Rapid Determination of Ultimate Strength of Eccentrically Loaded Bolt Groups

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The Eighth Edition *Manual of Steel Construction* of the American Institute of Steel Construction[1] bases its Tables X through XVIII for "Eccentric Loads on Fastener Groups" on an ultimate strength method.[2] In this method, relative rotation of the connected parts occurs about a given instantaneous center, causing, on each connector, a force which acts perpendicular to the ray joining that connector to the instantaneous center.

NOMENCLATURE

n = quantity of bolts
P_x = horizontal component of normalized applied force P
P_y = vertical component of normalized applied force P
x_o = x-coordinate of centroid
Y_o = y-coordinate of centroid
M_o = moment of P about centroid
J = polar moment of inertia
x, d_x = x-distance of bolt from centroid or instantaneous center
y, d_y = y-distance of bolt from centroid or instantaneous center
d = distance of bolt from instantaneous center
d_{max} = largest magnitude of d
Δ = delta, in load deformation expression
M_i = moment about instantaneous center of bolt force divided by R_{ult}
M_p = moment about instantaneous center of normalized applied force P
R_{ult} = ultimate bolt force
r_v = elastic solution for force on most highly stressed bolt
C_e = elastic solution for bolt coefficient

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R_x = x-component of bolt force
R_y = y-component of bolt force
F_{xy} = x-component of unbalanced force
F_{yy} = y-component of unbalanced force
F = magnitude of unbalanced force
C_u = bolt coefficient for ultimate strength solution
a_x = x-component of distance from centroid of bolt group to instantaneous center
a_y = y-component of distance from centroid of bolt group to instantaneous center
k = elastic coefficient which multiplies d to give bolt force
e = base of natural logarithms
P_{max} = elastic solution for maximum allowable eccentric load
r_{v1} = elastic solution for force on most highly stressed bolt due to normalized applied load
R_i = force on one bolt when ultimate load is applied
P_{ult} = the ultimate load

BACKGROUND

For high-strength bolted connections, the deformation Δ_{max} of the bolt furthest from that center is assumed to be 0.34 in., and all other deformations can be found by assuming them to be linearly proportional to their distances from the center. The load-deformation relationship is non-linear, and is

$$R = R_{ult}(1 - e^{-10\Delta})^{0.55}$$

so the force on each bolt is determined in magnitude, direction, and point of application. It is then a simple application of statics to find the applied force and moment which will produce a selected R_{ult}.

Unfortunately, the engineer's problem is never the one outlined above. Instead, he must solve the inverse problem, starting with the applied forces; even if the bolt arrangement is prescribed, the determination of the capacity of the connection must proceed by trial and error, since the location of the instantaneous center is unknown. For selected common arrangements of bolts, it is the purpose of the tables in the *Manual* to provide direct solutions in the form:

$$P = C \times r_v$$

where

P = permissible eccentric load on the bolt group
r_v = permissible load on one fastener
C = tabulated coefficients

Coefficients for a reasonable number of bolt arrangements and eccentric loads are tabulated.

Where unusual fastener group arrangements do not conform to those in the tables, the *Manual* suggests an elastic or modified elastic procedure. The elastic procedure generally produces conservative results; the modified elastic procedure can produce results which are unconservative compared to those of the ultimate strength method.* This paper discusses neither the theoretical development nor the experimental verification of any of the methods; the objective is to improve the efficiency of calculating the ultimate strength solution.

In this paper, a procedure is given for:

1. Directly finding the instantaneous center corresponding to elastic behavior of any bolt arrangement for any eccentric load.
2. Directly determining the corresponding coefficient relating permissible eccentric load on the group to permissible load on one bolt, for the elastic case.
3. Directly determining an *approximate* coefficient relating permissible eccentric load on the group to permissible load on one bolt, for ultimate strength.
4. Iterating to improve the approximate coefficient.

The procedure is only slightly more complex than the standard elastic procedure in which superposition is used, and is amenable to solution using a pocket calculator, if the number of bolts is not too large. A Fortran program has also been written, suitable for use on a micro-computer. A fair number of test cases has shown that:

(*a*) The permissible load for the elastic case is usually a lower bound for the correct ultimate strength.
(*b*) The approximate permissible load for the first ultimate strength determination is usually an upper bound for the correct ultimate strength.
(*c*) Each iteration usually comes closer to the true position of the instantaneous center, with the approximate ultimate strength coefficient descending monotonically.

* *The modified elastic procedure has been removed by the 8th Edition Manual Errata published in the Engineering Journal, Second Quarter, 1981.*

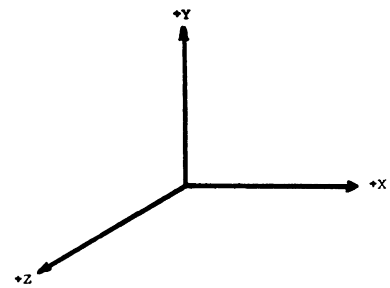

Fig. 1. Positive axes

(*d*) To achieve a precision of 1% in the coefficient, the number of iterations required is usually very small (two or three).

The procedure described herein should make it possible for engineers to estimate ultimate capacities of both groups with great rapidity. Note that when (*a*) and (*b*) above are true, the engineer can decide quickly whether to proceed with further iterations, or to settle for the alternative somewhat conservative elastic value.

LOCATING THE INSTANTANEOUS CENTER FOR THE ELASTIC CASE

Define positive directions for X, Y, and Z according to the right hand rule, as shown in Fig. 1. Thus, distances or forces to the right or up are positive, as are counter-clockwise moments. Meticulous attention to signs is required.

Textbooks on steel structures[3] contain proofs that the resisting forces developed on a bolt located at coordinate distances x and y from the centroid (x_o, y_o) are:

$$-\frac{P_x}{n}, -\frac{P_y}{n}, \frac{M_o y}{J}, -\frac{M_o x}{J}$$

where $J = \Sigma(x^2 + y^2)$ for all bolts (Fig. 2). Forces P_x and P_y act through the centroid, and M_o is the moment about the centroid.

The quantity M_o/J can be considered a mapping function which transforms distances to forces. From a new center, with coordinates a_x, a_y (Fig. 3), the forces on the bolt will be:

$$\text{In the } Y\text{-direction:} \quad -\frac{M_o}{J}(x - a_x)$$

$$\text{In the } X\text{-direction:} \quad +\frac{M_o}{J}(y - a_y)$$

To locate the instantaneous center, the values of a_x and a_y must be found such that:

$$-\frac{M_o}{J}(x - a_x) = -\frac{M_o}{J}x - \frac{P_y}{n}$$

and

$$+\frac{M_o}{J}(y - a_y) = \frac{M_o}{J}y - \frac{P_x}{n}$$

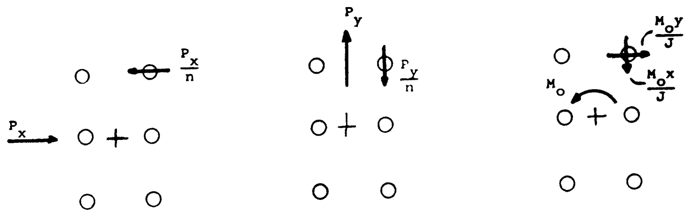

Fig. 2. Resisting forces

from which

$$a_x = -\frac{P_y}{n}\frac{J}{M_o}$$

$$a_y = \frac{P_x}{n}\frac{J}{M_o}$$

and the coordinates of the instantaneous center for the elastic case are

$$x_1 = x_o + a_x$$

$$y_1 = y_o + a_y$$

Therefore, the procedure is seen to begin familiarly, with calculations locating the centroid, determining the sum of the moments about the centroid, and obtaining the polar moment of inertia. The location of the instantaneous (elastic) center is found by adding algebraically the quantities a_x and a_y to the centroidal coordinates.

In what follows, applied load P is normalized to have unit magnitude.

DETERMINING THE ELASTIC COEFFICIENT (C_e)

Measuring d from the elastic instantaneous center (Fig. 4), the force on each bolt due to the normalized force is kd, and the moment of that force is kd^2. The sum of all the bolt moments is $k\Sigma d^2$ and this, plus M_p, the moment of the applied load about the center, must equal zero.

Therefore, the bolt furthest from the instantaneous center is the most highly stressed bolt, and it will sustain a load $r_{v1} = kd_{max} = (-M_p/\Sigma d^2)d_{max}$. Note that M_p is the moment due to normalized force P (of unit magnitude) relative to the instantaneous center. Therefore, the true maximum bolt force, r_v, will be $P_{max}r_{v1}$, yielding:

$$r_v = P_{max}r_{v1} = P_{max}(M_p/\Sigma d^2)d_{max}$$

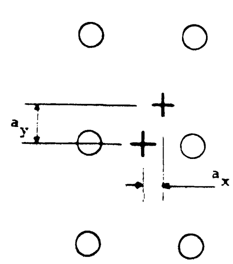

Fig. 3. Relocated center

Solving for P_{max},

$$P_{max} = -\frac{\Sigma^2}{M_p d_{max}} r_v = C_e r_v$$

Observe that elastic coefficient C_e is unsigned, with only its magnitude needed.

DETERMINING AN APPROXIMATE VALUE FOR THE ULTIMATE STRENGTH COEFFICIENT (C_{u1})

If the elastic instantaneous center is used as an approximate location for the instantaneous center for determining ultimate strength, the deformation at each bolt can be found from:

$$\Delta = 0.34\ (d_i/d_{max})$$

and the ratio of the force on each bolt (R) to the ultimate bolt capacity (R_{ult}) can be found from

$$R/R_{ult} = (1 - e^{-10\Delta})^{0.55}$$

M, the moment about the instantaneous center of each bolt's force fraction, is obtained by multiplying the force fraction by its distance. Total moment of all the bolts is:

$$R_{ult}(\Sigma M_i) = R_{ult}\Sigma(R_i/R_{ult})d_i$$

As noted in the elastic solution, M_p is due to a normalized force P. Therefore, the ultimate moment due to an applied load P_{ult} will be $P_{ult}M_p$. Summing moments:

$$P_{ult}M_p + R_{ult}\Sigma M_i = 0$$

$$P_{ult} = -(\Sigma M_i/M_p)R_{ult} = C_u R_{ult}$$

Coefficient C_u is unsigned, with only its magnitude needed. Call this first approximation C_{u1}.

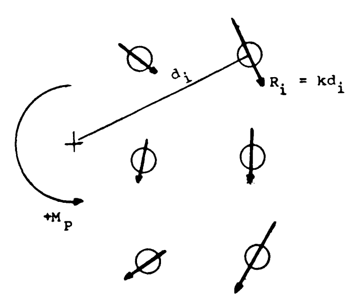

Fig. 4. Elastic resisting forces

ITERATING TO IMPROVE THE APPROXIMATE COEFFICIENT

When the instantaneous center has been correctly located, not only will the sum of the moments about that center be zero, but the vector sum of the forces will also be zero.

For the elastic case, the X and Y force components on each bolt can be found from $(+M_p/\Sigma d^2)d_y$ and $(-M_p/\Sigma d^2)d_x$, respectively. These will be found to sum to zero. (See Example 1, below.)

For the ultimate strength case, the X and Y force components on each bolt can be found as:

$$R_x = -\frac{d_y}{d}\frac{R}{R_{ult}}R_{ult}$$

$$R_y = \frac{d_x}{d}\frac{R}{R_{ult}}R_{ult}$$

When each of these is summed and added to the respective components of the applied load, the results will be:

$$F_{xx} = P_x + \Sigma R_x$$

$$F_{yy} = P_y + \Sigma R_y$$

$$\vec{F} = \vec{F}_{xx} + \vec{F}_{yy}$$

and the magnitude of force F will not be zero unless the instantaneous center is correctly located.

It is clear that the smaller the values for F_{xx} and F_{yy}, the closer the trial center is to the correct center. It has been found that components of a desirable shift from the previous trial center can usually be predicted from the same formulas used to locate the elastic instantaneous center. Thus, the coordinates of the next center should be:

$$x_2 = x_1 - \frac{F_{yy}J}{nM_0}$$

$$y_2 = y_1 + \frac{F_{xx}J}{nM_0}$$

From this point, the ultimate strength solution is repeated until the unbalanced forces are sufficiently close to zero. Successive values of the coefficient (C_{u2}, C_{u3}, etc.) should approach the correct ultimate strength value.

An h-square extrapolation can be used after the second iteration to predict the final value of C_u. Letting F_1 and C_{u1} be the values of unbalanced force and ultimate strength coefficient at the end of the first iteration, and F_2 and C_{u2} the values at the end of the second iteration, the formula is:

$$C_u = \frac{F_1{}^2C_{u2} - F_2{}^2C_{u1}}{F_1{}^2 - F_2{}^2}$$

Tests of several bolt groups have been made using the computer program; comparisons of extrapolated values with the results of the final iteration have shown generally good agreement.

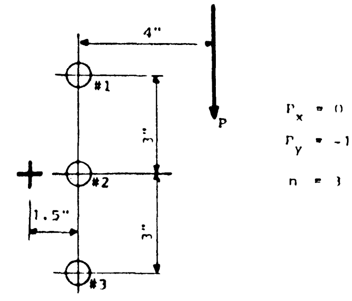

Fig. 5. Example 1

EXAMPLE 1

This simple example is chosen to illustrate the procedure, using manual calculations. Its results can be confirmed using Table X in the Seventh Edition AISC *Manual*[4] for the elastic coefficient, and using Table X in the Eighth Edition AISC *Manual* for the ultimate strength coefficient. (See Fig. 5.)

By inspection, the centroid of the bolt group coincides with bolt #2. The moment of the applied load about the centroid is $M_o = -4$ and $J = 2 \times 3^2 = 18$.

Locate the elastic instantaneous center:

$$a_x = -\frac{P_yJ}{nM_o} = \frac{1 \times 18}{3 \times (-4)} = -1.5$$

$$a_y = \frac{P_xJ}{nM_o} = 0$$

Determine the elastic coefficient (see Table 1.1):

$$M_p = -5.5$$

$$C_e = \frac{\Sigma d^2}{M_p d_{max}} = \frac{24.748}{5.5 \times 3.354} = 1.342$$

To demonstrate that the location of the instantaneous center is correct, bolt force components R_x and R_y are found for each bolt (see Table 1.1). To avoid having P_{max} appear in every term, choose $P_{max} = 1$. Then the calculations for bolt #1 are:

$$R_x = +\frac{M_p}{\Sigma d^2}d_y = \frac{-5.5}{24.748} \times 3 = -0.667$$

$$R_y = -\frac{M_p}{\Sigma d^2}d_x = \frac{+5.5}{24.748} \times 1.5 = +0.333$$

Table 1.1

	Distances from i.c.					
Bolt	d_x	d_y	d	d^2	R_x	R_y
1	1.5	3	3.354	11.249	−0.667	+0.333
2	1.5	0	1.500	2.250	0.0	+0.333
3	1.5	−3	3.354	11.249	+0.667	+0.333
Summation:				24.748	0.0	+0.999

Table 1.2

Bolt	Distances from i.c. d_x	d_y	d	Δ	R/R_{ult}	M	R_x	R_y
1	1.5	3	3.354	0.34	0.9815	3.292	−0.612	+0.306
2	1.5	0	1.500	0.1521	0.8731	1.310	0.0	+0.609
3	1.5	−3	3.354	0.34	0.9815	3.292	+0.612	+0.306
	Summation:					7.894	0.0	+1.221

and the total forces are obtained:

$$F_{xx} = P_x + \Sigma R_x = 0 + 0 = 0$$

$$F_{yy} = P_y + \Sigma R_y = -1 + 0.999 = -0.001 \text{ (say zero)}$$

For the elastic case, the F_{xx} and F_{yy} calculations are merely checks and are not necessary.

Determine the first ultimate strength coefficient (see Table 1.2):

$$\Delta_i = 0.34 \frac{d_i}{d_{max}}$$

$$\frac{R_i}{R_{ult}} = (1 - e^{-10\Delta_i})^{0.55}$$

$$M_i = \frac{R_i}{R_{ult}} d_i$$

$$M_p = -5.5$$

$$C_{u1} = \frac{\Sigma M_i}{M_p} = \frac{7.894}{5.5} = 1.435$$

Choosing $P_{ult} = 1$,

$$R_{ult} = -\frac{M_p}{\Sigma M_i} = \frac{+5.5}{7.894} = 0.697$$

$$R_{x1} = -\frac{d_{y1}}{d_1}\frac{R_1}{R_{ult}} R_{ult} = \frac{-3}{3.354} \times 0.9815 \times 0.697 = -0.612$$

$$R_{y1} = \frac{d_{x1}}{d_1}\frac{R_1}{R_{ult}} R_{ult} = \frac{1.5}{3.354} \times 0.9815 \times 0.697 = 0.306$$

Determine the unbalanced forces:

$$F_{xx} = P_x + \Sigma R_x = 0 + 0 = 0$$

$$F_{yy} = P_y + \Sigma R_y = -1 + 1.221 = +0.221$$

$$\vec{F} = \vec{F}_{xx} + \vec{F}_{yy} = 0.221$$

Locate the new instantaneous center:

$$-\frac{F_{yy}J}{nM_0} = \frac{-0.221 \times 18}{3 \times (-4)} = +0.332$$

$$\frac{F_{xx}J}{nM_0} = 0$$

New center is 0.332 in. right of the previous position. Repeat calculations (see Table 1.3).

$$M_p = -5.168$$

$$C_{u2} = \frac{\Sigma M_i}{M_p} = \frac{7.285}{5.168} = 1.410$$

$$R_{ult} = -\frac{M_p}{\Sigma M_i} = \frac{+5.168}{7.285} = 0.709$$

$$F_{xx} = 0$$

$$F_{yy} = +0.091$$

$$F = 0.091$$

The extrapolation formula predicts the final value of C_u as:

$$C_u = \frac{F_1^2 C_{u2} - F_2^2 C_{u1}}{F_1^2 - F_2^2} = \frac{(0.221)^2 (1.410) - (0.091)^2 (1.435)}{(0.221)^2 - (0.091)^2} = 1.405$$

Table 1.3

Bolt	Distances from i.c. d_x	d_y	d	Δ	R/R_{ult}	M	R_x	R_y
1	1.168	3	3.219	0.340	0.9815	3.159	−0.649	+0.252
2	1.168	0	1.168	0.1234	0.8275	0.967	0.0	+0.587
3	1.168	−3	3.219	0.340	0.9815	3.159	+0.649	+0.252
	Summation:					7.285	0.0	+1.091

Table 1.4

Bolt	Distances from i.c. d_x	d_y	d	Δ	R/R_{ult}	M
1	1.031	3	3.172	0.340	0.9815	3.113
2	1.031	0	1.031	0.1105	0.8015	0.826
3	1.031	−3	3.172	0.340	0.9815	3.113
Summation:						7.052

Locate new instantaneous center:

$$-\frac{F_{yy}J}{nM_o} = \frac{-0.091 \times 18}{3 \times (-4)} = +0.137$$

New center is 0.137 in. right of previous position. Repeat calculations (see Table 1.4).

$$M_p = -5.031$$

$$C_{u3} = \frac{\Sigma M}{M_p} = \frac{7.052}{5.031} = 1.402$$

The coefficient has stabilized at 1.40.

EXAMPLE 2

This example is chosen to illustrate a case not covered in the AISC tables, a bracket with an inclined load. The calculations are presented in "design office" form, with minimum explanation.

Centroid located by inspection.

$$J = 10 \times 3^2 = 90$$

$$M_o = (-0.6 \times 5) - (0.8 \times 20) = -19$$

Locate i.c.:

$$a_x = -\frac{P_yJ}{nM_o} = \frac{0.8\,(90)}{6\,(-19)} = -0.632$$

$$a_y = \frac{P_xJ}{nM_o} = \frac{0.6\,(90)}{6\,(-19)} = -0.474$$

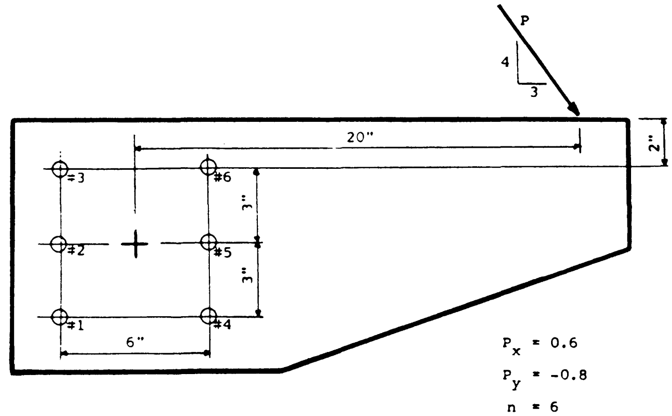

Fig. 6. Example 2

Do calculations in tabular form (see Table 2.1).

$$M_p = -0.6 \times 5.474 - 0.8 \times 20.632 = -19.790$$

$$C_e = \frac{\Sigma d^2}{M_p d_{max}} = \frac{93.744}{19.790 \times 5.026} = 0.942$$

$$C_{u1} = \frac{\Sigma M_i}{M_p} = \frac{22.207}{19.790} = 1.122$$

$$R_{ult} = -\frac{M_p}{\Sigma M_i} = \frac{+19.790}{22.207} = +0.891$$

$$F_{xx} = P_x + \Sigma R_x = 0.6 - 0.472 = 0.128$$

$$F_{yy} = P_y + \Sigma R_y = -0.8 + 0.348 = -0.452$$

$$\vec{F} = \vec{F}_{xx} + \vec{F}_{yy} = 0.470$$

Locate new i.c. and calculate Table 2.2:

$$-\frac{F_{yy}J}{nM_o} = \frac{+0.452 \times 90}{6 \times (-19)} = 0.357$$

$$\frac{F_{xx}J}{nM_o} = \frac{0.128 \times 90}{6 \times (-19)} = -0.101$$

$$M_p = -0.6 \times 5.575 - 0.8 \times 20.989 = -20.136$$

Table 2.1

Bolt	Distances from i.c. d_x	d_y	d	d^2	Δ	R/R_{ult}	M	R_x	R_y
1	−2.368	−2.526	3.462	11.988	0.234	0.9458	3.274	+0.615	−0.576
2	−2.368	+0.474	2.415	5.832	0.163	0.8870	2.142	−0.155	−0.775
3	−2.368	+3.474	4.204	17.676	0.284	0.9674	4.067	−0.712	−0.486
4	+3.632	−2.526	4.424	19.572	0.299	0.9720	4.300	+0.494	+0.711
5	+3.632	+0.474	3.663	13.416	0.248	0.9530	3.491	−0.110	+0.842
6	+3.632	+3.474	5.026	25.260	0.340	0.9815	4.933	−0.604	+0.632
Summation:				93.744			22.207	−0.472	+0.348

Table 2.2

Bolt	Distances from i.c. d_x	d_y	d	Δ	R/R_{ult}	M	R_x	R_y
1	−2.011	−2.425	3.150	0.200	0.9231	2.908	+0.643	−0.533
2	−2.011	+0.575	2.092	0.133	0.8446	1.767	−0.210	−0.735
3	−2.011	+3.575	4.102	0.260	0.9584	3.931	−0.756	−0.425
4	+3.989	−2.425	4.668	0.296	0.9712	4.534	+0.457	+0.751
5	+3.989	+0.575	4.030	0.256	0.9567	3.856	−0.124	+0.857
6	+3.989	+3.575	5.357	0.340	0.9815	5.258	−0.593	+0.661
Summation:						22.254	−0.583	+0.576

$$C_{u2} = \frac{\Sigma M_i}{M_p} = \frac{22.254}{20.136} = 1.105$$

$$R_{ult} = -\frac{M_p}{\Sigma M_i} = \frac{+20.136}{22.254} = +0.905$$

$$F_{xx} = P_x + \Sigma R_x = +0.6 - 0.583 = 0.017$$

$$F_{yy} = P_y + \Sigma R_y = -0.8 + 0.576 = -0.224$$

$$\vec{F} = \vec{F}_{xx} + \vec{F}_{yy} = 0.225$$

Force has decreased and C_u appears stable.

Extrapolation formula predicts:

$$C_u = \frac{F_1{}^2 C_{u2} - F_2{}^2 C_{u1}}{F_1{}^2 - F_2{}^2}$$

$$= \frac{(0.470)^2\,(1.105) - (0.225)^2\,(1.122)}{(0.470)^2 - (0.225)^2}$$

$$= 1.100$$

Use $C_u = 1.10$.

SUMMARY

In any problem to be solved by iteration, one can hope for:

1. A good place to start.
2. An efficient algorithm for improving each successive trial.

The procedure presented appears to offer both these characteristics.

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