Simplified Design for Torsional Loading of Rolled Steel Members

PHILIP H. LIN

When a structural member is loaded in such a manner that the plane of loading does not pass through the shear center, the member is twisted about its longitudinal axis and is said to be in torsion. In the energy-related industries, there are many occasions in which torsion is severe enough to control the design of a structural member. It is therefore very important that a structural engineer be able to recognize a torsion situation and apply a simple and fairly accurate solution to achieve the design.

Rolled steel members under uniform and nonuniform torsion have been studied analytically and experimentally by many investigators. For various loading and boundary conditions, solutions to differential equations have been derived, and for a limited number of cases, charts have been made for use by the design engineer. Unfortunately, because the differential equation solution is rather mechanical and time-consuming and, moreover, the designer cannot clearly visualize what is happening in working with the hyperbolic functions for ϕ, an improper design may result without the designer's knowledge. Thus, this "exact" method is really suited only for analysis.

APPROXIMATE SOLUTION

Warping torsion is the out-of-plane effect that arises when the flanges are laterally displaced during twisting and is identical to bending from laterally applied loads. As such, torsion causes flexural normal stresses across the flange width (see Fig. 1) as well as shear stresses that are normally not of significance. Torsion boundary conditions, then, are analogous to lateral bending boundary conditions. Therefore, design of a beam to include torsion is usually achieved by transforming torsion into ordinary bending, which is accomplished by substituting a lateral force P_H times beam section depth h for the applied torsional moment T on a beam section (see Fig. 2). The force P_H can then be treated as a lateral load acting on the flange of the beam. This substitution, however, either overestimates or underestimates the lateral shear force and, consequently, either overestimates or underestimates the lateral bending that causes warping normal stresses. In some situations it is so excessively conservative as to be practically useless.

Philip H. Lin is Senior Structural Engineer, The Ralph M. Parsons Company, Pasadena, Calif.

FLEXURAL ANALOGY METHOD WITH β MODIFIER

It is imperative that a simple, understandable, yet fairly accurate solution be derived by modifying the aforementioned approximate solution.

In considering the beam in Fig. 2, the substitution system will indicate constant shear over one-half the span, but the true distribution of lateral shear contributing to lateral deflection is only that part due to warping. See Fig. 3(A2).

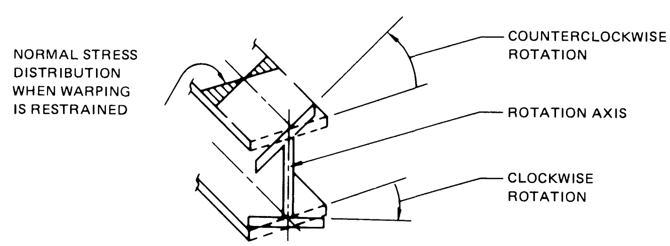

Fig. 1. Warping of cross section

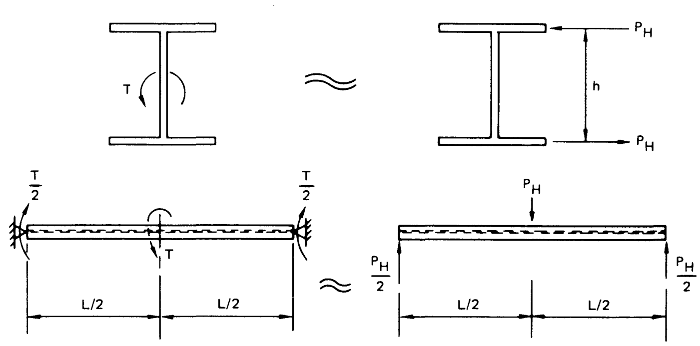

Fig. 2. Substitution of torsion and ordinary bending

Figures 3(A1), 3(A2), and 4(A) show that the normal shear distribution V_f is the sum of absolute values of St. Venant torsional shear V_s and warping torsional shear V_w. The relationship between the flexural analogy and the true torsion problem is thus illustrated. It is interesting to see that the pattern of the torsional moment diagram in Fig. 3(B) is very close to that of the flexural moment diagram in Fig. 4(B). If one could correctly assess the correlation factor between the two, design for torsion could be greatly simplified. The correlation factor could be designated as modifier β. Mathematically, the torsional moment at any point is equal to the flexural moment at that point times the modifier β at that point, i.e., at point **X**, or:

$$M_T = \beta \times M_o$$

where β is primarily a function of beam torsion properties, boundary conditions at the ends of the beam, and the distance X to the end of the beam.

For the problem in Fig. 3, if the beam is free to rotate after a torque T is applied, the beam will displace a twisting angle ϕ and a lateral displacement u, as shown in Fig. 5. From geometry,

$$u = \phi\left(\frac{h}{2}\right)$$

Differentiating twice with respect to X, one obtains:*

$$u'' = \phi''\left(\frac{h}{2}\right) \qquad (1)$$

From the conventional beam theory:

$$EI_f u'' = -M_f \qquad (2)$$

where I_f is the moment of inertia about the y-axis of one beam flange, and M_f is the lateral bending moment acting on one flange. Substituting Eq. (1) into Eq. (2), one obtains:

$$M_f = -\frac{h}{2} EI_f \phi''$$

or

$$M_f = -\frac{EC_w}{h}\phi'' \qquad (3)$$

where

$$C_w \text{ (warping constant)} = \frac{h^2}{2} I_f$$

The mathematical expressions of the twist angle ϕ for various cases have been derived and can be obtained from the references. For the case considered in this problem, ϕ at point **X** is:

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* $u'' = \frac{du^2}{dx^2}$; $\phi'' = \frac{d\phi^2}{dx^2}$

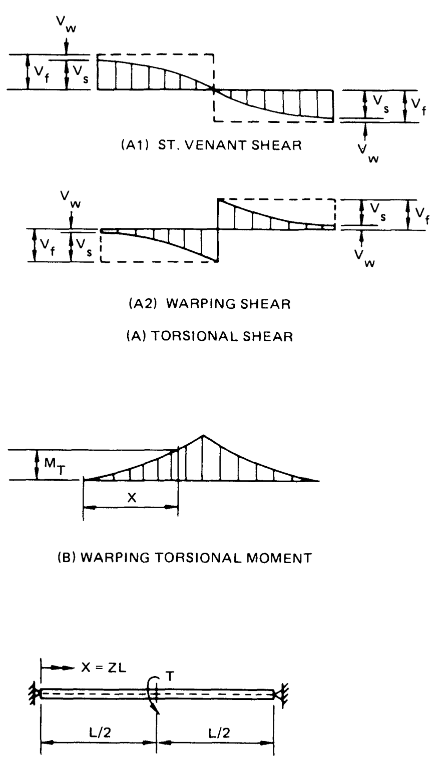

Fig. 3. Torsionally simply supported beam subjected to concentrated torsion T at midspan

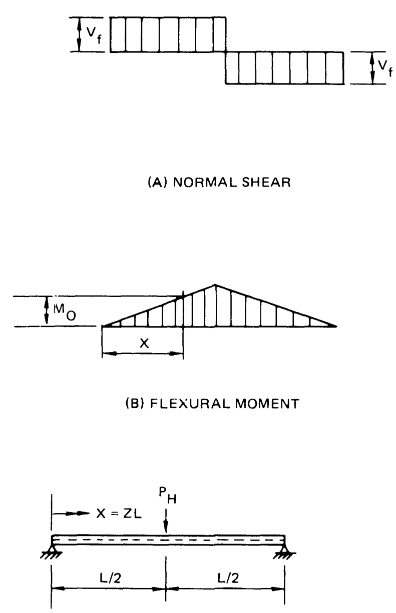

Fig. 4. Simply supported beam subjected to concentrated lateral load P_H at midspan

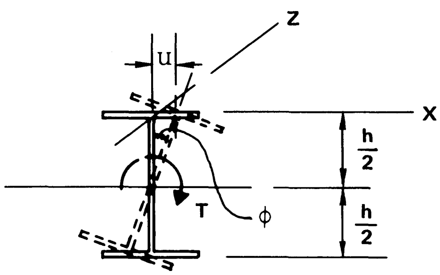

Figure 5

$$\phi = \frac{T}{2GJ\lambda}\left(\lambda X - \frac{2\sinh \lambda L/2}{\sinh \lambda L}\right)\sinh \lambda X \qquad (4)$$

and its second derivative is

$$\phi'' = -\left(\frac{T\lambda}{GJ}\right)\left(\frac{\sinh \lambda L/2}{\sinh \lambda L}\right)\sinh \lambda X \qquad (5)$$

Substituting the expression for ϕ'' into Eq. (3), noting that

$$\lambda^2 = \frac{GJ}{EC_w}$$

one obtains

$$M_f = \left(\frac{T}{h}\right)\left(\frac{\sinh \lambda L/2}{\lambda \sinh \lambda L}\right)\sinh \lambda X \qquad (6)$$

The bending moment M_o at point **X** on one flange of the beam due to T/h at midspan is:

$$M_o = \frac{1}{2}\left(\frac{T}{h}\right)X$$

or

$$\frac{T}{h} = \frac{2M_0}{X}$$

Substituting the expression for T/h into Eq. (6), and replacing M_f with M_T:

$$M_T = \frac{2M_o}{X}\left(\frac{\sinh \lambda L/2}{\lambda \sinh \lambda L}\right)\sinh \lambda X \qquad (7)$$

Let

$$M_T = \beta \times M_o \qquad (8)$$

Comparing Eqs. (7) and (8), one can conclude that:

$$\beta = \frac{(2\sinh \lambda L/2)(\sinh \lambda X)}{\lambda X \sinh \lambda L} \qquad (9)$$

where $\lambda = \sqrt{GJ/EC_w}$. The terms J and C_w are torsion and warping constants of the steel sections which can be obtained from the 1970 AISC *Manual of Steel Construction* for rolled shape sections. For built-up open web sections, formulas for J and C_w can be found in Example 4.

Similarly, the formulas for β in other cases can be readily derived. Computer programs were developed to produce values of β for most practical design applications. These β values are tabulated in Appendix Tables 1–6. The general formulas in those tables for torsional moment M_T are the formulas for the moment of a simple beam subjected to the converted lateral force times β. The converted lateral force is equal to torsion divided by the depth of the beam, i.e., T/h or m/h. For example, the torsional moment at midspan of the beam in Fig. 3 is equal to β at that point times the simple beam moment. Thus,

$$M_T = \beta \times M_o$$

where

$$M_o = \frac{(T/h)L}{4}$$

TORSIONAL STRESSES

In the design of a wide flange beam section, there are three kinds of shear stress, shown in Fig. 6, that will arise from torsional loading:

1. Shear stresses V_s in the web and flanges due to St. Venant torsion.
2. Shear stresses V_w in the flanges due to lateral bending or warping torsion.
3. Shear stresses u due to normal bending of the flanges.

In most practical steel design situations, the shear stress contributions are normally not of significance; therefore, only the normal stresses from vertical and torsional bending are computed and combined in order to check the design adequacy of a beam. This combined normal stress, which is the maximum at the flange tips, can be expressed as:

$$f = \frac{M}{S_x} + \frac{2M_T}{S_y}$$

where M and M_T are flexural and torsional moments, respectively, and S_x and S_y are the section moduli with respect to the X-X and Y-Y axes, respectively.

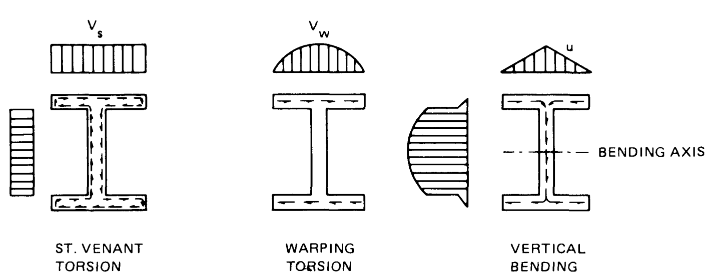

Fig. 6. Distribution of shear stress in wide flange section

ILLUSTRATIVE EXAMPLES

Four examples are presented to illustrate the application of this method. It is assumed that the deflection of the beam is acceptable without calculation. The result of each example has been closely compared and checked by the exact solution.

Example 1

Given:
Select a wide flange section of A36 structural steel beam to carry a 9-kip concentrated load on a simply supported span of 15 ft. Apply the load eccentrically 6 in. from the web, and assume that the ends of the beam are torsionally supported, that the maximum combined allowable stress is equal to 0.6 F_y, and that the beam weight is disregarded. See Fig. 7.

Solution:
The maximum flexural bending moment M at the load point is:

$$M = \left(\frac{1}{4}\right)(9)(15) = 33.75 \text{ kip-ft}$$

The maximum torsional moment M_T at the load point is:

$$M_T = \frac{1}{4}\left(\frac{4.5}{h}\right)(15\beta) = 16.88\left(\frac{\beta}{h}\right)$$

For steel,

$$\frac{G}{E} = \frac{E}{2E(1+\mu)} = \frac{1}{2(1+0.3)} = 0.3846$$

and

$$\lambda = \sqrt{\frac{GJ}{EC_w}} = 0.62\sqrt{\frac{J}{C_w}}$$

Try W10X54:

$h = 10\frac{1}{8}$ in. = 0.83 ft
$J = 1.84$ in.^{4} $\qquad S_x = 60.4$ in.^{3}
$C_w = 2{,}350$ in.^{6} $\qquad S_y = 20.7$ in.^{3}
$\lambda = 0.01735$ in.$^{-1}$ $\qquad \lambda L = 3.12$

For $\alpha = 0.5$, $Z = 0.5$, and $\lambda L = 3.12$, enter Table 1 to obtain $\beta = 0.58886$. Then,

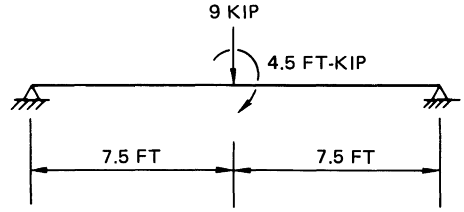

Fig. 7. Example 1

$$M_T = 16.88\left(\frac{0.58886}{0.83}\right)$$
$$= 11.97 \text{ kip-ft}$$

The maximum combined normal stress, f, at the load point is:

$$f = \frac{M}{S_x} + \frac{2M_T}{S_y} = \frac{(33.75)12}{60.4} + \frac{(24)(11.97)}{20.7}$$
$$= 20.59 \text{ kips/in.}^2 \quad \textbf{o.k.}$$

Use W10X54.

Example 2

Given:
Compute the stresses and design a beam of A36 structural steel to carry a 10-ton concentrated load eccentrically 6 in. to the plane of the web, as shown in Fig. 8. Assume that the beam is fixed at both ends, that the maximum combined allowable stress is equal to $0.6F_y$, and that the beam weight is disregarded.

Solution:
The maximum flexural bending moment M at the load point and the ends is

$$M = \frac{1}{8}(20)(10) = 25 \text{ kip-ft}$$

The maximum torsional moment M_T at the load point and the ends is:

$$M_T = \left(\frac{1}{8}\right)\left(\frac{10}{h}\right)(10\beta) = 12.5\left(\frac{\beta}{h}\right)$$

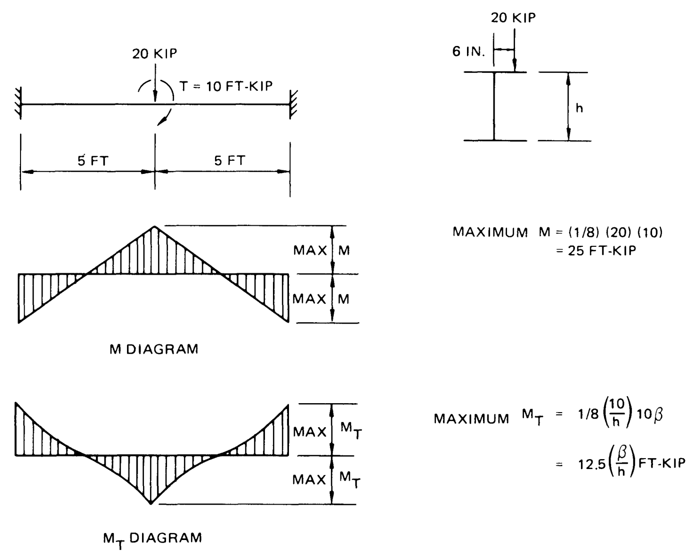

Fig. 8. Example 2

Try W12×53:

$$h = 12.06 \text{ in.} = 1.0 \text{ ft} \qquad S_x = 70.7 \text{ in.}^3$$
$$J = 1.59 \text{ in.}^4 \qquad S_y = 19.2 \text{ in.}^3$$
$$C_w = 3170 \text{ in.}^6 \qquad \lambda = 0.01389 \text{ in.}^{-1}$$
$$\lambda L = (0.01389)(10)(12) = 1.67$$

For $\alpha = 0.5$, $Z = 0$ or 0.5, and $\lambda L = 1.67$, enter Table 2 to obtain $\beta = 0.94253$. Then,

$$M_T = 12.5 \left(\frac{0.94253}{1.0}\right) = 11.78 \text{ kip-ft}$$

and the maximum combined normal stress f at the load point and the ends is

$$f = \frac{(25)(12)}{70.7} + \frac{(24)(11.78)}{19.2} = 4.24 + 14.73$$

$$= 18.97 \text{ kip/in.}^2 < 0.6F_y \quad \textbf{o.k.}$$

Use W12×53

Example 3

Given:
Design an A36 structural steel beam to carry a uniform load of 2 kips/ft on a simply supported span of 22 ft. Apply the load eccentrically 6 in. to the plane of the web, and assume that the ends of the beam are torsionally supported and that the maximum combined allowable stress is equal to $0.6F_y$. See Fig. 9.

Solution:
The uniform load (plus the estimated beam weight) is:

$$w = 2 + 0.1 \text{ (estimated beam weight)}$$
$$= 2.1 \text{ kips/ft}$$

The uniform torsion m is:

$$m = (2)\left(\frac{6}{12}\right) = 1 \text{ kip-ft/ft}$$

The maximum flexural bending moment M at midspan is:

$$M = \frac{wL^2}{8} = 127.1 \text{ kip-ft}$$

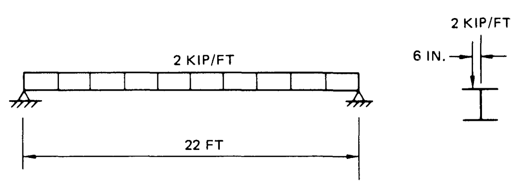

Fig. 9. Example 3

The maximum torsional moment M_T at midspan is:

$$M_T = \left(\frac{(m/h)L^2}{8}\right)(\beta) = \left(\frac{60.5}{h}\right)(\beta)$$

Try W14×103:

$$h = 14.25 \text{ in.} = 1.188 \text{ ft}$$
$$J = 6.02 \text{ in.}^4 \qquad S_x = 164 \text{ in.}^3$$
$$C_w = 18{,}900 \text{ in.}^6 \qquad S_y = 57.6 \text{ in.}^3$$
$$\lambda = 0.011065 \text{ in.}^{-1} \qquad \lambda L = 2.92$$

For $\lambda L = 2.92$ and $Z = 0.5$, enter Table 3 to obtain $\beta = 0.52562$. Then,

$$M_T = \left(\frac{60.5}{1.188}\right)(0.52562) = 26.77 \text{ kip-ft}$$

The maximum combined normal stress f at midspan is

$$f = \frac{(127.1)(12)}{164} + \frac{(24)(26.77)}{57.6}$$
$$= 20.45 \text{ kip/in.}^2 < 0.6\,F_y \quad \textbf{o.k.}$$

Use W14×103

Example 4

Given:
Design an A36 structural steel beam to support a piece of equipment having concentrated loads of 300 kips and 400 kips at the base of its support legs as shown below. The beam will be loaded 3 in. eccentrically to the plane of its web because of thermal expansion during equipment operation. Assume that the ends of the beam are torsionally simply supported, that the allowable stress equals $0.6\,F_y$, and that the beam weight is disregarded. See Fig. 10.

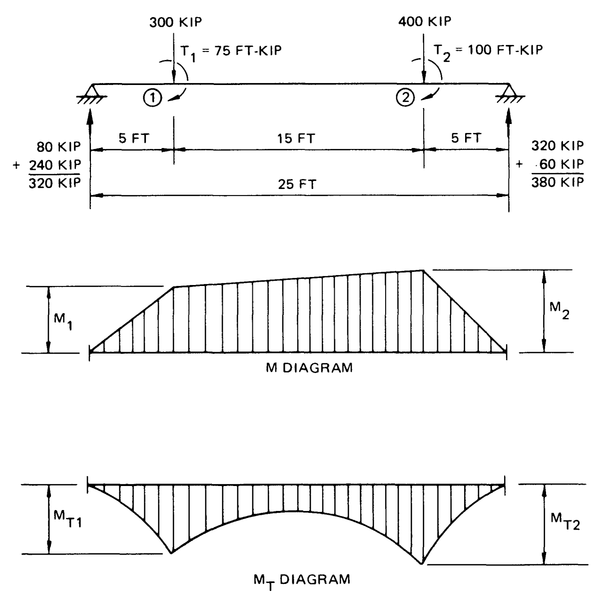

Fig. 10. Example 4

Solution:
The maximum flexural bending moment M_2 at load point ② is:

$$M_2 = (320 + 60)(5) = 1900 \text{ kip-ft}$$

The maximum torsional moment M_{T2} at load point ② is:

$$\text{Pseudo reaction } R_r = (75)\left(\frac{5}{25}\right) + (100)\left(\frac{20}{25}\right) = 15 + 80$$

$$M_{T2} = \frac{(15)(\beta_1) + (80)(\beta_2)}{h}$$

Try W36X300:

$h = 36.72$ in. $= 3.06$ ft $S_x = 1{,}110$ in.^{3}
$J = 64.2$ in.^{4} $S_y = 156$ in.^{3}
$C_w = 398{,}000$ in.^{6} $\lambda = 0.007874$ in.$^{-1}$
$\lambda L = (0.007874)(25)(12) = 2.36$

For $\alpha_1 = 0.2$, $\lambda L = 2.36$, and $Z = 0.8$, enter Table 1 to obtain $\beta_1 = 0.49353$.

For $\alpha_2 = 0.8$ (or 0.2), $\lambda L = 2.36$, and $Z = 0.8$ (or 0.2), enter Table 1 to obtain $\beta_2 = 0.79862$.

Then,

$$M_{T2} = \frac{(15)(0.49353) + (80)(0.79862)}{3.06} = 23.3 \text{ kip-ft}$$

The maximum combined normal stress f at load point ② is:

$$f = \frac{(1900)(12)}{1110} + \frac{(24)(23.3)}{156} = 24.13 \text{ kips/in.}^2 > 0.6F_y \quad \textbf{n.g.}$$

Try a built-up section (see Fig. 11):

Section properties* are:

$$J = \left(\frac{1}{3}\right)[(2bt_f^3) + (\bar{h}t_w^3)] = \left(\frac{1}{3}\right)[(2)(18)(2^3) + (34)(1^3)] = 107.3 \text{ in.}^4$$

$$C_w = \left(\frac{1}{24}\right) t_f\bar{h}^2b^3 = \left(\frac{1}{24}\right)(2)(34)^2(18)^3 = 561{,}816 \text{ in.}^6$$

* *For other sections, see Ref. 6, Table A-3, "Properties of Sections", p. 503.*

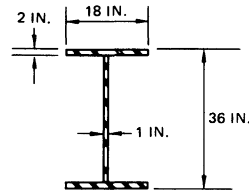

Figure 11

$S_x = 1{,}308$ in.^{3}
$S_y = 216$ in.^{3}
$\lambda = 0.0085683$ in.$^{-1}$
$\lambda L = (0.0085683)(25)(12) = 2.57$

For $\lambda L = 2.57$ and following the previous procedures, obtain $\beta_1 = 0.44223$ and $\beta_2 = 0.77405$. Then,

$$M_{T2} = \frac{(15)(0.44223) + (80)(0.77405)}{3.0} = 22.85 \text{ kip-ft}$$

and

$$f = \frac{(1{,}900)(12)}{1{,}308} + \frac{(24)(22.85)}{216} = 19.97 \text{ kip/in.}^2 < 0.6F_y \quad \textbf{o.k.}$$

Use: Built-up Section

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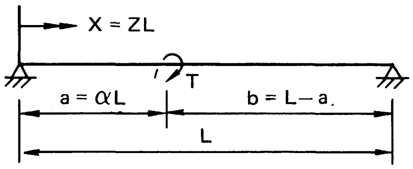
APPENDIX

Table 1. β Values for Simply Supported Beam Subjected to Concentrated Torsion T



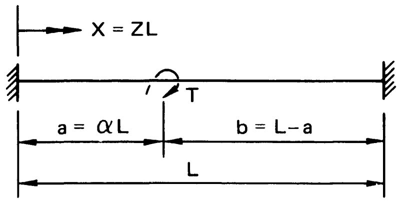
$$M_T = \beta \times M_O$$

$$M_O = \frac{(T/h)bX}{L} \quad \text{for } 0 < X \leqslant a$$

$$M_O = \frac{(T/h)a(L-X)}{L} \quad \text{for } a \leqslant X < L$$

		Z								
λL	α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0.1	0.97215	0.94621	0.92367	0.90441	0.88830	0.87525	0.86518	0.85803	0.85376
	0.2	0.94621	0.95094	0.92829	0.90893	0.89274	0.87963	0.86951	0.86232	0.85803
1.0	0.3	0.92367	0.92829	0.93603	0.91650	0.90018	0.88696	0.87675	0.86951	0.86518
	0.4	0.90441	0.90893	0.91650	0.92717	0.91066	0.89728	0.88696	0.87963	0.87525
	0.5	0.88830	0.89274	0.90018	0.91066	0.92423	0.91066	0.90018	0.89274	0.88830
	0.1	0.90737	0.82421	0.75509	0.69828	0.65238	0.61626	0.58904	0.57005	0.55883
	0.2	0.82421	0.84075	0.77024	0.71230	0.66547	0.62863	0.60086	0.58149	0.57005
2.0	0.3	0.75509	0.77024	0.79590	0.73602	0.68764	0.64957	0.62087	0.60086	0.58904
	0.4	0.69828	0.71230	0.73602	0.77004	0.71943	0.67959	0.64957	0.62863	0.61626
	0.5	0.65238	0.66547	0.68764	0.71943	0.76159	0.71943	0.68764	0.66547	0.65238
	0.1	0.83383	0.69234	0.58217	0.49686	0.43150	0.38237	0.34671	0.32255	0.30856
	0.2	0.69234	0.72373	0.60856	0.51939	0.45106	0.39970	0.36243	0.33717	0.32255
3.0	0.3	0.58217	0.60856	0.65415	0.55830	0.48485	0.42965	0.38958	0.36243	0.34671
	0.4	0.49686	0.51939	0.55830	0.61572	0.53472	0.47384	0.42965	0.39970	0.38237
	0.5	0.43150	0.45106	0.48485	0.53472	0.60343	0.53472	0.48485	0.45106	0.43150
	0.1	0.76451	0.57599	0.44036	0.34281	0.27295	0.22347	0.18933	0.16709	0.15456
	0.2	0.57599	0.62269	0.47606	0.37060	0.29508	0.24159	0.20468	0.18064	0.16709
4.0	0.3	0.44036	0.47606	0.53942	0.41993	0.33435	0.27374	0.23192	0.20468	0.18933
	0.4	0.34281	0.37060	0.41993	0.49566	0.39464	0.32311	0.27374	0.24159	0.22347
	0.5	0.27295	0.29508	0.33435	0.39464	0.48201	0.39464	0.33435	0.29508	0.27295
	0.1	0.70230	0.47911	0.33192	0.23450	0.16995	0.12735	0.09969	0.08253	0.07319
	0.2	0.47911	0.54026	0.37428	0.26443	0.19164	0.14360	0.11241	0.09306	0.08253
5.0	0.3	0.33192	0.37428	0.45209	0.31941	0.23148	0.17346	0.13578	0.11241	0.09969
	0.4	0.23450	0.26443	0.31941	0.40804	0.29572	0.22159	0.17346	0.14360	0.12735
	0.5	0.16995	0.19164	0.23148	0.29572	0.39465	0.29572	0.23148	0.19164	0.16995
	0.1	0.64703	0.39947	0.25051	0.16031	0.10540	0.07189	0.05159	0.03970	0.03349
	0.2	0.39947	0.47356	0.29697	0.19005	0.12494	0.08522	0.06116	0.04707	0.03930
6.0	0.3	0.25051	0.29697	0.38590	0.24695	0.16236	0.11074	0.07947	0.06116	0.05159
	0.4	0.16031	0.19005	0.24695	0.34411	0.22623	0.15430	0.11074	0.08522	0.07189
	0.5	0.10540	0.12494	0.16236	0.22623	0.33168	0.22623	0.16236	0.12494	0.10540
	0.1	0.55424	0.28016	0.14387	0.07501	0.04065	0.02280	0.01357	0.00885	0.00661
	0.2	0.28016	0.37470	0.19241	0.10086	0.05437	0.03050	0.01815	0.01183	0.00885
8.0	0.3	0.14387	0.19241	0.29517	0.15472	0.08340	0.04678	0.02784	0.01815	0.01357
	0.4	0.07541	0.10086	0.15472	0.25997	0.14013	0.07860	0.04678	0.03050	0.02280
	0.5	0.04065	0.05437	0.08340	0.14013	0.24983	0.14013	0.08340	0.05437	0.04065

Table 2. β Values for Fixed Supported Beam Subjected to Concentrated Torsion T



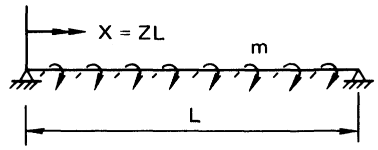
$$M_T = \beta \times M_O$$

$$M_O = \frac{(T/h)b^2}{L^2}\left[\frac{(3a+b)X}{L} - a\right] \quad \text{for } 0 \leqslant X \leqslant a$$

$$M_O = \frac{(T/h)a^2}{L^2}\left[\frac{(a+3b)(L-X)}{L} - b\right] \quad \text{for } a \leqslant X \leqslant L$$

		Z										
λL	α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	0.1	0.99393	1.01349	0.99576	0.98126	0.96970	0.96032	0.94891	1.00914	0.97195	0.97369	0.98105
	0.2	0.98888	0.97751	0.99859	0.98375	0.97197	0.96272	0.95351	1.06920	0.97080	0.97220	0.97925
1.0	0.3	0.98482	0.97741	1.02017	0.98805	0.97596	0.96674	0.95908	0.89245	0.97038	0.97164	0.97842
	0.4	0.98176	0.97512	0.96244	0.97962	0.98176	0.97238	0.96563	0.95448	0.97041	0.97198	0.97856
	0.5	0.97967	0.97318	0.96995	0.96995	0.97318	0.97967	0.97318	0.96995	0.96995	0.97318	0.97967
	0.1	0.97681	1.05024	0.98242	0.92861	0.88682	0.85376	0.81440	1.02256	0.89511	0.90248	0.93021
	0.2	0.95784	0.91567	0.99354	0.93808	0.89519	0.86232	0.83034	1.23012	0.89098	0.89696	0.92335
2.0	0.3	0.94286	0.91540	1.07008	0.95455	0.91001	0.87675	0.84978	0.61889	0.88951	0.89481	0.92008
	0.4	0.93170	0.90713	0.86168	0.92322	0.93174	0.89728	0.87297	0.83390	0.88972	0.89594	0.92038
	0.5	0.92423	0.90018	0.88830	0.88830	0.90018	0.92423	0.90018	0.88830	0.88830	0.90018	0.92423
	0.1	0.95133	1.10121	0.95880	0.85105	0.77082	0.70976	0.63957	1.01190	0.78717	0.80456	0.86139
	0.2	0.91258	0.82802	0.98312	0.87072	0.78737	0.72585	0.66818	1.38297	0.77925	0.79353	0.84709
3.0	0.3	0.88274	0.82786	1.12560	0.90536	0.81696	0.75332	0.70355	0.28876	0.77659	0.78906	0.83999
	0.4	0.86102	0.81191	0.92573	0.84235	0.86119	0.79316	0.74667	0.67516	0.77734	0.79088	0.83992
	0.5	0.84687	0.79872	0.77529	0.77529	0.79872	0.84687	0.79872	0.77529	0.77529	0.79872	0.84687
	0.1	0.92057	1.15655	0.92450	0.75935	0.64296	0.55883	0.46651	0.95849	0.66758	0.69878	0.78863
	0.2	0.85954	0.72939	0.96608	0.79085	0.66784	0.58149	0.50446	1.44848	0.65610	0.68191	0.76558
4.0	0.3	0.81401	0.72983	1.16409	0.84731	0.71300	0.62087	0.55226	0.00090	0.65249	0.67476	0.75365
	0.4	0.78185	0.70651	0.58326	0.74992	0.78229	0.67959	0.61229	0.51475	0.65415	0.67680	0.75234
	0.5	0.76159	0.68764	0.65238	0.65238	0.68764	0.76159	0.68764	0.65238	0.65238	0.68764	0.76159
	0.1	0.88713	1.20870	0.88050	0.66304	0.51946	0.42189	0.32072	0.86361	0.55123	0.59816	0.72107
	0.2	0.80395	0.83115	0.94231	0.70650	0.55137	0.44881	0.36273	1.40383	0.53705	0.57603	0.68892
5.0	0.3	0.74413	0.63271	1.17341	0.78615	0.61046	0.49674	0.41688	0.19608	0.53288	0.56624	0.67170
	0.4	0.70332	0.60358	0.45358	0.65622	0.70417	0.57068	0.48755	0.37551	0.53563	0.56792	0.66833
	0.5	0.67863	0.58060	0.53508	0.53508	0.58060	0.67863	0.58060	0.53508	0.53508	0.58060	0.67863
	0.1	0.85285	1.25322	0.82890	0.56909	0.40962	0.30856	0.21038	0.74241	0.44675	0.50901	0.88260
	0.2	0.74924	0.54004	0.91282	0.62345	0.44648	0.33717	0.25159	1.27115	0.43092	0.48268	0.62165
6.0	0.3	0.67769	0.54308	1.15210	0.72590	0.51649	0.38958	0.30626	0.29970	0.42655	0.47064	0.59922
	0.4	0.63067	0.51012	0.34529	0.56771	0.63203	0.47384	0.38089	0.26572	0.43031	0.47153	0.59330
	0.5	0.60343	0.48485	0.43150	0.43150	0.48485	0.60343	0.48485	0.43150	0.43150	0.48485	0.60343
	0.1	0.78579	1.31352	0.71302	0.40357	0.24207	0.15456	0.08135	0.49016	0.28401	0.36917	0.57297
	0.2	0.64917	0.38888	0.84276	0.47400	0.28226	0.18064	0.11281	0.89486	0.26767	0.33811	0.51678
8.0	0.3	0.56255	0.39511	1.04015	0.61645	0.36379	0.23192	0.15760	0.31766	0.26350	0.32328	0.48545
	0.4	0.50987	0.35975	0.19395	0.41711	0.51237	0.32311	0.22583	0.12642	0.26840	0.32225	0.47434
	0.5	0.48201	0.33435	0.27295	0.27295	0.33435	0.48201	0.33435	0.27295	0.27295	0.33435	0.48201

Table 3. β Values for Simply Supported Beam Subjected to Uniform Torsion m



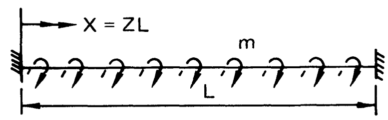
$M_T = \beta \times M_O$

$$M_O = \frac{(m/h)\,(L - X)X}{2}$$

for $0 < X < L$

λL	Z 0.1	0.2	0.3	0.4	0.5
0.5	0.97785	0.97643	0.97541	0.97481	0.97460
1.0	0.91744	0.91218	0.90843	0.90619	0.90545
1.5	0.83317	0.82265	0.81520	0.81076	0.80928
2.0	0.74039	0.72423	0.71287	0.70613	0.70389
2.5	0.65022	0.62882	0.61390	0.60510	0.60219
3.0	0.56864	0.54278	0.52493	0.51447	0.51103
3.5	0.49773	0.46831	0.44825	0.43657	0.43274
4.0	0.43737	0.40525	0.38364	0.37117	0.36710
4.5	0.38644	0.35239	0.32980	0.31689	0.31270
5.0	0.34355	0.30819	0.28509	0.27204	0.26782
5.5	0.30732	0.27118	0.24795	0.23496	0.23079
6.0	0.27657	0.24005	0.21697	0.20422	0.20015
6.5	0.25029	0.21372	0.19101	0.17862	0.17469
7.0	0.22768	0.19131	0.16914	0.15719	0.15341
8.0	0.19100	0.15557	0.13476	0.12383	0.12042
9.0	0.16274	0.12870	0.10946	0.09961	0.09657
10.0	0.14045	0.10804	0.09041	0.08160	0.07892

Table 4. β Values for Fixed Supported Beam Subjected to Uniform Torsion m

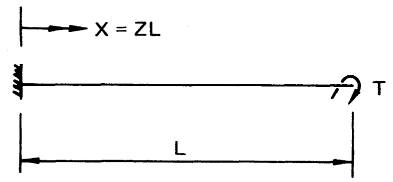


$M_T = \beta \times M_O$

$$M_O = \frac{(m/h)}{12}(6LX - L^2 - 6X^2) \text{ for } 0 \leqslant X < L$$

λL	Z 0.0	0.1	0.2	0.3	0.4	0.5
0.5	0.99586	0.99319	0.97602	0.99486	0.99315	0.99276
1.0	0.98372	0.97325	0.90614	0.97977	0.97312	0.97159
1.5	0.96441	0.94159	0.79627	0.95572	0.94137	0.93806
2.0	0.93911	0.90031	0.65504	0.92419	0.90005	0.89449
2.5	0.90925	0.85185	0.49246	0.88688	0.85167	0.84360
3.0	0.87625	0.79867	0.31858	0.84556	0.79878	0.78810
3.5	0.84145	0.74306	0.14241	0.80185	0.74371	0.73049
4.0	0.80598	0.68694	0.02869	0.75717	0.68839	0.67284
4.5	0.77070	0.63178	0.18923	0.71261	0.63433	0.61674
5.0	0.73628	0.57869	0.33555	0.66902	0.58261	0.56332
5.5	0.70317	0.52837	0.46559	0.62699	0.53391	0.51329
6.0	0.67164	0.48124	0.57851	0.58689	0.48863	0.46703
6.5	0.64184	0.43751	0.67443	0.54895	0.44693	0.42467
7.0	0.61381	0.39720	0.75410	0.51327	0.40878	0.38617
8.0	0.56301	0.32646	0.86951	0.44871	0.34260	0.32004
9.0	0.51869	0.26768	0.93594	0.39275	0.28845	0.26667
10.0	0.48006	0.21916	0.96485	0.34454	0.24437	0.22383

Table 5. β Values for Cantilever Beam Subjected to Concentrated Torsion T at Free End

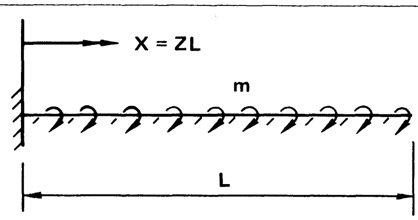


$$M_T = \beta \times M_O$$

$$M_O = \left(\frac{T}{h}\right)(L - X) \quad \text{for } 0 \leq X < L$$

	Z									
λL	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	0.92423	0.91705	0.91066	0.90504	0.90018	0.89609	0.89274	0.89015	0.88830	0.88719
1.0	0.76159	0.73915	0.71943	0.70229	0.68764	0.67540	0.66547	0.65782	0.65238	0.64913
1.5	0.60343	0.56651	0.53412	0.50763	0.48485	0.46608	0.45106	0.43959	0.43150	0.42669
2.0	0.48201	0.43446	0.39464	0.36155	0.33435	0.31237	0.29508	0.28204	0.27295	0.26758
2.5	0.39465	0.34000	0.29572	0.26002	0.23148	0.20898	0.19164	0.17879	0.16995	0.16478
3.0	0.33168	0.27246	0.22623	0.19023	0.16236	0.14100	0.12494	0.11329	0.10540	0.10082
3.5	0.28519	0.22310	0.17654	0.14164	0.11556	0.09621	0.08208	0.07205	0.06539	0.06158
4.0	0.24983	0.18600	0.14013	0.10714	0.08340	0.06641	0.05437	0.04606	0.04065	0.03760
4.5	0.22217	0.15737	0.11284	0.08214	0.06094	0.04632	0.03631	0.02961	0.02534	0.02297
5.0	0.19998	0.13476	0.09193	0.06369	0.04500	0.03261	0.02444	0.01913	0.01584	0.01404
5.5	0.18181	0.11655	0.07564	0.04986	0.03353	0.02315	0.01656	0.01242	0.00992	0.00859
6.0	0.16666	0.10163	0.06274	0.03935	0.02518	0.01655	0.01129	0.00810	0.00624	0.00526
7.0	0.14286	0.07882	0.04403	0.02499	0.01448	0.00862	0.00534	0.00349	0.00248	0.00198
8.0	0.12500	0.06241	0.03155	0.01620	0.00849	0.00458	0.00257	0.00153	0.00100	0.00074
10.0	0.10000	0.04088	0.01692	0.00711	0.00305	0.00135	0.00062	0.00030	0.00016	0.00011

Table 6. β Values for Cantilever Beam Subjected to Uniform Torsion m



$$M_T = \beta \times M_O$$

$$M_O = \left(\frac{1}{2}\right)\left(\frac{m}{h}\right)(L - X)^2 \quad \text{for } 0 \leq X < L$$

	Z									
λL	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	0.94302	0.93101	0.91735	0.90115	0.88093	0.85399	0.81501	0.75150	0.62606	0.25159
1.0	0.81930	0.78157	0.73937	0.68996	0.62878	0.54769	0.43045	0.23918	0.13965	1.27375
1.5	0.69584	0.63327	0.56509	0.48686	0.39133	0.26564	0.08421	0.21245	0.80271	2.57824
2.0	0.59693	0.51556	0.42985	0.33412	0.21935	0.06979	0.14563	0.49910	1.20683	3.34978
2.5	0.52147	0.42694	0.33124	0.22773	0.10639	0.04990	0.27457	0.64493	1.39244	3.67467
3.0	0.46322	0.35961	0.25924	0.15455	0.03492	0.11720	0.33548	0.69745	1.43515	3.70964
3.5	0.41697	0.30712	0.20556	0.10373	0.00945	0.15140	0.35486	0.69474	1.39522	3.57921
4.0	0.37924	0.26512	0.16459	0.06791	0.03644	0.16549	0.35043	0.66207	1.31247	3.36549
4.5	0.34776	0.23077	0.13269	0.04233	0.05228	0.16769	0.33324	0.61505	1.21125	3.11839
5.0	0.32104	0.20221	0.10744	0.02388	0.06096	0.16311	0.30999	0.56286	1.10567	2.86650
5.5	0.29805	0.17814	0.08720	0.01050	0.06503	0.15488	0.28458	0.51067	1.00338	2.62506
6.0	0.27805	0.15766	0.07083	0.00079	0.06613	0.14491	0.25924	0.46120	0.90818	2.40135
7.0	0.24497	0.12489	0.04656	0.01127	0.06342	0.12385	0.21291	0.37469	0.74397	2.01522
8.0	0.21877	0.10014	0.03012	0.01725	0.05778	0.10440	0.17451	0.30554	0.61356	1.70595
10.0	0.18000	0.06615	0.01105	0.02046	0.04524	0.07407	0.11961	0.20914	0.43069	1.26211