# Analysis of Vierendeel Frames for Deflections

A. ABDUL-SHAFI

A quick and simple method is described herein for the solution of deflections in symmetrical vierendeel type structures such as multistory one-bay frames and vierendeel trusses. The method takes shear and flexural effects into consideration. A solution for moments,[1] that can be conveniently utilized in the evaluation of deflections, is also briefly described. The deflections are solved for by use of two factors: one factor is associated with fixed end moments in chord elements, and the other is related to the distributed moments at both ends of the element.

### SOLUTION FOR MOMENTS

The solution for moments is basically similar and equal in simplicity to that in Ref. 2, but it is more convenient to use when deflections are also required. Moments are determined in a single cycle of conventional moment distribution. This is achieved by applying a measured amount of over-relaxation at each joint, such that the moment equilibrium at any given joint would not be disturbed by the distribution of moments at other joints. Simple relaxation factors are used for this purpose.

As in Ref. 2, the analysis of symmetrical vierendeel frames, Figs. 1 and 2, is carried out by considering only one half of the frame. Stiffnesses of members are also determined in the same manner and they are defined again here for clarity. Thus, the cantilever stiffness[3] of any chord element **ij** is determined by Eq. (1), in which $h$ = length of chord element, and the carryover factor is $-1$. The stiffness of any web member (cross member) **ii'** is defined by Eq. (2), in which $L$ = length of member.

$$K_{ij} = EI/h \tag{1}$$

$$K_{ii'} = 6EI/L \tag{2}$$

The distribution factors are determined as in the conventional process; they are defined by Eq. (3), in which $D_{ij}$ = distribution factor at end **i** of member **ij**.

$$D_{ij} = K_{ij}/\Sigma K_i \tag{3}$$

*A. Abdul-Shafi is Associate Professor of Civil Engineering, South Dakota State University, Brookings, South Dakota.*

**Relaxation Factors**—These factors facilitate the use of a single cycle moment distribution solution; they are defined by Eqs. (4), in which $f_i$ = relaxation factor at any joint **i** covering the unbalanced moment at **i**; $f_i^{i-1}$ = relaxation factor at **i** covering the effect of moment distribution at joint **i** $- 1$; and $f_i^{i+1}$ = relaxation factor at **i** covering the effect of moment distribution at joint **i** $+ 1$.

$$\begin{aligned} f_i &= 1/(1 - f_{i+1}^{i} D_{i+1,i}) \\ f_i^{i-1} &= f_i D_{i-1,i} \\ f_i^{i+1} &= f_i D_{i+1,i} \end{aligned} \tag{4}$$

**Fixed End Moments**—Sidesway fixed end moments at ends of any chord element **ij** are defined by Eq. (5), in which $F$ = total shear force acting on the panel under consideration and $h$ = length of panel.

$$FEM_{ij} = FEM_{ji} = Fh/4 \tag{5}$$

### SOLUTIONS FOR DEFLECTION

Two deflection factors are considered in this solution. One factor determines the deflection at end **i** of chord element, Fig. 3a, due to a unit clockwise fixed-end moment caused by sidesway. Taking shear and flexural effects into consideration, it can be shown that such a factor is defined by Eq. (6), in which $G$ = modulus of elasticity in shear, $A_w$ = web area of flanged members (depth of web is taken as equal to total depth of section), and $\lambda$ = form factor (= 1 for flanged sections). The minus sign in Eq. (6) indicates upward deflection at end **i** (corresponding to a clockwise moment), Fig. 3a. The other factor determines the deflection at end **i** of a cantilever, caused by a unit distributed moment, Fig. 3b. This factor is defined by Eq. (7). In substituting for the different values in Eqs. (6) and (7), it should be noted that either inches and kips or feet and kips can be used for units. The resulting values of the deflection factors, in both cases, are the same (i.e., in./kip-in. or ft/kip-ft). Thus, the dimension for the computed deflections is governed only by the dimension of the moments in the frame: if the moments in the frame were in kip-in., the resulting deflections would be in inches, and if the moments were in kip-ft, the deflections would be in feet.

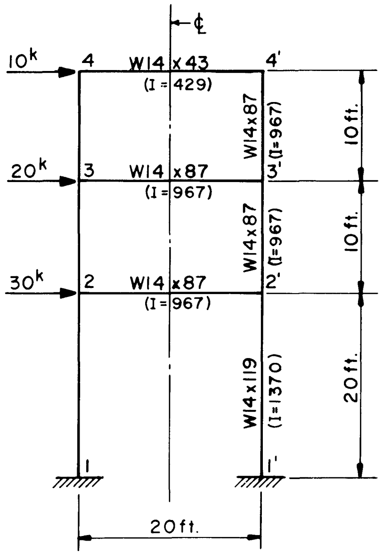


*Fig. 1. Example 1*

$$D^{f}_{i,i-1} = -\left[\frac{L^2}{6EI} + 2\phi\right] \tag{6}$$

where

$$\phi = \lambda/GA_w$$

$$D^{d}_{i,i-1} = \frac{L^2}{2EI} \tag{7}$$

**Example 1**—The frame in Fig. 1 is considered in this example. Considering the left half of the frame, stiffnesses of members are determined by Eqs. (1) and (2):

$$K_{43} = K_{32} = E(967)/120 = 8.0583E$$

$$K_{21} = E(1370)/240 = 5.7083E$$

$$K_{44'} = 6E(429)/240 = 10.725E$$

$$K_{33'} = K_{22'} = 6E(967)/240 = 24.175E$$

The distribution factors are determined by Eq. (3):

$D_{44'} = 10.725E/(10.725E + 8.0583E) = 0.571;$
$D_{43} = 0.429;\quad D_{34} = D_{32} = 0.200;\quad D_{33'} = 0.600;$
$D_{23} = 0.2124;\quad D_{21} = 0.1504;\quad D_{22'} = 0.6372$

Starting with joint **4**, and moving in a sequential order towards joint **2**, the relaxation factors are determined by Eq. (4). Thus:

$f_4 = 1/(1 - 0) = 1$
$f_4^3 = f_4 D_{32} = 0.200$

$f_3 = 1/(1 - f_4^3 D_{43}) = 1.0939$
$f_3^2 = f_3 D_{23} = 0.2323; f_3^4 = f_3 D_{43} = 0.4693$

$f_2 = 1/(1 - f_3^2 D_{32}) = 1.0487$
$f_2^1 = f_2 D_{12} = 0;\quad f_2^3 = f_2 D_{32} = 0.2097$

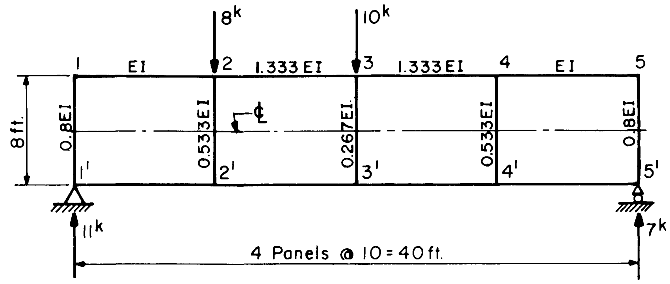


*Fig. 2. Example 2*

The three relaxation factors at each joint are shown in a triangular formation on line 1 in Table 1. Fixed end moments are determined by Eq. (5):

$$FEM_{43} = FEM_{34} = 10(10)/4 = 25 \text{ kip-ft}$$

$$FEM_{32} = FEM_{23} = 30(10)/4 = 75 \text{ kip-ft}$$

$$FEM_{21} = FEM_{12} = 60(20)/4 = 300 \text{ kip-ft}$$

The fixed end moments are shown on line 3 in Table 1. Starting on line 4 in the table, the operation for determining the moments to be distributed at each joint is shown. At any joint **i**, the value on line 4 is obtained by multiplying the relaxation factor $f_i$ (see line 1 in table) by the sum of fixed end moments at **i**. The remaining part of this operation is self explanatory. The moments to be distributed at each joint are indicated on line 5 in the table. These moments are distributed on line 7. The values shown in boxes, on line 7, may be temporarily ignored at this stage. Final moments are shown on line 10.

The solution for deflection is started on line 11 of Table 1. Shear effect has been taken into consideration in the solution. The deflection factors, on line 11, are determined by Eqs. (6) and (7) (using $E$ = 29,000 ksi and $G$ = 12,000 ksi):

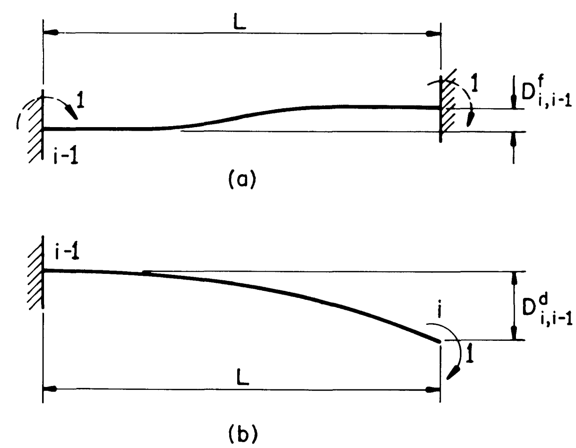


*Fig. 3. Deflection factors*

## Table 1. Solution of Example 1

| Joint | 1 | | 2 | | | 3 | | 4 | |
|---|---|---|---|---|---|---|---|---|---|
| 1. Relaxation Factors $f_i$ | 0.0 | | 1.0487 | | | 1.0939 | | 1.0 | |
| $f_i^{i-1}$ $f_i^{i+1}$ | 0.0 | 0.0 | | 0.2097 | 0.2323 | | 0.4693 | 0.20 | |
| 2. Member | 1-2 | 2-1 | 2-2' | 2-3 | 3-2 | 3-3' | 3-4 | 4-3 | 4-4 |
| 3. F.E.M. (Kip-ft.) | -300.00 | -300.00 | | -75.00 | -75.00 | | -25.00 | -25.00 | |
| 4. (Σ F.E.M.) x $f_i$ | 0.00 | | -393.26 | | | -109.39 | | -25.00 | |
| | | | | | | - 11.73 | = $f_3^4$x | -25.00 | |
| Determination of Moments to be Distributed | | | - 25.40 | = $f_2^3$x | | -121.12 | | | |
| | | | -418.66 | x$f_3^2$ = | | - 97.25 | | | |
| | | | | | | -218.37 | x$f_4^3$ = | -43.67 | |
| 5. Moments to be Distr. | 0.00 | | -418.66 | | | -218.37 | | -68.67 | |
| 6. Distribution Factors | 0.00 | .1504 | .6372 | .2124 | .200 | .600 | .200 | .4290 | .571 |
| 7. Distribution | $\boxed{\Sigma DM_{2,1} = 62.97}$ | 62.97 | 266.77 | 88.92 $\boxed{\Sigma DM_{3,2} = 132.59}$ | 43.67 | 131.02 | 43.67 $\boxed{\Sigma DM_{4,3} = 73.13}$ | 29.46 | 39.21 |
| 8. Carry-Over | -62.97 | 0.00 | | -43.67 | -88.92 | | -29.46 | -43.67 | |
| 9. F.E.M. (line 3) | -300.00 | -300.00 | | -75.00 | -75.00 | | -25.00 | -25:00 | |
| 10. Final Moments (Kip-ft) | -362.97 | -237.03 | 266.77 | -29.75 | -120.25 | 131.02 | -10.79 | -39.21 | 39.21 |
| 11. Deflection Factors[a] $D_{i,\ i-1}^f$ | 0.00 | | -2.618 | | | -1.139 | | -1.139 | |
| $D_{i,\ i-1}^d$ | 0.00 | | 7.249 | | | 2.567 | | 2.567 | |
| 12. F.E.M.$_{i,i-1}$ x $D_{i,\ i-1}^f$ | 0.00 | | 785.40 | | | 85.43 | | 28.48 | |
| 13. $\Sigma DM_{i,i-1}$ x $D_{i,\ i-1}^d$ | 0.00 | | 456.47 | | | 340.36 | | 187.72 | |
| 14. | - | | 0.00 | | | 1241.87 | | 1667.66 | |
| 15. Deflection (ft.) = Σ lines 12, 13 & 14 | 0.00 | | 1241.87 | | | 1667.66 | | 1883.86 | |

[a]Lines 11 through 15 are scaled by a factor of $10^{-4}$. For true deflections, line 15 should be multiplied by $10^{-4}$.

## Table 2. Solution of Example 2

| Joint | 1 | | | | 2 | | | | 3 | | | | 4 | | | | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. Relaxation $f^i$ | 1.0247 | | | | 1.0685 | | | | 1.0656 | | | | 1.0231 | | | | 1.0 |
| Factors $f_i^{i-1}$ $f_i^{i+1}$ | - | 0.1618 | | 0.1527 | | 0.3053 | | 0.2243 | | 0.2243 | | 0.2923 | | 0.1462 | | 0.1579 | - |
| 2. Member | 1-1' | 1-2 | | 2-1 | 2-2' | 2-3 | | 3-2 | 3-3' | 3-4 | | 4-3 | 4-4' | 4-5 | | 5-4 | 5-5' |
| 3. F.E.M. (Kip-ft.) | | -27.50 | | -27.50 | | -7.50 | | -7.50 | | 17.50 | | 17.50 | | 17.50 | | 17.50 | |
| 4. Σ F.E.M. x $f_i$ | -28.18 | | | | -37.40 | | | | 10.66 | | | | 35.81<br>2.56<br>38.37 | | | | 17.50<br>17.50 |
| Determination of Moments to be Distributed | -5.10<br>-33.28 | | | | 5.88<br>-31.52<br>-5.08<br>-36.60 | | | | 8.61<br>19.27<br>-8.21<br>11.06 | | | | 3.23<br>41.60 | | | | 6.57 |
| 5. Moments to be Distr. | -33.28 | | | | -36.60 | | | | 11.06 | | | | 41.60 | | | | 24.07 |
| 6. Distribution Factors | .8571 | .1429 | | .1579 | .6316 | .2105 | | .2857 | .4286 | .2857 | | .2105 | .6316 | .1579 | | .1429 | .8571 |
| 7. Distribution | 28.52 | 4.76 | $\boxed{10.54}$ | 5.78 | 23.12 | 7.70 | $\boxed{4.55}$ | -3.16 | -4.74 | -3.16 | $\boxed{-11.92}$ | -8.76 | -26.27 | -6.57 | $\boxed{-10.01}$ | -3.44 | -20.63 |
| 8. Carry-Over | | -5.78 | | -4.76 | | 3.16 | | -7.70 | | 8.76 | | 3.16 | | 3.44 | | 6.57 | |
| 9. F.E.M. (line 3) | | -27.50 | | -27.50 | | -7.50 | | -7.50 | | 17.50 | | 17.50 | | 17.50 | | 17.50 | |
| 10. Final Moments (Kip-ft) | 28.52 | -28.52 | | -26.48 | 23.12 | 3.36 | | -18.36 | -4.74 | 23.10 | | 11.90 | -26.27 | 14.37 | | 20.63 | -20.63 |
| 11. Deflection $D^f_{i,\,i-1}$ | 0.0 | | | | -16.67 | | | | -12.50 | | | | -12.50 | | | | -16.67 |
| Factors[a] $D^d_{i,\,i-1}$ | 0.0 | | | | 50.00 | | | | 37.50 | | | | 37.50 | | | | 50.00 |
| 12. F.E.M.$_{i,i-1}$ x $D^f_{i,i-1}$ | 0.0 | | | | 458.33 | | | | 93.75 | | | | -218.75 | | | | -291.67 |
| 13. ΣDM$_{i,i-1}$ x $D^d_{i,i-1}$ | 0.0 | | | | 526.75 | | | | 170.48 | | | | -446.81 | | | | -500.35 |
| 14. | - | | | | 0.00 | | | | 985.08 | | | | 1249.31 | | | | 583.75 |
| 15. Σ lines 12, 13 & 14 | 0.0 | | | | 985.08 | | | | 1249.31 | | | | 583.75 | | | | -208.27 |
| 16. Rot. Correction = 208.27 x Xi/40 | 0.0 | | | | 52.07 | | | | 104.14 | | | | 156.20 | | | | 208.27 |
| 17. Deflection = Σ lines 15, 16 | 0.0 | | | | 1037.15 | | | | 1353.45 | | | | 739.95 | | | | 0.0 |

[a]Lines 11 through 17 are scaled by a factor of 144 x $(EI)^{-1}$. For true deflections line 17 should be multiplied by $144(EI)^{-1}$.

$$D^f_{43} = D^f_{32} = -\left[\frac{(120)^2}{6E\ (967)} + \frac{2}{(12000)(14 \times 0.42)}\right]$$
$$= -0.0001139$$

$$D^f_{21} = -\left[\frac{(240)^2}{6E\ (1370)} + \frac{2}{(12000)(14.5 \times 0.57)}\right]$$
$$= -0.0002618$$

$$D^d_{43} = D^d_{32} = \frac{(120)^2}{2E\ (967)} = 0.0002567$$

$$D^d_{21} = \frac{(240)^2}{2E\ (1370)} = 0.0007249$$

On line 12, the value at any joint **i** is determined by multiplying the factor $D^f_{i,i-1}$ by the fixed end moment at end **i** of span **i**, **i − 1**. Values on line 13 are obtained by multiplying the factor $D^d_{i,i-1}$ by the sum of distributed moments at both ends of span ***i, i* − 1** (see sum of distributed moments, $\Sigma DM_{i,i-1}$, in boxes on line 7 in the table). Final deflections are indicated on line 15.

In the above example it should be noted that if the supports of the frame were hinged, instead of being fixed, no change in the solution procedure is required. Computations are influenced only by the change in distribution factors. In the case of hinged supports the distribution factor at the base is equal to 1. In the given example, where the supports are fixed, the distribution factor is zero.

It is also obvious that the solution in Ref. 2 can be utilized for determining deflection according to the method described herein. For this purpose the sum of distributed moments (in the two cycles of moment distribution) at the ends of each chord element should be first determined.

**Example 2**—The vierendeel truss, Fig. 2, is considered in this example. It is analyzed by considering joints along the top chord members. Stiffnesses, distribution factors, and fixed-end moments are computed as in Example 1. The relaxation factors are computed by starting at joint **5** and moving towards joint **1**. In this example, shear was ignored in computing the deflection factors [by disregarding the term in which $\phi$ appears in Eq. (6)]. The solution is shown in Table 2. All steps in the table, from starting point through line 15, are similar to those in Table 1. A deflection correction is applied on line 16 since joint **5** should have zero deflection. The truss is rotated about its left support and the correction at any joint **i**, at a distance $X_i$ from the left end, is determined as indicated on the left end of line 16. Final deflections are on line 17.

## CONCLUSION

The method presented is simple and efficient, and it takes shear and flexural effects into consideration. Due to its limited number of operations, the method also lends itself easily to the various instruments of computation.


## ACKNOWLEDGEMENT

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