Buckling of One-Story Frames

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Two buckling modes can occur in building frames: (1) column buckling of one or more individual members and (2) frame buckling. The two modes can be investigated independently. In the first mode, the column is supported against lateral movements at both ends, but bows between the ends. The surrounding frame provides the lateral support at the ends. In the second mode, the top of the frame moves laterally with respect to the bottom. This paper deals primarily with frame buckling, but also discusses the interrelationship between the two modes. Specifically, it presents a conservative method of determining the frame buckling load from an ordinary (first-order) analysis and suggests compatible methods of checking column buckling of individual members.

Methods of calculating the buckling load of frames have been discussed in papers too numerous to mention here. However, the state-of-the-art is adequately covered in two recent books: the Council on Tall Buildings' volume on steel design[1] and the Structural Stability Research Council's guide to structural stability.[2] These two references describe the two main methods of considering frame buckling: effective-length methods and $P\Delta$ methods.

To account for the frame buckling mode, the effective-length methods use effective lengths greater than the actual lengths of the columns. Charts give the effective lengths for single frames that are symmetric with respect to both loading and geometric properties, but the application of these charts is difficult and confusing for more complex frames. Effective-length methods can be used for loading cases involving gravity (vertical) loads alone or gravity loads in combination with wind (horizontal) loads. For the latter case, however, these methods do not account for the $P\Delta$ moments caused by the interaction of vertical and horizontal loads. The $P\Delta$ methods account for these $P\Delta$ moments in combined loading cases, but are usually not applied to gravity loading alone, because symmetrical gravity loading of symmetrical frames creates no $P\Delta$ moments. As pointed out by Yura,[3] the $P\Delta$ method can give unconservative results for frames subjected to small horizontal loads in combination with large vertical loads.

The frame stiffness method presented in this paper is a form of the $P\Delta$ method and is based on the work in Refs. 1 and 2. It includes several new concepts that permit it to be applied consistently to one-story frames subjected to either gravity loads alone or gravity loads in combination with lateral loads. Thus, it avoids the unconservative results pointed out by Yura.[3]

The paper is presented in three parts. The first is a concise description of the concept. The second gives proposed design procedures and the third part is a series of four appendices that give derivations and justifications.

DESCRIPTION OF CONCEPT

Actual Frame Stiffness—The frame buckling load depends primarily on the lateral stiffness of the frame. As shown in Fig. 1b, this stiffness, s, is defined as the horizontal force, H, divided by the resulting lateral deflection, Δ_o. The frame stiffness depends on the dimensions, material properties, and cross-sectional properties of the frame, and can be determined from an ordinary (first-order) analysis.

The lateral deflection, or sidesway, results from two sources: bending of the individual members and axial length changes of the columns. For frames of normal proportions with no diagonal bracing, the effect of member bending dominates. For the example in Fig. 1a, the deflection caused by axial length changes is less than 1% of that caused by member bending. Therefore, the effects of axial length changes are often neglected in older methods of structural analysis such as moment distribution. However, the effects of axial length changes are usually included automatically in finite-element methods.

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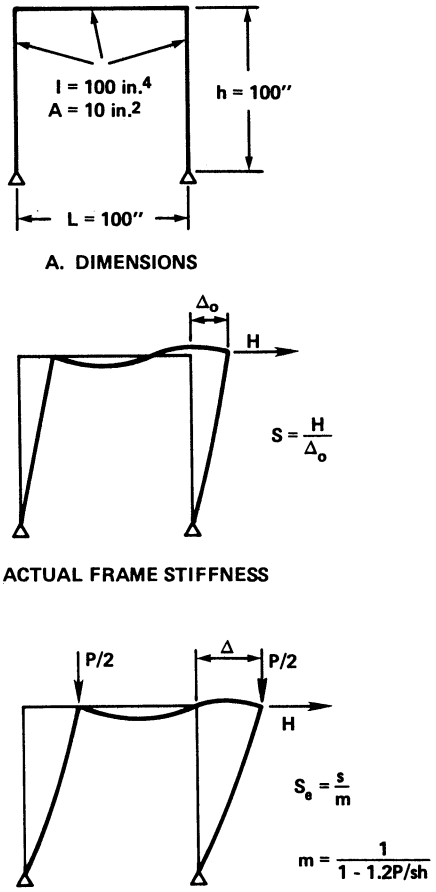

Fig. 1. Frame stiffness

Effective Frame Stiffness—Gravity (vertical) loads reduce the effective frame stiffness;[4] that is, they reduce the lateral load required to cause a given lateral deflection. This reduction results from the $P\Delta$ effect, which is explained in detail in Appendix A. Briefly, the sideward shift of the gravity loads as the lateral load is applied causes additional moments that increase the lateral deflection by a magnification factor, m. Thus, the frame stiffness is reduced by the factor, m, as illustrated in Fig. 1c and given by:

$$s_e = s/m \tag{1}$$

The magnification factor m is defined by:

$$m = \frac{1}{1 - \dfrac{1.2\,P}{sh}} \tag{2}$$

in which P is the total gravity load, h is the frame height, and s is the frame stiffness. Thus, m depends on both the gravity load and the properties of the frame. Equation (2) is derived in Appendix A. The factor 1.2 in the equation is a correction factor that accounts for the nonlinear variation of the additional ($P\Delta$) moments in the columns of the frame. This proposed factor, which is derived in Appendix B, has not been included in previous $P\Delta$ approaches.[1,2] However, a similar factor was used by LeMessurier in an approximate second-order analysis method.[5]

Critical Load—As the gravity load on a frame increases, the effective lateral stiffness, s_e, of the frame decreases as defined by Eq. (1). When the effective stiffness is 0, the frame cannot resist sideward buckling and the frame has reached its critical frame buckling load, P_{cr}. This occurs when the term $1.2P/sh$ in Eq. (2) equals 1. Therefore,

$$P_{cr} = \frac{sh}{1.2} \text{ for } P_{cr} \le P_y/2 \tag{3}$$

A transition curve is proposed to define the critical frame buckling load when it exceeds $P_y/2$, where P_y is the *total* yield load of all columns in the frame. This transition curve, which is developed in Appendix C, is

$$P_{cr} = P_y - \frac{0.3P_y^2}{sh} \text{ for } P_{cr} \ge P_y/2 \tag{4}$$

The curve is intended to account for the detrimental effects of residual stresses and geometric imperfections at higher loads. As explained in Appendix C, it is analogous to the transition curve used by AISC[6] for the buckling of individual columns. P_{cr} in Eq. (3) is analogous to the Euler buckling load for an individual column.

As discussed in Appendix C, the following factors of safety are consistent with those used by AISC[6] for columns and, therefore, are reasonable for frame buckling:

$$F.S. = 23/12 \text{ for } s \le 0.6P_y/h \tag{5}$$

$$F.S. = \frac{5}{3} + \frac{3}{8}\left(\frac{0.6P_y}{sh}\right) - \frac{1}{8}\left(\frac{0.6P_y}{sh}\right)^3 \text{ for } s \ge 0.6P_y/h \tag{6}$$

Factors Affecting Critical Load.—By using Eqs. (3) and (4), the critical frame buckling load can easily be calculated from the frame stiffness, which can be determined from an ordinary (first-order) analysis as discussed earlier. The equations apply to all types of one-story frames, including multi-bay frames, and frames with diagonal cross bracing.

The frame buckling load, P_{cr}, is the *total* gravity load required to cause frame buckling. It does not depend on the distribution of load. For example, the buckling load for the frame in Fig. 1 is the same whether all the load is applied to one column, half is applied to each column, or the load is applied to the beam somewhere between the columns. This is true because the tops of all columns must move sidewards the same amount when the frame buckles. Thus, the bending stiffness of all columns is mobilized.

The bending stiffness (moment of inertia divided by length) of both the beam and columns affects the frame stiffness and hence the frame buckling load. The cross-

sectional areas of the columns and beam affect the frame buckling load only through their effect on frame stiffness, which is very small, as discussed earlier. The areas of diagonal bracing members, on the other hand, have a major effect on frame stiffness and buckling.

Correction Factor—Equations (3) and (4) include the correction factor 1.2 discussed earlier. As indicated in Appendix B, it is conservative by amounts that are usually less than 10%. As a result, the critical frame buckling loads from Eq. (3) are also slightly conservative. This is confirmed in Appendix D, which compares critical frame buckling loads computed from Eq. (3) with theoretical loads (eigenvalues) from finite-element analyses. The comparison was made for 19 different frames. As expected, the values from Eq. (3) are conservative by amounts ranging from 0 to 14%.

Yielding Instability—The critical buckling load is not affected by lateral loads or by an initial lateral deflection, which could result from construction tolerances. It is a function only of the gravity load and frame properties as defined in Eqs. (3) and (4). Lateral load and initial deflection, however, do affect a related type of frame buckling—yielding instability. Such instability results from $P\Delta$ moments that are not considered in an ordinary (first-order) analysis.

$P\Delta$ Effect—The $P\Delta$ moments are caused by a lateral deflection, Δ_o, of the tops of the columns with respect to their bottoms. The lateral deflection results from three sources: (*1*) lateral deflection caused by lateral load, (*2*) lateral deflection caused by gravity load, and (*3*) construction tolerances. The gravity loads cause a lateral deflection only if they are placed asymmetrically or if the frame itself is asymmetric.

As explained in Appendix A, the $P\Delta$ moments can be calculated by applying a fictitious horizontal load to the top of the frame. This load equals $P\Delta_o/h$. The final moments are obtained by multiplying the moments from this load by the magnification factor from Eq. (2). These moments are conservative at the ends of the beams and columns, but are usually slightly unconservative elsewhere along the columns, as indicated in Appendix A. The $P\Delta$ moments are directly proportional to the total Δ_o from all three sources.

Plastic-Hinge Formation—The effect of the $P\Delta$ moments is illustrated in Fig. 2. The figure shows the lateral deflection, Δ, of a frame as the gravity load, P, is increased while the lateral load, H, is held constant. A similar curve would result if the frame had an initial deflection of Δ_o, but no lateral load. The deflection for either case equals $m\Delta_o$, and m is given by Eq. (2). The curve approaches P_{cr} asymptotically. As the deflection becomes large, however, the $P\Delta$ moments in combination with the moments due to

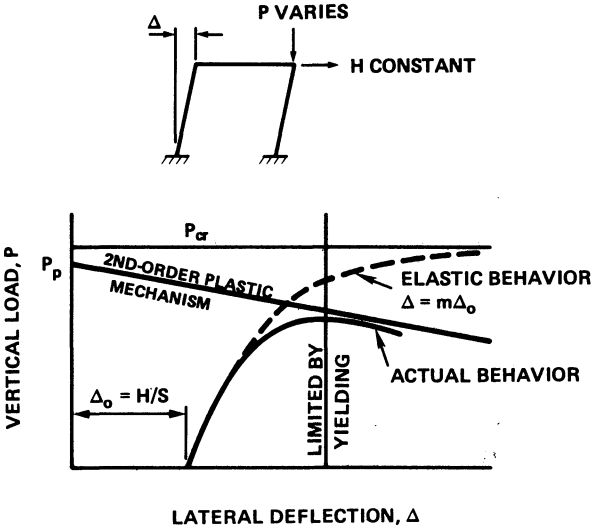

Fig. 2. Load deflection curve showing yielding instability

loading cause large stresses and yielding. Eventually, plastic hinges form and may limit the ultimate load to a value well below P_{cr}.[1,2]

This type of failure can be prevented by including the $P\Delta$ moments in the design of the individual beams and columns. Ordinary design procedures would then assure that excessive yielding does not occur.

PROPOSED DESIGN PROCEDURE

Any frame can be designed to resist buckling by checking three failure modes: (*1*) critical buckling of the frame, (*2*) yielding instability of the frame, and (*3*) buckling of the individual columns. The three checks can be made independently and are discussed separately.

The lateral stiffness of the frame, which is needed in making these checks, can be determined from an ordinary (first-order) analysis in which the frame is subjected to a unit horizontal load at the top. The frame stiffness, s, is the reciprocal of the horizontal deflection, Δ, of the top. The moments caused by the unit load are also used in the checks.

Critical Buckling Load—The allowable critical frame-buckling load is given by:

$$P_{cr} = \frac{12}{23}\left[\frac{sh}{1.2}\right] \text{ for } P_{cr} \le P_y/2 \qquad (7)$$

$$P_{cr} = \left[\frac{1}{\frac{5}{8} + \frac{3}{8}\left(\frac{0.6P_y}{sh}\right) - \frac{1}{8}\left(\frac{0.6P_y}{sh}\right)^3}\right] \times \left[P_y - \frac{0.3P_y^2}{sh}\right]$$

$$\text{for } P_{cr} \ge P_y/2 \qquad (8)$$

in which h is the frame height, P_y is the *total* yield load of all the columns in the frame, and P_{cr} is the *total* gravity load necessary to cause frame buckling.

If the actual P is greater than P_{cr}, the frame must be stiffened. This can be done by increasing the moments of inertia of the beams and/or columns in any combination that provides the required stiffness, s. Thus, proportioning the structure to resist critical frame buckling is exactly the same as proportioning it to resist drift.

P_{cr} is independent of any lateral load or initial lateral deflection. The equations for P_{cr} are analogous to the AISC[6] equations for column buckling. In fact, if applied to a single cantilever column (unsupported at the top), Eqs. (7) and (8) gives answers very close to those from the AISC column buckling equations [Eqs. (1.5-1) and (1.5-2) from the AISC Specification]. The first term in Eqs. (7) and (8) is a factor of safety analogous to the variable AISC factor.[6] Similarly, the second term in Eq. (8) is based on an empirical curve similar to that used by AISC to account for the effects of residual stresses and geometric imperfections. The second term in Eq. (7) is analogous to the Euler buckling equation for a single cantilever column. Equations (7) and (8) control the frame buckling check when the lateral load and initial lateral deflection due to construction inaccuracies are small; otherwise, yielding instability discussed in the following paragraphs controls.

Yielding Instability—$P\Delta$ moments in combination with the moments due to loading may cause excessive yielding and failure of a frame at gravity loads well below P_{cr}. This type of failure, which is referred to as *yielding instability*, can be prevented by including the $P\Delta$ moments in the design of all individual members. Because the $P\Delta$ effect is nonlinear with respect to P, the $P\Delta$ moments must be determined at a gravity load β times the design (working) load and then divided by β. The resulting moments can then be treated as any other moments in a working stress design. The load factor 1.7 used in the plastic design part of the AISC Specification should be adequate for β, but some designers may prefer to use a higher value.

The $P\Delta$ moments in all members can be obtained by multiplying the moments caused by the unit horizontal load discussed earlier by the factor $mP\Delta_o/h$, in which P is the total gravity load on the frame. Axial loads in the columns due to the $P\Delta$ effect can be obtained from the unit-horizontal-load case in the same way. The magnification factor m is given by

$$m = \frac{1}{1 - \dfrac{1.2\,\beta P}{sh}} \qquad (9)$$

The initial deflection Δ_o is the sum of the deflections from three sources: (1) lateral deflection caused by the factored lateral load βH, (2) lateral deflection caused by the factored gravity load βP, and (3) the lateral deflection caused by construction inaccuracies. The parts of Δ_o caused by βH and βP can be calculated by an ordinary (first-order) analysis. The gravity load, βP, causes lateral deflection only if it is placed asymmetrically, or if the frame itself is asymmetric.

Column Buckling Load—If frame buckling is prevented by the procedures discussed earlier, an individual column can buckle only by bowing between its ends. Therefore, the effective length of the column cannot exceed its actual length and the effective-length factor, K, can be conveniently taken as 1.0. Alternatively, a lower factor that accounts for the rotational restraint provided by the beams could be used.

The columns are usually subjected to end moments as well as axial loads. Consequently, interaction equations, such as Eq. (1.6-1a) in the AISC Specification, are usually needed to design the columns. These interaction equations include an axial-load term and a bending term. The bending term in the AISC equation includes a magnification factor that is similar to the one for frames, but magnifies the deflection of the central part of the column with respect to its ends. The effective-length factor, K, should be taken as 1 in calculating this magnification factor as well as in calculating the axial-load term. The AISC bending term also includes a factor C_m that accounts for various end-moment ratios. The formula given by AISC for "restrained compression members in frames braced against joint translation" should be used in calculating C_m. The $P\Delta$ moments, of course, must be included in checking the individual columns.

SUMMARY

A simple, conservative $P\Delta$ method has been proposed for designing one-story frames against frame buckling; that is, sidesway-type buckling. The critical frame buckling load is given by a simple equation involving the lateral frame stiffness, which can be conveniently calculated by ordinary (first-order) methods. A factor of safety must be provided against frame buckling regardless of whether lateral loads are applied or not. To prevent yielding instability, all members must also be designed for $P\Delta$ moments that can be calculated from an ordinary (first-order) analysis of a frame by applying a magnification factor that again involves the lateral frame stiffness. The individual columns must be checked for column buckling; that is, bowing between ends that are held against lateral movement by the frame. An effective-length factor of 1 should be used in this check.

The equation for the critical frame buckling load and the magnification factor include a factor to account for differences between the actual and assumed distribution of moment when the $P\Delta$ effect is represented by a fictitious lateral load. As a result, these equations are conservative by amounts that are usually less than 10%. An empirical buckling-strength curve is used to account for the effects

of residual stresses and geometric imperfections on the critical frame buckling load. This curve is similar to the widely used column-strength curve for columns.

The proposed method can be applied consistently for both gravity loading and combined gravity and lateral loading. It conservatively predicts frame behavior in a manner that is easily understood and requires only ordinary (first-order) analyses. The author is investigating ways of extending the method to multistory frames.

SYMBOLS

β = load factor
C_m = factor in AISC interaction equation
Δ = lateral deflection
Δ_o = first-order or initial lateral deflection
E = modulus of elasticity
$F.S.$ = factor of safety
h = story height
H = horizontal force
I = moment of inertia
K = effective-length factor
l = bay width
L = member length (beam or column)
m = magnification factor
M = moment
P = vertical load
P_y = frame yield load; sum of yield loads of all individual columns
r = radius of gyration
s = frame stiffness
s_e = effective frame stiffness
σ = stress
A = area

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APPENDIX A. $P\Delta$ EFFECT

The sideward shift of the gravity loads as a lateral load is applied to a frame causes additional moments that increase the lateral deflection. This behavior, which is referred to as the $P\Delta$ effect, is illustrated in Fig. A1. For simplicity, a frame with pinned bases and a rigid beam (infinite moment of inertia) is considered, but the principles discussed apply to any one-story frame.

Fictitious Lateral Load—The gravity loads in the example, if applied alone, cause no moment because they are placed directly over the columns as illustrated in Fig. A1b. The horizontal force, H, if applied alone, causes lateral deflection, moments, and forces as illustrated in Fig. A1a. If P and H are applied together, the sideward shift of P by the amount Δ_o causes additional moments and forces due to P as shown in Fig. A1c. The moments and forces due to H do not change. The additional moments and forces caused by the shift of P are called $P\Delta$ moments and forces.

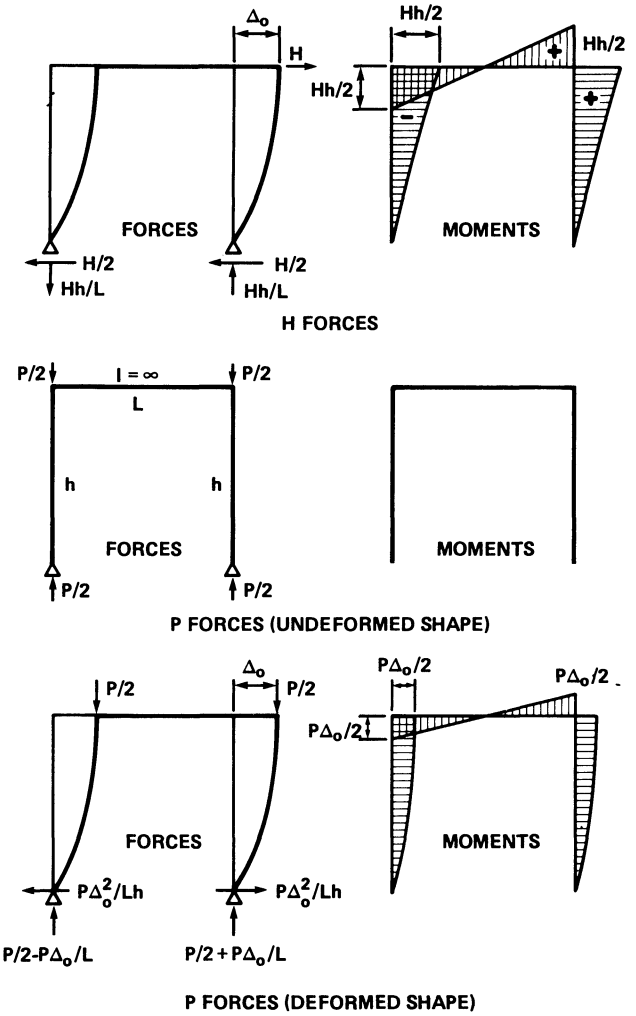

Fig. A1. Moments and forces on frame

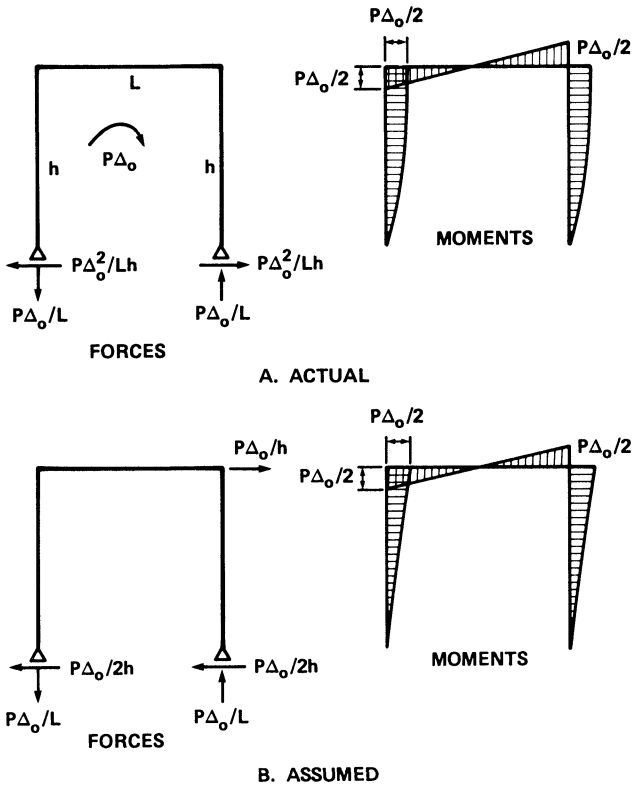

Fig. A2. PΔ moments and forces on frame

They can be calculated by subtracting the moments and forces for the undeformed shape (Fig. A1b) from those for the deformed shape (Fig. A1c). The $P\Delta$ moments and forces for the example are shown in Fig. A2a.

The $P\Delta$ moments in Fig. A2a cause an additional lateral deflection and shift of P, which in turn cause more moment and a further shift of P. The final deflection, moments, and forces can be determined by an iterative process in which the additional deflections and moments get smaller with each succeeding cycle. To facilitate this process, it is common to approximate the $P\Delta$ effect by applying a fictitious lateral load to the frame as shown in Fig. A2b. The fictitious load equals $P\Delta_o/h$ and causes moments similar to, but not exactly the same as, those caused by the actual $P\Delta$ effect. The difference is discussed in *Appendix B.*

Series Solution of Iterative Process—The iterative process of calculating the final deflection and moments is as follows. The initial lateral deflection caused by the lateral load, H, without the gravity load, P, is Δ_o. The fictitious lateral load for the first cycle is $P\Delta_o/h$ and the additional deflection caused by this fictitious load is:

$$\Delta_1 = \left(\frac{P}{sh}\right)\Delta_o \qquad \text{(A1)}$$

This additional deflection, Δ_1, causes an additional deflection:

$$\Delta_2 = \left(\frac{P}{sh}\right)\Delta_1 = \left(\frac{P}{sh}\right)^2\Delta_o \qquad \text{(A2)}$$

and for the nth cycle

$$\Delta_n = \left(\frac{P}{sh}\right)^n\Delta_o \qquad \text{(A3)}$$

The ratio of the final deflection to the initial deflection will be called the magnification factor, m, and is given by:

$$m = \frac{\Sigma\Delta_n}{\Delta_o} = \Sigma\frac{\Delta_o}{\Delta_o} + \frac{\Delta_1}{\Delta_o} \cdots\cdots \frac{\Delta n}{\Delta_o} \qquad \text{(A4)}$$

Substituting the expression for Δ into this series gives:

$$m = \sum_o^\infty 1 + \frac{P}{sh} + \left(\frac{P}{sh}\right)^2 \cdots\cdots + \left(\frac{P}{sh}\right)^n \qquad \text{(A5)}$$

and the sum of the series is

$$m = \frac{1}{1 - \dfrac{P}{sh}} \text{ when } P/sh < 1 \qquad \text{(A6)}$$

The correct final deflection can be obtained without iterations by simply multiplying the initial deflection by the magnification factor from Eq. (A6). Thus,

$$\Delta = m\Delta_o = \frac{mH}{s} \qquad \text{(A7)}$$

The moments and vertical forces caused by the sideward shift of P are similar to the moments and vertical forces caused by H and can be calculated by multiplying the latter moments and vertical forces by m. Thus, the final moments, forces, and deflections as well as the effective frame stiffness can be calculated very simply and directly from the frame stiffness determined by an ordinary (first-order) analysis.

APPENDIX B. CORRECTION FACTOR

The actual $P\Delta$ moments in the columns of a frame differ from those caused by the assumed fictitious lateral load discussed in *Appendix A.* As illustrated by the example in Figs. A1 and A2, the assumed moments vary linearly along the column, whereas the actual moments vary nonlinearly in accord with the deflected shape. Thus, the actual moment is larger over most of the column height and this larger moment causes more lateral deflection. The actual and assumed beam moments are the same.

In Fig. B1, the method of virtual work is used to calculate the ratio of the actual deflection to the assumed deflection (deflection resulting from the assumed fictitious lateral load) for the example frame. The ratio is 1.2.

The moments in the beam did not enter into the calculation, because the stiffness of the beam is infinite and consequently no energy is stored in the beam. However, if

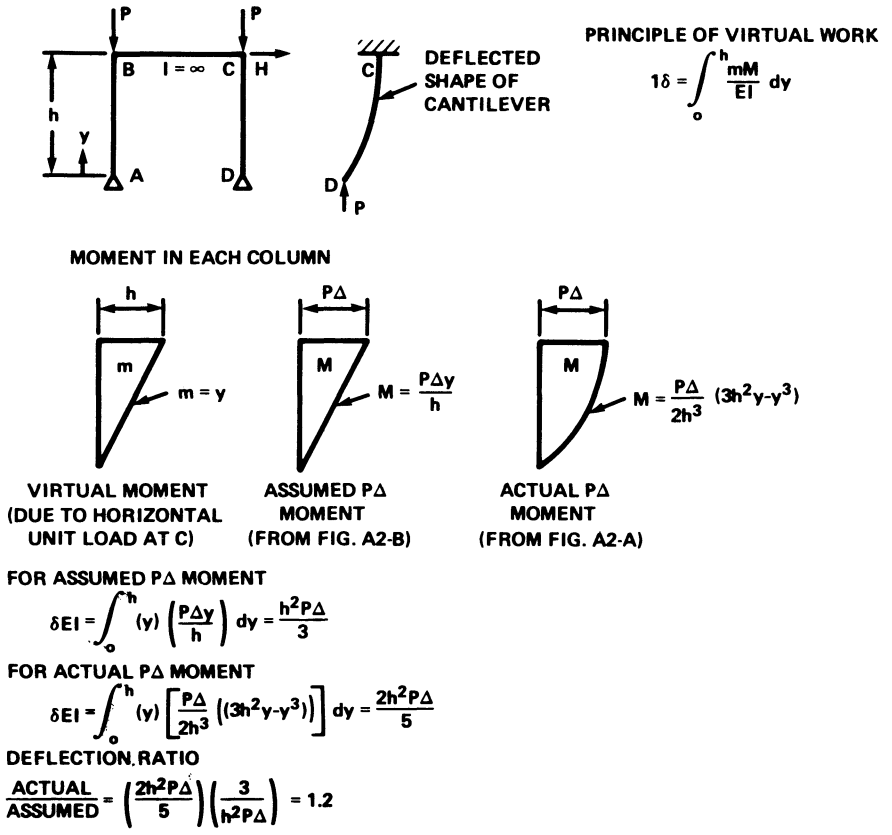

Fig. B1. Derivation of correction factor

the beam stiffness is not infinite, the energy stored in the beam must be included. Since the actual and assumed moments are the same in the beam, this would result in a lower value of the actual/assumed deflection ratio and this value would depend on the relative stiffness of the beam and columns.

The variation of the actual/assumed deflection ratio with relative beam/column stiffness for frames with fixed and pinned bases is shown in Fig. B2. The member stiffness is defined as its moment of inertia, I, divided by its length, L. For the fixed frame, the ratio is 1.2 when the beam stiffness is infinite. It is again 1.2 when the beam stiffness is 0, because the beam stores no energy. Between these limits, the ratio is smaller and reaches a minimum value of 1.125 at a beam/column stiffness ratio of 0.75. For the pinned frame, the ratio is 1.2 when the beam stiffness is infinite, and decreases to 1.0 when the beam stiffness is 0. When the beam stiffness is close to 0, the column curvature is small and the corresponding $P\Delta$ moments are almost linear. For other types of one-story frames, such as those with multiple bays or with cross bracing, 1.2 is again the maximum value. Therefore, it is appropriate to use this value as a correction factor in design applications.

The correction factor can be included in the magnification factor in the following way. The additional deflection for each cycle of the iterative process is:

$$\Delta_n = \left(\frac{1.2\,P}{sh}\right)\Delta_{n-1} \tag{B1}$$

Therefore, the corrected magnification factor is:

$$m = \frac{1}{1 - \dfrac{1.2P}{sh}} \quad \text{when } 1.2P/sh < 1 \tag{B2}$$

This equation is conservative by amounts ranging from 0 to 6% for fixed frames, and 0 to 20% for pinned frames. The values above 11% are for pinned frames with a beam stiffness less than the column stiffness. Such frames have little lateral stiffness and therefore are usually not practical.

The correction factor for the example covered in Fig. B1 can also be derived by comparing the frame-stiffness buckling load with the Euler buckling loads for the two columns in the frame. When the frame buckles, each column acts as an upside-down cantilever with the rigid beam as a base. The Euler buckling load for each column is:

$$P_{crE} = \pi^2 EI/4h^2 \tag{B3}$$

and the corresponding total buckling load for the frame is twice this amount. The lateral deflection caused by a lateral load, V, at the end of a cantilever is:

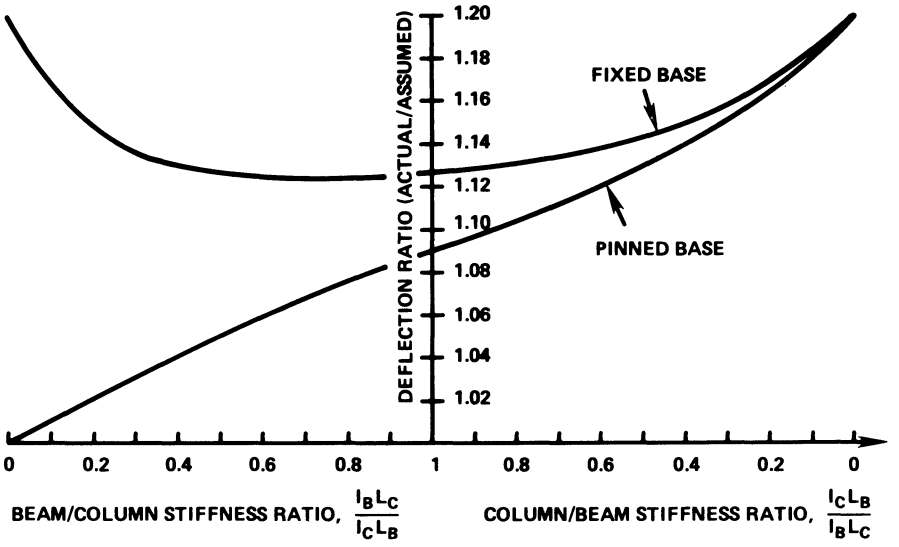

Fig. B2. Deflection ratio as a function of stiffness ratio for frame

$$\Delta = Vh^3/3EI \tag{B4}$$

therefore, the lateral stiffness of the frame is:

$$s = 2V/\Delta = 6EI/h^3 \tag{B5}$$

Thus, the frame-stiffness buckling load is:

$$P_{cr} = sh/C = 6EI/Ch^2 \tag{B6}$$

in which C is the correction factor. Equating the two buckling loads [Eq. (B5) and Eq. (B3) times 2] gives:

$$C = 12/\pi^2 = 1.216 \tag{B7}$$

APPENDIX C. BUCKLING-STRENGTH CURVE

The critical frame buckling load defined by Eq. (3) of the text is analogous to the Euler buckling load of an individual column in that it does not account for the detrimental effects of residual stresses and various geometric imperfections. The empirical curve shown in Fig. C1a is widely used[2] to account for these detrimental effects in columns. The empirical curve is a parabola that is tangent to the Euler curve at a stress equal to half the yield stress of the column. At its other end, the curve is tangent to a horizontal line at the yield stress.

A similar empirical curve can be used for frame buckling when P_{cr} exceeds $0.5P_y$. Such a curve is shown in Fig. C1b and is defined by the equation

$$P_{cr} = P_y - \frac{0.3P_y{}^2}{sh} \text{ for } P_{cr} > P_y/2 \tag{C1}$$

P_y is the *total* yield load of all columns in the frame. The factor $1/\sqrt{s}$ is used as the abscissa in Fig. C1b to make the plot analogous to that in Fig. C1a.

The variable factor of safety specified[6] by the AISC is generally used with the buckling-strength curve for individual columns. This factor varies from 1.92 at $P_{cr} = 0.5P_y$ to 1.67 at $P_{cr} = P_y$. For frame buckling a similar variation is provided by the equation

$$F.S. = \frac{5}{3} + \frac{3}{8}\left(\frac{0.6P_y}{sh}\right) - \frac{1}{8}\left(\frac{0.6P_y}{sh}\right)^3$$

when $s > 0.6P_y/h$ (C2)

$$F.S. = 23/12 \text{ when } s \le 0.6P_y/h \tag{C3}$$

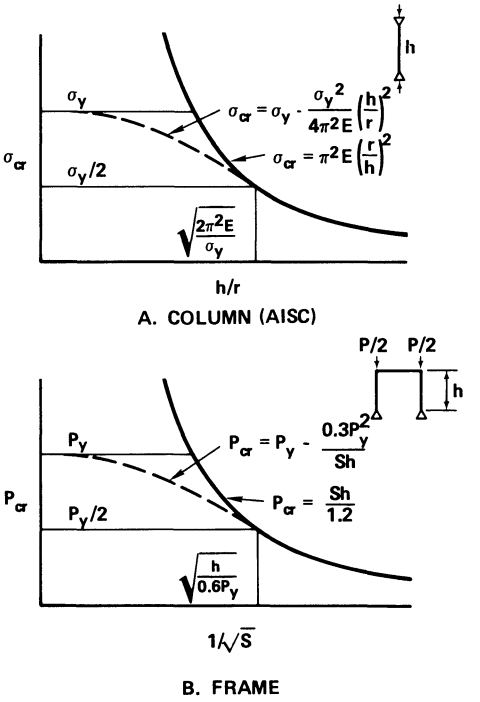

Fig. C1. Buckling strength curves

Table—D1. Frame Buckling Loads (see Fig. D1)

Column Bases	Moment of Inertia, in.⁴ C1	C2	C3	B1	B2	Area Ratio* A/I	Area of D1 (in.²)	Frame Stiffness (s) (kip/in.)	Buckling Load Proposed Method	Exact** Method	Proposed Exact
Fixed	100	100	—	0	—	∞	0	17.40	1,450	1,450	1.000
	100	100	—	50	—	∞	0	39.77	3,314	3,530	0.939
	100	100	—	100	—	∞	0	48.72	4,060	4,319	0.940
	100	100	—	200	—	∞	0	56.55	4,712	4,944	0.953
	100	100	—	∞	—	∞	0	69.60	5,800	5,800	1.000
	150	50	—	100	—	∞	0	45.36	3,780	3,900	0.969
	100	100	—	100	—	∞	50	5,175	431,250	458,790	0.940
	100	100	—	100	—	0.1	0	48.22	4,018	4,277	0.940
Pinned	100	100	—	50	—	∞	0	8.70	725	829	0.875
	100	100	—	100	—	∞	0	11.60	967	1,065	0.908
	100	100	—	200	—	∞	0	13.92	1,160	1,234	0.940
	100	100	—	∞		∞	0	17.40	1,450	1,450	1.000
	150	50	—	100	—	∞	0	10.38	865	890	0.972
	100	100	—	100	—	∞	50	5,138	428,167	471,862	0.907
	100	100	—	100	—	0.1	0	11.50	958	1,057	0.907
Fixed	100	100	100	100	200	∞	0	75.93	6,328	6,505	0.973
	100	100	100	100	200	∞	2	280.99	23,416	24,706	0.948
Pinned	100	100	100	100	200	∞	0	18.16	1,514	1,527	0.992
	100	100	100	100	200	∞	2	228.22	18,602	21,615	0.861

* Area ratio for all members except D1.

** Eigenvalue with a single sidesway degree of freedom assumed.

Note: Dash indicates member does not exist.

APPENDIX D. ACCURACY OF PROPOSED METHOD OF CALCULATING THE CRITICAL BUCKLING LOAD

Equation (3) of the text, which gives the critical buckling load of a frame, is not exact because of differences between the actual and assumed $P\Delta$ moments. Table D1 compares the critical buckling load from this equation with the more exact theoretical buckling load described below. The comparison was made for 19 different frames.

The frame stiffness and the more exact theoretical buckling load for each frame were determined by using the general-purpose finite-element program ANSYS.[7] For these analyses, each column was divided into 10 elements. The theoretical buckling load was determined by the eigenvalue procedure in this program. A single degree of freedom was assumed. When applied to a single cantilever column with no bracing at the top, the procedure gives a buckling load of about 1.5% above that given by the Euler buckling equation. Both buckling loads neglect the very small effects of member shortening by assuming that the cross sectional areas are infinite except in two cases.

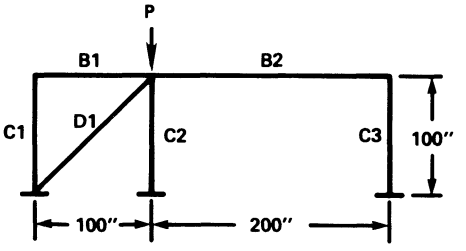

Fig. D1. Frame configuration, Table D1

For the frames considered, the frame buckling load from the proposed method ranged from 0 to 14% below the corresponding more exact buckling load. The average difference was 5.5%.