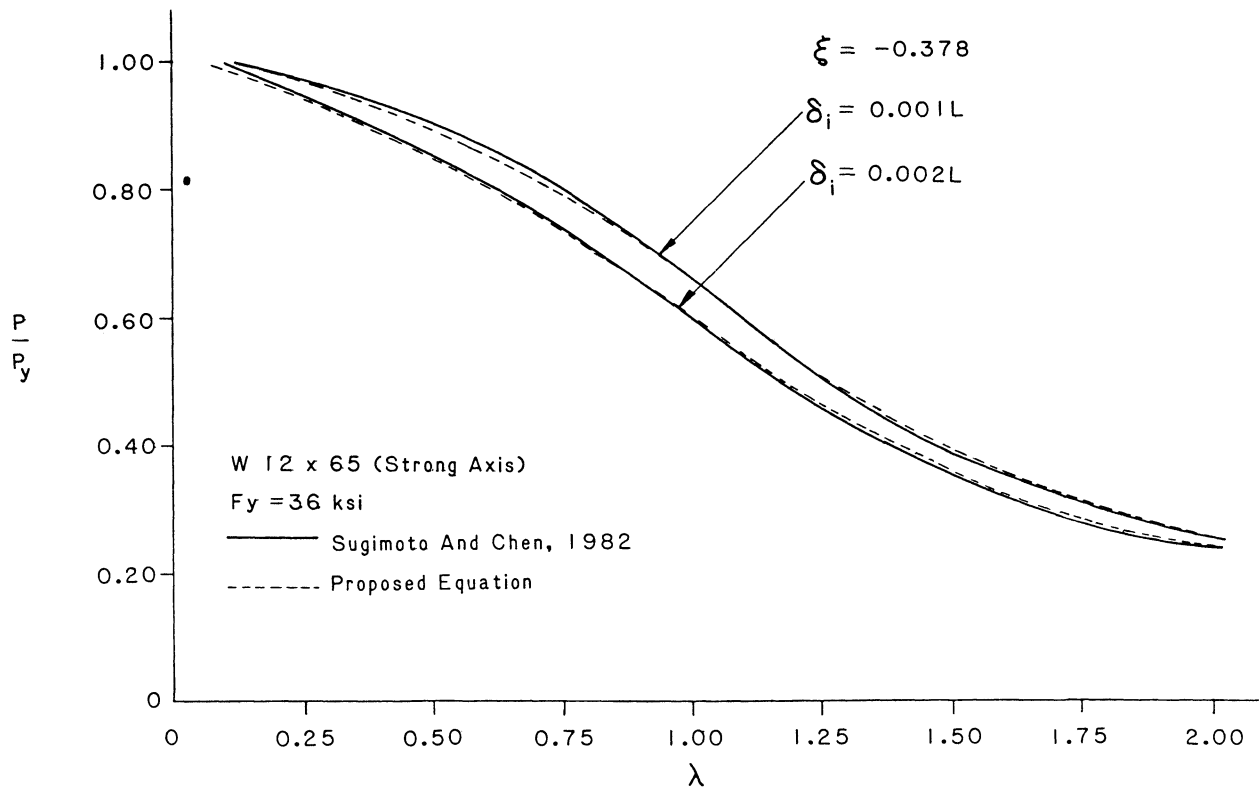


Fig. 9. Comparison of proposed equation with computer model by Sugimoto and Chen

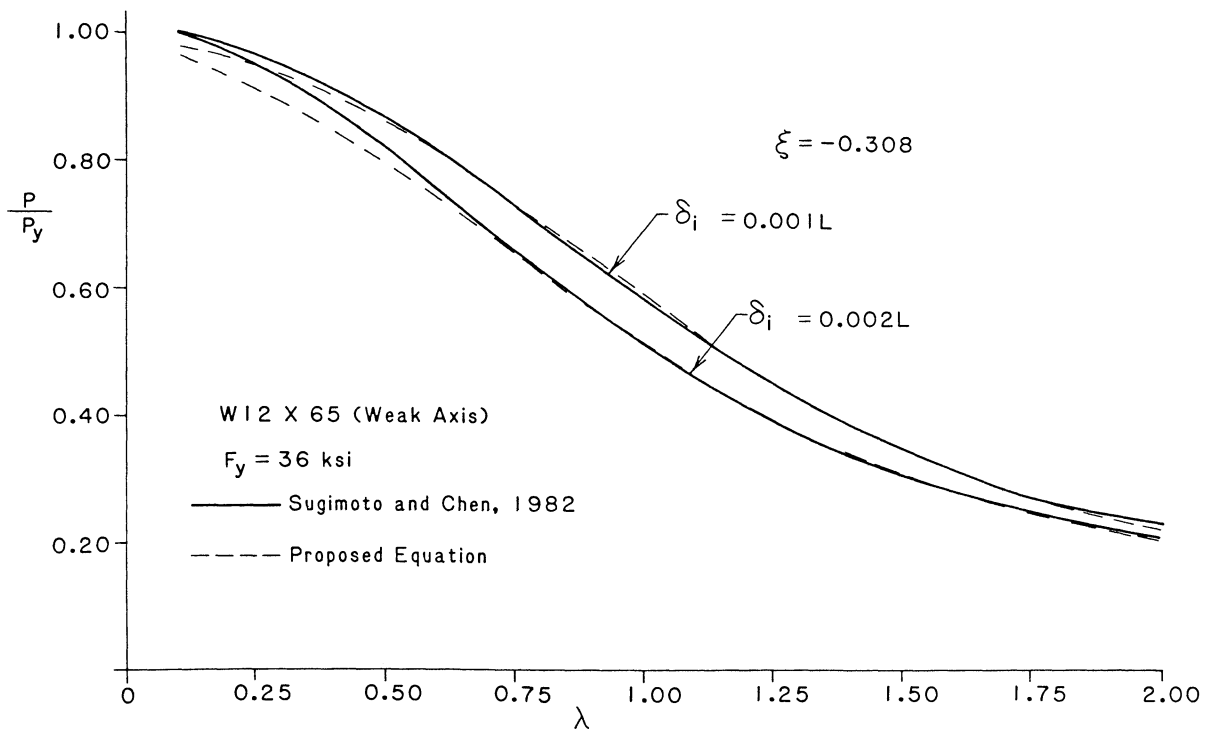


Fig. 10. Comparison of proposed equation with computer model by Sugimoto and Chen

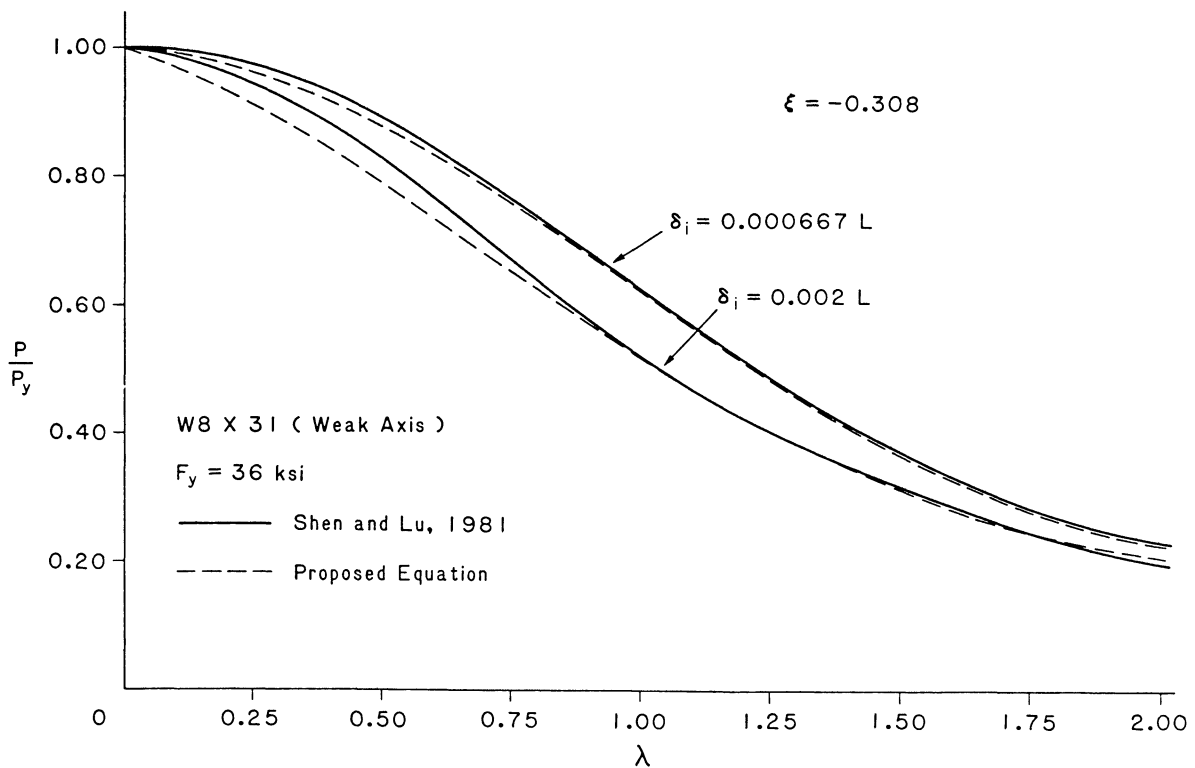


Fig. 11. Comparison of proposed equation with computer model by Shen and Lu

and shapes is not advisable unless a more appropriate expression for \hat{E} is available and new values of ξ are computed for these other shapes.

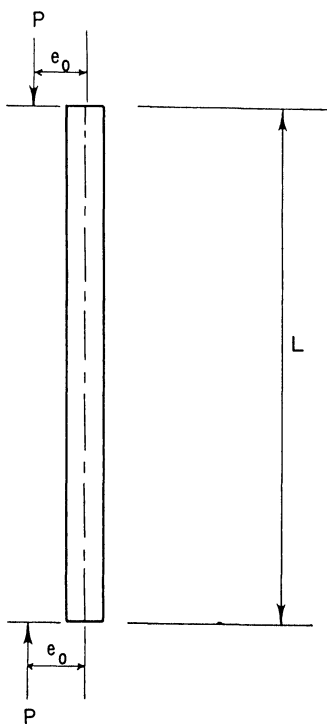


Fig. 12. Physical model of a column with load eccentricity e_o

Figures 9 and 10 show a comparison of the column curves generated using the proposed equation and a computer model developed by Sugimoto and Chen.¹⁶ The column is a W12x65 shape bent about the strong and weak axes respectively. Two sets of imperfections corresponding to an initial out-of-straightness at mid-height equal to $0.001L$ and $0.002L$ are shown. It can be seen that good correlation between the proposed equation and the computer model is observed.

Figure 11 shows the column curves of a W8x31 section bent about the weak axis. The two solid curves were obtained by Shen and Lu¹⁵ using a computer model for initially crooked columns. The two dashed curves were obtained by the authors using the proposed column equation. Two different values of initial crookedness were used ($0.000667L$ and $0.002L$). Again, good approximations are observed.

ECCENTRICALLY LOADED COLUMNS

The physical model for a column with load eccentricity is shown in Fig. 12. The difference between a column with a load eccentricity e_o and an initial crookedness δ_i is that the expression for the moment magnification factor changes somewhat. Figure 13 shows a comparison of these two moment magnification factors.

Based on a numerical study, the authors found the magnification factor for load eccentricity can be closely

represented by the following approximation

$$\sec \frac{\pi}{2} \sqrt{\frac{P}{P_E}} \approx \frac{1 + 0.25 \left(\frac{P}{P_E} \right)}{1 - \frac{P}{P_E}} \quad (33)$$

This approximation is plotted in Fig. 13 as (Δ) points. It can be seen that an excellent agreement is obtained.

The failure criterion for an eccentrically loaded column has the form

$$\frac{P}{P_t} + \frac{Pe_o \sec \frac{\pi}{2} \sqrt{\frac{P}{P_E}}}{M_{mc}} = 1 \quad (34)$$

Using Eq. 12, 33 and 34, we obtain

$$\frac{P}{P_t} + \frac{Pe_o \left[1 + 0.25 \left(\frac{P}{P_E} \right) \right]}{\left(1 - \frac{P}{P_E} \right) Z \bar{\sigma}_y} = 1 \quad (35)$$

or

$$\left(\frac{P}{P_y} \right) \left(\frac{P_y}{P_E} \right) \left(\frac{P_E}{P_t} \right) + \frac{Pe_o \left[1 + 0.25 \left(\frac{P}{P_y} \right) \left(\frac{P_y}{P_E} \right) \right]}{\left(1 - \frac{P}{P_y} \frac{P_y}{P_E} \right) \left(\frac{Z}{S} \right) S \left(\frac{\bar{\sigma}_y}{\sigma_y} \right) \sigma_y} = 1 \quad (36)$$

or

$$\left(\frac{P}{P_y} \right) \lambda^2 \hat{E} + \frac{\left(\frac{P}{P_y} \right) \left[1 + 0.25 \left(\frac{P}{P_y} \right) \lambda^2 \right]}{\left(1 - \frac{P}{P_y} \lambda^2 \right)} \bar{\eta} = 1 \quad (37)$$

where

$$\bar{\eta} = \frac{e_o A}{f \hat{\sigma}_y S} \quad (38)$$

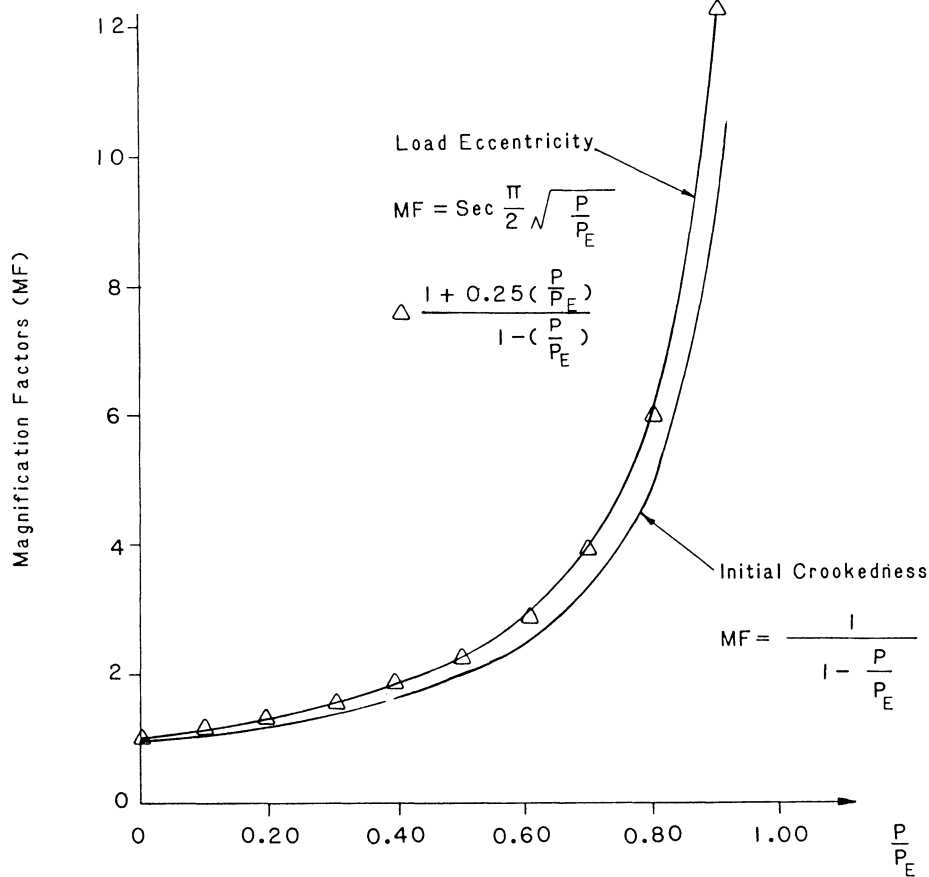


Fig. 13. Comparison of moment magnification factors for initial crookedness with load eccentricity

Solving Eq. 37 for P/P_y gives

$$P/P_y = \left[\bar{\eta} + (1 + \hat{E}) \lambda^2 - \sqrt{[\bar{\eta} + (1 + \hat{E}) \lambda^2]^2 - 4 [\hat{E} \lambda^4 - 0.25 \bar{\eta} \lambda^2]} \right] / (2\hat{E} \lambda^4 - 0.5 \bar{\eta} \lambda^2) \quad (39)$$

Equation 39 can be used to generate column curves for columns with load eccentricity.

Figure 14 shows a comparison of Eq. 39 to the well-known secant formula for eccentrically loaded columns.

The secant formula is expressed³ as

$$\sigma_{max} = \frac{P}{A} \left[1 + \frac{e_o c}{r^2} \sec \left(\frac{KL}{2r} \sqrt{\frac{P}{AE}} \right) \right] \quad (40)$$

where

- σ_{max} = maximum fiber stress
- P = applied load
- A = cross-sectional area
- e_o = load eccentricity
- c = distance from neutral axis to extreme fiber
- r = radius of gyration
- KL = effective length of the column

If we let $\sigma_{max} = \sigma_y$, the yield stress of the material, and express Eq. 40 in terms of λ , we get

$$\frac{P}{P_y} = \left[1 + \left(\frac{e_o c}{r^2} \right) \sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_y}} \lambda \right) \right]^{-1} \quad (41)$$

P/P_y was evaluated from this equation by trial and error and the result is plotted as a solid curve in Fig. 14. The dashed curve was evaluated in Eq. 39. The column used is a S3x5.7 section bent about its weak axis. Good agreement is obtained.

COLUMN WITH INITIAL CROOKEDNESS AND LOAD ECCENTRICITY

The ultimate strength of this column (Fig. 15) is

$$\frac{P}{P_t} + \frac{P \delta_i}{\left(1 - \frac{P}{P_E}\right) M_{mc}} + \frac{P e_o \sec \frac{\pi}{2} \sqrt{\frac{P}{P_E}}}{M_{mc}} = 1 \quad (42)$$

Following the same procedure as above, the non-dimensional load P/P_y can be expressed as

$$\frac{P}{P_y} = \frac{q - \sqrt{q^2 - 4p}}{2p} \quad (43)$$

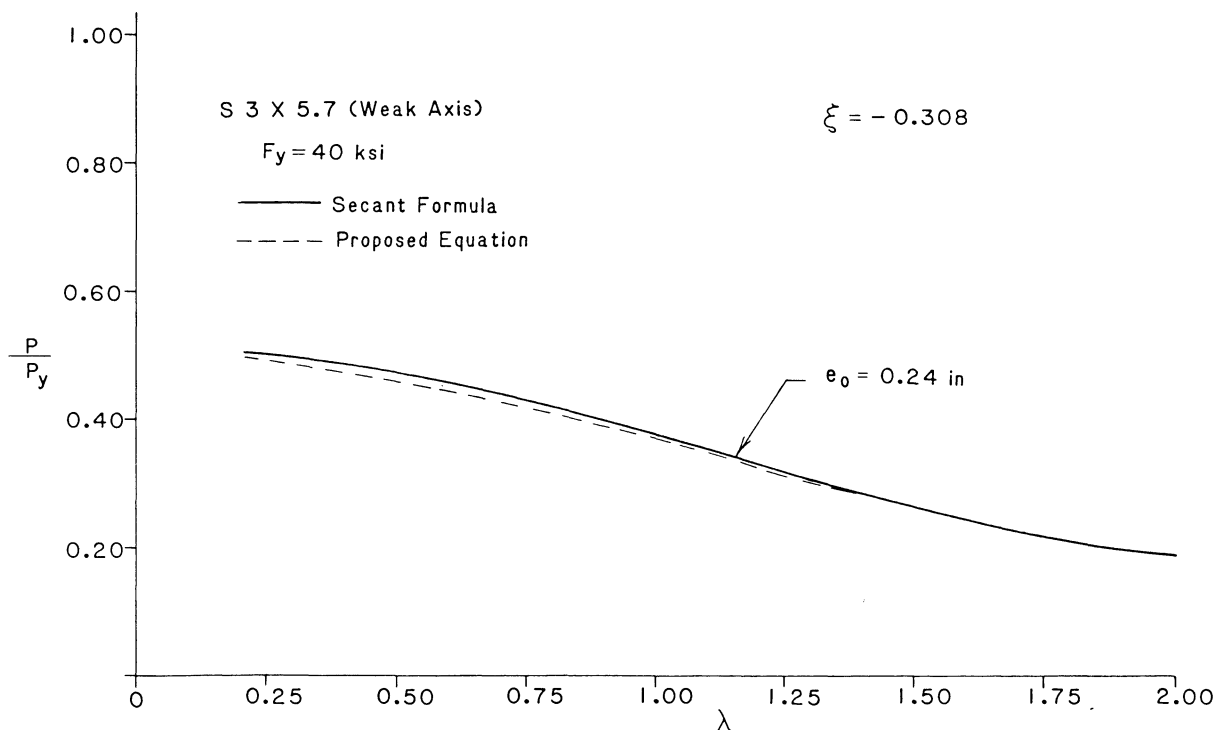


Fig. 14. Comparison of proposed equation with the secant formula

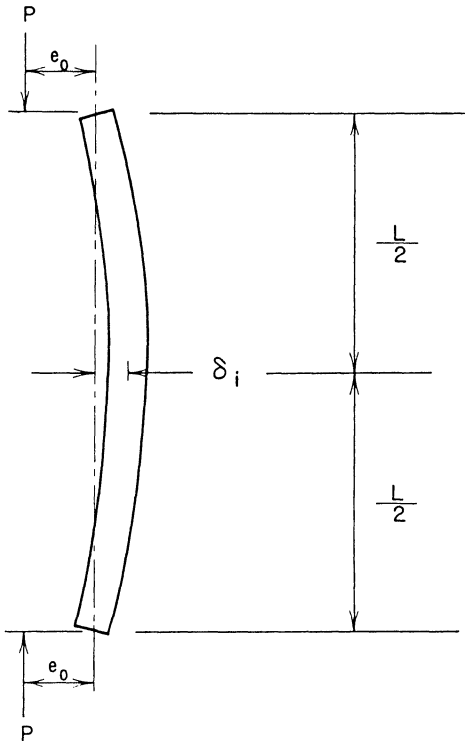


Fig. 15. Physical model for a column with initial crookedness δ_i and load eccentricity e_o

in which

$$p = \hat{E} \lambda^4 - 0.25 \bar{\eta} \lambda^2$$

$$q = \hat{\eta} + \bar{\eta} + (1 + \hat{E}) \lambda^2$$

\hat{E} , $\hat{\eta}$, $\bar{\eta}$ are defined as before. Note that if $\bar{\eta} = 0$, Eq. 43 reduces to Eq. 22 and if $\hat{\eta} = 0$, Eq. 43 reduces to Eq. 39.

Equation 43 was programmed in the UNIX system of the Engineering Computer Network (ECN) at Purdue University. The small size and minimal storage required for this program make it possible to store it in a pocket programmable calculator or a home computer. A brief flow chart, the program listing together with sample input and output are shown in detail in Appendix B.

APPLICATION TO DESIGN

In this section, different formats currently in use or proposed in the AISC Specification will be discussed briefly. The design of columns using the current¹¹ and proposed column formulas according to the formats will then be given.

Design Formats

The basic philosophy behind any design format is that the load applied to any structural member can be resisted by the strength of the member with a guaranteed margin of safety. The discrimination between different formats

is the manner for which the factor of safety is defined and applied.

- *Allowable Stress Design (AISC, Part I)*
The ASD has the general format of

$$\frac{R_n}{F.S.} \geq \sum_{i=1}^j Q_{ni} \quad (44)$$

where

$$\begin{aligned} R_n &= \text{nominal resistance (in stress form)} \\ Q_n &= \text{nominal load effects (in stress form)} \\ F.S. &= \text{factor of safety} \end{aligned}$$

In the ASD format, the factor of safety is applied to the nominal resistance of the structural member. Since F.S. is always greater than unity, the nominal resistance is scaled down. An acceptable design is one such that the 'scaled down' nominal resistance is still greater than the applied nominal loads.

- *Plastic Design (AISC, Part II)*
The Plastic Design has the following format

$$R_n \geq \sum_{i=1}^j \gamma_i Q_{ni} \quad (45)$$

where

$$\begin{aligned} R_n &= \text{nominal resistance (in load form)} \\ Q_n &= \text{nominal load effect (in load form)} \\ \gamma &= \text{load factors (greater than unity)} \end{aligned}$$

In this format, the factor of safety is applied to the applied loads as load factors. An acceptable design is one such that the nominal resistance of the structural member is greater than the factored loads applied to the member.

- *Load and Resistance Factor Design (AISC, Part III, proposed)*

The LRFD has the following general format

$$\phi R_n \geq \sum_{i=1}^j \gamma_i Q_{ni} \quad (46)$$

where

$$\begin{aligned} R_n &= \text{nominal resistance (in load form)} \\ Q_n &= \text{nominal load effects (in load form)} \\ \phi &= \text{resistance factor (less than unity)} \\ \gamma &= \text{load factor (greater than unity)} \end{aligned}$$

The LRFD format has the features of both the ASD and Plastic Design formats in that "safety factors" are applied to both the resistance and load terms. However, unlike the previous two formats, the resistance and load factors in the LRFD format were evaluated using a probabilistic ap-

proach. The margin of safety is reflected in the terms of β which is referred to as the safety index and, like ϕ and γ , is evaluated based on the theory of probability. An acceptable design is one such that the probability of failure of the structural member falls below a tolerable level.

Design Examples

Design a column 20 ft long, pinned at both ends with sidesway inhibited, to carry a load of 80 kips and a live load of 140 kips. Use A36 steel. Because of the manufacturing and fabrication processes, the column has an initial crookedness at mid-height in the strong direction or in the y - z plane (camber) of 0.12 in., and an initial crookedness at mid-height in the weak direction or x - z plane (sweep) of 0.24 in. (Fig. 16).

• ASD Format

Since there is no bracing in the weak direction, weak axis buckling will control.

$$\sum_{i=1}^j Q_{ni} = 80 + 140 = 220 \text{ kips}$$

Required resistance is

$$\frac{R_n}{\text{F.S.}} \geq 220 \text{ kips}$$

Recall that by setting $\hat{\eta} = 0$, the proposed column equa-

tion reduces exactly to the CRC basic strength curve. Thus, column tables in the AISC Manual¹¹ can be used for the design.

Section	P
W10x60	243 kips
W12x58	230 kips

$$\text{Use W12x58} \left(\frac{R_n}{\text{F.S.}} = 230 > 220, \text{ o.k.} \right)$$

• Plastic Design Format

Factored load

$$\begin{aligned} \sum_{i=1}^2 \gamma_i Q_{ni} &= 1.7 (80 + 140) \\ &= 374 \text{ kips} \end{aligned}$$

Required resistance (AISC Specification, Sect. 2.4)

$$R_n = 1.7AF_a \geq 374 \text{ kips}$$

Section	$P = AF_a$	$R_n = 1.7 AF_a = 1.7P$
W10x60	243 kips	413 kips
W12x58	230 kips	391 kips

$$\text{W12x58} (R_n = 391 > 374, \text{ o.k.})$$

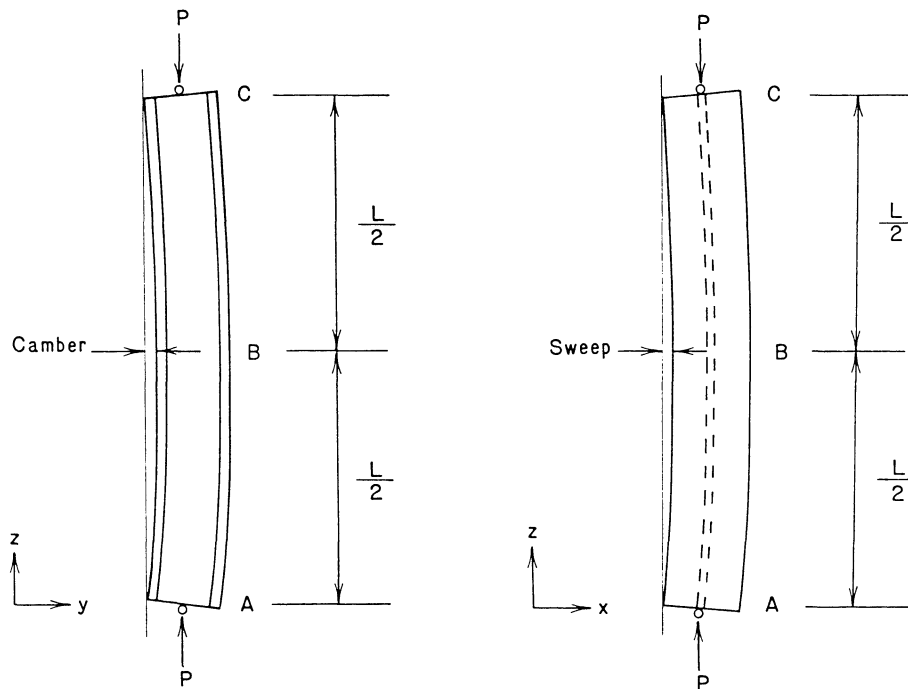


Fig. 16. A pin-ended column with initial crookedness

Check slenderness ratio requirement in plastic design for the W12x58 section if the column were required to develop a plastic hinge.

$$\left(\frac{KL}{r_y} = \frac{(1)(20)(12)}{2.51} = 95.6 \right) < (C_c = 126.1)$$

Use W12x58

• **LRFD format (proposed)**

$$\sum_{i=1}^2 \gamma_i Q_{ni} = 1.2(80) + 1.6(140) = 320 \text{ kips}$$

Required resistance is

$$\phi R_n = \phi P > 320 \text{ kips}$$

This resistance can be obtained from the computer program (Appendix B) once a trial section has been picked. The initial crookedness is input as ρ (the coefficient of initial out-of-straightness) where

$$\rho = \frac{\delta_i}{L} = \frac{0.24}{(20)(12)} = 0.001$$

for weak axis buckling.

Section	$K_y L_y / r_y$	λ	P/P_E	P	ϕP
W10x60	93.385	1.047	0.569	361	307
W12x65	79.470	0.891	0.656	451	383

Use W12x65 ($\phi P = 383 > 320$ o.k.)

A ϕ value of 0.85 was used in the above citation. The same value of ϕ was recommended by the authors in a previous paper⁹ in the design of end-restrained columns using the LRFD format.

Example 2:

Same as Example 1, except that bracing is now provided at mid-height perpendicular to the web line at point B in Fig. 16.

• **ASD Format**

$$K_y L_y = 10 \text{ ft}$$

From column tables

Section	P	r_x / r_y
W10x45	232 kips	2.15
W12x45	228 kips	2.65

Use W10x45 ($R_n/F.S. = 232 > 220$, o.k.)

• **Plastic Design Format**

Since the Plastic Design Format for columns can be

obtained by multiplying 1.7 on both sides of the ASD Format, the same section, i.e. W10x45 will be chosen. However, the slenderness ratio requirement for plastic design of columns still needs to be checked if a plastic hinge is required to develop in the column.

$$\left(\frac{K_y L_y}{r_y} = \frac{(1)(10)(12)}{2.01} = 59.7 \right) < (C_c = 126.1)$$

Use W10x45

• **LRFD Format**

Since the column is now braced at mid-height in the weak direction, a new value for the initial crookedness in the weak direction for segment AB (or BC) needs to be found. This can be done by assuming the initial crookedness of the column is a half sine-wave, i.e.

$$\delta(z) = \delta_i \sin \frac{\pi z}{L} = 0.24 \sin \frac{\pi z}{L}$$

the initial crookedness at mid-length of segment AB (Fig. 17) is

$$\begin{aligned} \delta_{iAB} &= \delta(z)|_{z=L/4} = \frac{\delta_i}{2} \\ &= 0.24 \sin \frac{\pi(L/4)}{L} = \frac{0.24}{2} \\ &= 0.05 \text{ in.} \end{aligned}$$

and so

$$\rho_{AB} = \frac{\delta_{iAB}}{L_{AB}} = \frac{0.05}{(10)(12)} = 0.00042$$

By assuming the line of action of P is co-linear with the chord AB as indicated in Fig. 17, the computer program can be used to evaluate the resistance of any trial column. With

$$\begin{aligned} \rho_{\text{strong axis bending}} &= \frac{\delta_i}{L} = \frac{0.12}{(20)(12)} = 0.0005 \\ \rho_{\text{weak axis bending}} &= \rho_{iAB} = 0.00042 \end{aligned}$$

Assume weak axis buckling controls and use the computer program.

Section	$K_y L_y / r_y$	λ	P/P_y	P	ϕP
W10x45	59.701	0.670	0.834	399	339
W8x40	58.824	0.660	0.839	353	300

W10x45 ($\phi P = 339 > 320$, o.k.)

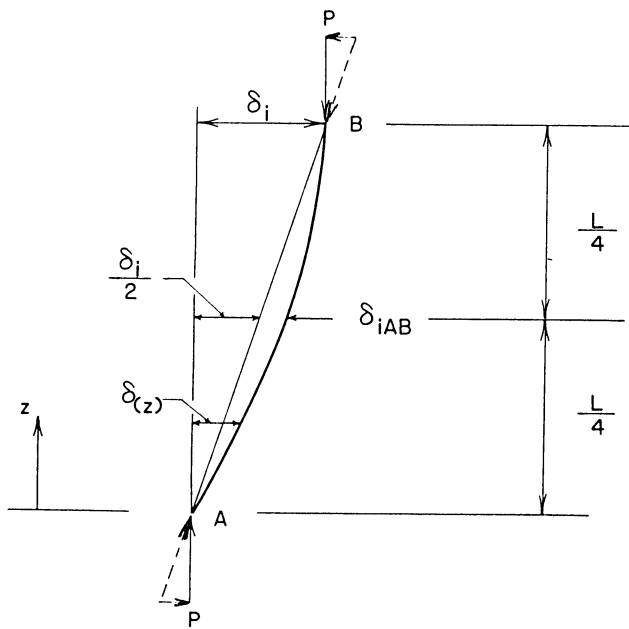


Fig. 17. Initial out-of-straightness δ_{iAB} of segment AB

Check strong axis buckling for the W10x45 section:

Section	$K_x L_x / r_x$	λ	P/P_y	P	ϕP
W10x45	55.556	0.623	0.870	416	354

Since $354 > 339$, weak axis buckling controls, assumption, **o.k.**

Use W10x45

Note the effect of initial crookedness can be accounted for explicitly by using the proposed column equation in the LRFD Format. This is important because LRFD is a form of limit state design. In a limit state approach to design, all factors important in affecting the ultimate strength of the structural member should be considered explicitly in the design.

SUMMARY AND CONCLUSIONS

Current design specification for columns makes use of the CRC basic strength curve which was developed using the tangent modulus concept. To account for geometrical imperfections, the use of a factor of safety is inevitable. However, this implicit way to account for geometrical imperfections in columns is not desirable in a limit state approach to design (e.g. LRFD). Although column curves for the initially crooked columns can be generated by the load-deflection method, the complexity in the numerical procedures involved makes this approach very undesirable in design.

The proposed column equation was developed to remedy this. This equation has the following features:

- realistic in nature
- versatile in application
- simple to use
- comparable to existing column formulas

It is well known that residual stresses (in steel columns), material nonlinearity (in aluminum columns) and initial out-of-straightness have a detrimental effect on column strength. These effects are identified explicitly in the proposed column equation in the terms \bar{E} and ρ respectively. Furthermore, parameters defining the length, size and shape of the columns (λ, c, r, f), as well as the yield stress of the material F_y , effect of plasticity ξ , and load eccentricity e_o are also explicitly identified in the equation in the terms $\hat{\eta}$ and $\bar{\eta}$. Therefore, it is realistic in nature.

It is versatile in the sense it is flexible enough to allow for further updating or revision deemed necessary in future years when the characteristics of new materials, geometry and end conditions are refined.

Also simple to use, the computer program was written especially to serve this purpose. Because of the advent of computer technology, computer-aided design will undoubtedly gain popularity in the near future.

Last, but not the least, the proposed equation is very comparable to existing column design formulas. By setting $\hat{\eta}$ (and/or $\bar{\eta}$) equal to zero, this equation reduces exactly to the CRC basic strength curve which is now contained in the AISC Specification.¹¹ The proposed equation can also be used to approximate the SSRC Multiple Column Curves which are recommended for use in the SSRC Guide.⁸

The validity of the proposed approach is further demonstrated by comparing it with three different elaborate computer models (Sugimoto and Chen,¹⁶ Shen and Lu¹⁵ and Batterman and Johnston¹). There is good agreement between the proposed equation and these computer models.

Although the proposed column equation was developed for pin-ended columns, it can be used in conjunction with the effective length factor K for columns with other end conditions, or with the K -nomographs (alignment charts) for columns in frames. This equation can also be used as the end points for beam-column interaction equations in the design of beam-columns. Further, the proposed equation can be used to approximate closely the secant column strength formula used widely in earlier years of column design.

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APPENDIX A—NOMENCLATURE

A	= cross-sectional area
c	= distance from neutral axis of cross section to extreme fiber
C_c	= $\sqrt{2 \pi^2 E / F_y}$
e_o	= load eccentricity
E	= elastic modulus
E_t	= tangent modulus
\hat{E}	= modulus ratio
f	= shape factor
F_{cr}	= critical stress
F_y	= yield stress
F.S.	= factor of safety
I	= moment of inertia
I_e	= moment of inertia of elastic part of the cross section
K	= effective length factor
L	= length of the column
M	= secondary moment
M_{mc}	= average flow moment
M_{pc}	= plastic moment reduced for the presence of axial force
n	= hardening parameter
P	= axial force
P_E	= Euler load
P_t	= tangent modulus load
P_y	= yield load
Q_n	= nominal load effect
r	= radius of gyration
R_n	= nominal resistance
S	= elastic section modulus
Z	= plastic section modulus
β	= safety index
δ_i	= initial mid-height out-of-straightness
$\hat{\eta}$	= initial crookedness imperfection parameter
$\bar{\eta}$	= load eccentricity imperfection parameter
γ	= load factor
λ	= non-dimensional slenderness ratio
ϕ	= resistance factor
ρ	= coefficient of initial crookedness
$\sigma_{0.2}$	= 0.2% offset yield stress
σ_{max}	= maximum stress
σ_y	= yield stress
$\bar{\sigma}_y$	= flow stress
$\hat{\sigma}_y$	= stress ratio
ξ	= plasticity parameter

APPENDIX B

Flow Chart :

