A Simple Formula for the Polar Second Moment of Area of a Regular Skew-Symmetric Bolt Group

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Some years ago the authors worked together for a London firm of consulting engineers on the design of long span bridges. There they encountered the problem of calculating the polar second moment of area of very large skew-symmetric bolt groups. At that time, the second author derived the results which appear as Eq. (5). Recently, the first author has simplified those results to obtain Eq. (10). The purpose of this paper is to derive both results.

APPLICATION

Consider a joint consisting of a number of bolts and carrying a moment M about the centroid of the bolt group. If it is assumed that the bolts alone deform, then the load carried by a bolt, of cross section A, at distance r from the centroid, may be obtained as $M(Ar/I_0)$, where I_0 is the polar second moment of area of all the bolts about the centroid of the bolt group. The formula is widely used in the design of bolted joints. For large bolt groups, the calculation of I_0 from first principles is tedious. The derivation given here allows considerable economy of effort.

DERIVATION OF FORMULA

Consider the regular skew-symmetric bolt group shown in Fig. 1. The array consists of m bolts at pitch a in one direction and n bolts at pitch b in the other direction, the sides being of length $(m-1)a$ and $(n-1)b$, respectively. The cross-sectional area of each bolt, A, is constant so the centroid 0 may be located by inspection.

The polar second moment of area about the centroid is

$$I_0 = A \sum r_i^2 \tag{1}$$

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where r_i is the radial distance from the centroid to the i^{th} bolt, and the summation is over the $m \times n$ bolts.

Consider two bolts spaced a distance x either side of the center line, as shown in the definition diagram in Fig. 2. From the cosine rule,

$$r_1^2 = x^2 + y^2 - 2xy \cos\left(\frac{\pi}{2} + \theta\right) = x^2 + y^2 + 2xy \sin\theta$$

$$r_2^2 = x^2 + y^2 - 2xy \cos\left(\frac{\pi}{2} - \theta\right) = x^2 + y^2 - 2xy \sin\theta \tag{2}$$

Therefore,

$$r_1^2 + r_2^2 = 2(x^2 + y^2) \tag{3}$$

and the dependence on θ, the angle of skew, is removed. All bolts not on the axes shown in Fig. 1 can be paired in this manner. Thus, I_0 is given by

$$I_0 = A(\Sigma x^2 + \Sigma y^2) \tag{4}$$

where the summations are over all the bolts and are as follows:

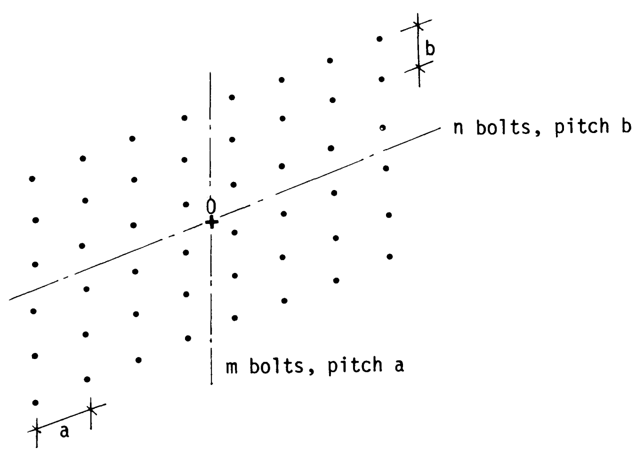

Fig. 1. Definition diagram for a skew-symmetric bolt group

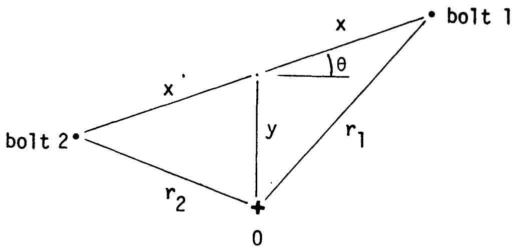

Fig. 2. Coordinates for a representative pair of bolts

(i) when m is odd:

$$\Sigma x^2 = na^2 \cdot 2\left[\left(\frac{m-1}{2}\right)^2 + \left(\frac{m-3}{2}\right)^2 + \ldots 1^2 + 0\right]$$

(ii) when m is even:

$$\Sigma x^2 = na^2 \cdot 2\left[\left(\frac{m-1}{2}\right)^2 + \left(\frac{m-3}{2}\right)^2 + \ldots + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]$$

and Σy^2 is obtained in terms of mb^2 and n in a similar way. Collecting these terms yields

$$I_0 = 2A\ na^2 S_m + mb^2 S_n \tag{5}$$

where

$$S_m = 1^2 + 2^2 + \ldots + \left(\frac{m-1}{2}\right)^2 \quad \text{for } m \text{ odd}$$
$$= \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \ldots + \left(\frac{m-1}{2}\right)^2 \quad \text{for } m \text{ even}$$

with similar expressions for S_n.

To proceed further with Eq. (5) requires a result from the theory of summation of series. One well known result is that for the sum of the squares of the first p positive integers, which may be written as

$$\sum_{i=1}^{p} i^2 = \frac{1}{6}p(p+1)(2p+1) \tag{6}$$

Interpretation of this is straightforward when m (say) is *odd,* for then p may be replaced with $(m-1)/2$, and so

$$S_m = \frac{1}{24}m(m^2-1) \tag{7}$$

with a similar expression for S_n.

When m is even, it is best to rearrange S_m into the following form:

$$S_m = \frac{1}{4}[1^2 + 2^2 + \ldots + (m-2)^2 + (m-1)^2] - \left[1^2 + 2^2 + \ldots + \left(\frac{m-4}{2}\right)^2 + \left(\frac{m-2}{2}\right)^2\right] \tag{8}$$

Comparing this to Eq. (6), it is readily shown that when m is *even*

$$S_m = \frac{1}{24}m(m-1)(2m-1) - \frac{1}{24}m(m-1)(m-2)$$
$$= \frac{1}{24}m(m^2-1) \tag{9}$$

with a similar expression for S_n.

But Eqs. (7) and (9) are identical, so that Eq. (5) reduces to the simple expression

$$I_0 = \frac{mnA}{12}[(m^2-1)a^2 + (n^2-1)b^2] \tag{10}$$

for the polar second moment of area of a regular skew-symmetric bolt group. This general result holds regardless of whether m and n are odd or even. There is clearly an analogy here with the polar second moment of area of a lamina.

The general case of a skew-symmetric bolt group includes the more common case where $\theta = 0$. The results are thus equally applicable to a regular rectangular bolt group.