A Practical Method of Second Order Analysis

Part 2—Rigid Frames

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In Part 1 of this series,[1] a method was developed for performing second order analyses of pin jointed systems. A key concept of the method involved the lateral stiffness parameter P_L, which was defined as the force applied at the end of a member or subsystem that produces unit rotational displacement of the member or subsystem. Repeating an example from Part 1, in Fig. 7(a) the cantilever **AB** with a height H is loaded laterally with the force V. Figure 7(b) shows the cantilever's deflected shape and reactions, point **B** having moved Δ_{ov} horizontally under the force V. Figure 7(c) graphically defines P_L as the value of V which makes $\Delta_{ov} = H$. Algebraically this may be stated:

$$\frac{\Delta_{ov}}{H} = \frac{V}{P_L} \qquad (9)^*$$

or

$$P_L = \frac{VH}{\Delta_{ov}} \qquad (1)$$

For the cantilever **AB** (ignoring shear strains):

$$\Delta_{ov} = \frac{VH^3}{3EI} \qquad (2a)$$

and

$$P_L = \frac{3EI}{H^2} \qquad (4)$$

This definition will be applied in performing a second order analysis of the simple structure shown in Fig. 8. A second order analysis is necessary because of the presence of both load V and load P. The deflection Δ_{ov} with V acting alone [Fig. 7(b)] will increase to Δ_{pv} when both P and V act. Figure 9 shows the deflected shape of the structure with the reactions and internal forces.

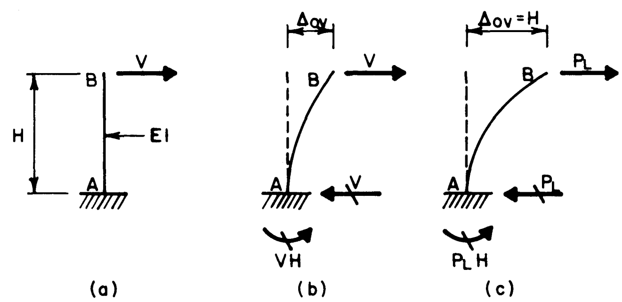

Figure 7

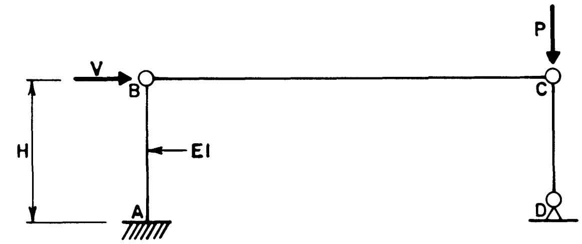

Figure 8

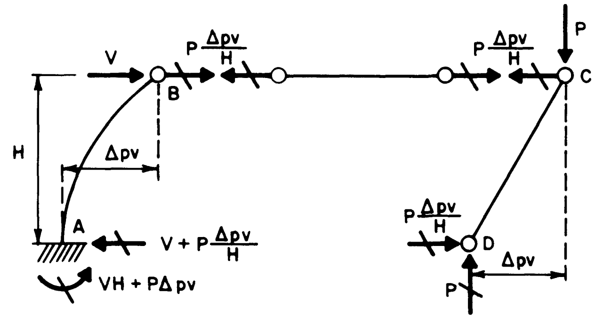

Figure 9

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* *Equations (1) through (28) are from Part 1 (Ref. 1).*

Force $P(\Delta_{pv}/H)$ is necessary at **C** and **D** for the equilibrium of member **CD** in its deflected position. Because this force exists at **C** and **D**, it must also exist at **A** and **B**, leading to the increased moment $VH + P\Delta_{pv}$ at **A**.

The deflection Δ_{pv} of cantilever **AB** may be found by:

$$\Delta_{pv} = \left(V + P\frac{\Delta_{pv}}{H}\right)\frac{H^3}{3EI}$$

Substituting $P_L = 3EI/H^2$:

$$\frac{\Delta_{pv}}{H} = \frac{\left(V + P\frac{\Delta_{pv}}{H}\right)}{P_L} \tag{10}$$

Solving for Δ_{pv}/H, the second order rotational displacement:

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P} \tag{12}$$

Equation (12) shows that rotational displacement becomes infinite when P equals the lateral stiffness parameter P_L, so that the critical buckling load is:

$$P_{cr} = P_L \tag{13}$$

The ratio of second order internal forces to first order internal forces was defined in Part 1 as the amplification factor *A.F.* The ratio of second order to first order base moments at **A** is:

$$A.F. = \frac{VH + P\Delta_{pv}}{VH} = 1 + \frac{P\Delta_{pv}}{VH} \tag{14}$$

Substituting Eq. (12) into Eq. (14) and simplifying:

$$A.F. = \frac{1}{1 - (P/P_L)} \tag{15}$$

Readers of Part 1 will recognize that this analysis is exactly the same as the analysis of a similar pin jointed structure (Fig. 2, Part 1). This is true as long as there is no local axial load in member **AB**. Member **AB** functions as a spring and could be replaced by a pin jointed A-frame or, indeed, by a horizontal spring at point **B** with a spring constant P_L/H. Equations (12), (13), and (15) would remain applicable.

This simple case is no longer valid when member **AB** has an axial load. Figure 10 is similar to Fig. 8, except that the load P acts on **AB** instead of **CD**. The reactions and deformations of member **AB** are shown in Fig. 11(a). Figure 11(b) shows the first order moment diagram and Fig. 11(c) shows the moment diagram from second order effects alone. This second order moment diagram has exactly the same shape as the deformed structure. Now, if V is very large and P is very small, this shape will be approximated by the equation for the deflection of an end loaded cantilever. (See Ref. 2, pg. 2-205.) Figure 11(c) is based on this assumption and Fig. 11(d) shows the portion of the $P\Delta_{pv}$ moment diagram which is outside of the right triangle with base $P\Delta_{pv}$ and height H.

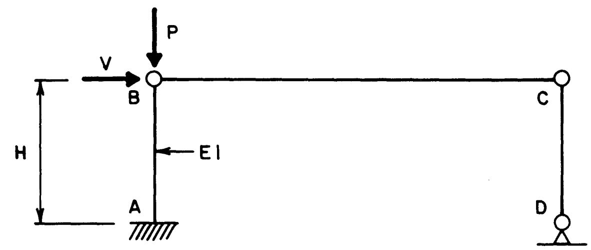


Figure 10

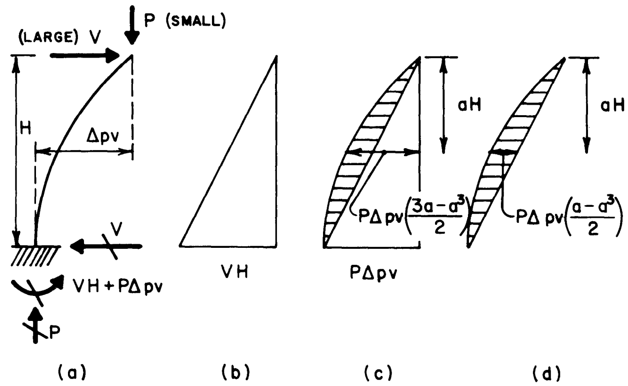


Figure 11

The deflection Δ_{pv} may be calculated by the moment area method as follows:

$$\Delta_{pv} = \left(\frac{VH}{EI}\frac{H}{2}\frac{2H}{3}\right) + \left(\frac{P\Delta_{pv}}{EI}\frac{H}{2}\frac{2H}{3}\right) + \left(\frac{P\Delta_{pv}}{EI}H^2\right)\int_0^1\left(\frac{a^2 - a^4}{2}\right)da$$

This expression may be written in terms of P_L as:

$$\Delta_{pv} = \frac{VH}{P_L} + \frac{P\Delta_{pv}}{P_L} + \frac{P\Delta_{pv}}{P_L}\left[3\int_0^1\left(\frac{a^2 - a^4}{2}\right)da\right]$$

The bracketed term is:

$$3\int_0^1\left(\frac{a^2 - a^4}{2}\right)da = \frac{1}{5} = 0.2$$

Substituting and solving for (Δ_{pv}/H):

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P - 0.2P} \tag{29}$$

This expression is very similar to Eq. (12), except for the additional term $0.2P$ in the denominator. The term $0.2P$ is only correct, however, if P is vanishingly small.

On the other hand, if V is assumed very small, as P is very large and approaches the buckling load, the deflected shape of the cantilever will approach a sine curve, as shown

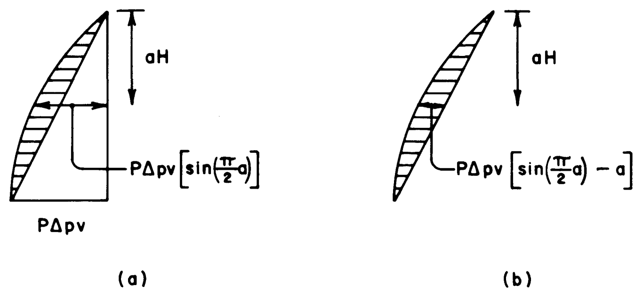


Figure 12

by Euler. Figure 12(a) shows the second order moment diagram based on a sine curve and Fig. 12(b) shows the portion outside the right triangle.

Again calculating the deflection Δ_{pv}:

$$\Delta_{pv} = \left(\frac{VH}{EI}\frac{H}{2}\frac{2H}{3}\right) + \left(\frac{P\Delta_{pv}}{EI}\frac{H}{2}\frac{2H}{3}\right) + \frac{P\Delta_{pv}}{EI}H^2\int_0^1 a\left(\sin\frac{\pi}{2}a - a\right)da$$

Solving the integral and substituting P_L:

$$\Delta_{pv} = \frac{VH}{P_L} + \frac{P\Delta_{pv}}{P_L} + \frac{P\Delta_{pv}}{P_L}\left(\frac{3}{(\pi/2)^2} - 1\right)$$

from which

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P - \left[\frac{3}{(\pi/2)^2} - 1\right]P} \tag{30}$$

The term in brackets is now 0.216, about 8% higher than 0.2. It is proposed that the rotational displacement equation be written as:

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P - C_L P} \tag{31}$$

where C_L is a constant. For the cantilever column, it has been shown that C_L actually varies between 0.2 and 0.216 as P varies from zero to the Euler load P_e. This is because of the change in shape of the deflection curve with increasing axial load. Since the higher value is conservative for all values of P, it can be adopted as the basis for approximating the second order rotational displacement Δ_{pv}/H.

If the numerator and denominator of the bracketed terms in Eq. (30) are multiplied by EI/H^2, the bracketed expression becomes:

$$\left[\frac{3\dfrac{EI}{H^2}}{\left(\dfrac{\pi}{2}\right)^2\dfrac{EI}{H^2}} - 1\right]$$

This expression will be recognized as being identical with:

$$\left[\frac{P_L}{P_e} - 1\right]$$

where P_e is Euler's load for the cantilever column. Equation (31) may be generalized for any column as:

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P - \left[\frac{P_L}{P_e} - 1\right]P} \tag{32}$$

Inspection of the right hand denominator shows that it equals zero when $P = P_e$, which means that deflection becomes infinite for any V at the buckling load. This, of course, must be the case for a correct second order analysis.

In general, for any column which can have rotational displacement:

$$C_L = \left[\frac{P_L}{P_e} - 1\right] \tag{33}$$

Introducing a stiffness factor, β, which accounts for the elastic end restraints on a column,

$$P_L = \beta\frac{EI}{H^2} \tag{34}$$

$$P_e = \left(\frac{\pi}{K}\right)^2\frac{EI}{H^2} \tag{35}$$

where K = effective length factor. Therefore,

$$C_L = \left[\frac{\beta K^2}{\pi^2} - 1\right] \tag{36}$$

The values of β and K are dependent on the end conditions of the column, defined by the following parameter:[2]

$$G = \frac{\Sigma\dfrac{EI_c}{L_c}}{\Sigma\dfrac{EI_g}{L_g}}$$

For the cantilever column, $G_{TOP} = \infty$, $G_{BOT} = 0$, $\beta = 3$, $K = 2$, and $C_L = 0.216$.

Returning to Fig. 11(a), substituting Eq. (31) into Eq. (14) and simplifying:

$$A.F. = \frac{1}{1 - \dfrac{P}{P_L - C_L P}} \tag{37}$$

Also, when the denominator of either Eq. (31) or Eq. (37) is zero, the column becomes unstable. Using Eq. (31):

$$P_L - P_{cr} - C_L P_{cr} = 0$$

$$P_{cr} = \frac{P_L}{1 + C_L} \tag{38}$$

This is, of course, simply a restatement of Eq. (33). However, Eq. (38) will acquire a broader significance when applied to a whole system.

At this point it is appropriate to examine the accuracy of Eqs. (31) and (37). So far, the only approximation that has been made is to assume C_L a constant instead of a variable over the range 0.200 to 0.216. An exact elastic solution for Δ_{pv}/H and the *A.F.* for Fig. 11(a) may be derived by writing a differential equation whose solution yields Eqs. (39) and (40):

$$\frac{\Delta_{pv}}{H} = \frac{V}{P}\left[\frac{\tan\alpha}{\alpha} - 1\right] \tag{39}$$

$$A.F. = \frac{\tan\alpha}{\alpha} \tag{40}$$

where $$\alpha = \sqrt{\frac{PH^2}{EI}} = \sqrt{\frac{3P}{P_L}}$$

A comparison of Eqs. (31) and (37) to these exact solutions for a range of P's is given in Table 2.

The importance of the factor C_L is also shown in Table 2 by comparing the *A.F.* based on Eq. (15), which omits the C_L correction, to the exact solution.

Summarizing the principles of the practical method:

Pin Jointed "A" Frames		Individual Columns	
$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P}$	(12)	$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P - C_L P}$	(31)
$A.F. = \frac{1}{1 - \frac{P}{P_L}}$	(15)	$A.F. = \frac{1}{1 - \frac{P}{P_L - C_L P}}$	(37)
$P_{cr} = P_L$	(13)	$P_{cr} = \frac{P_L}{(1 + C_L)}$	(38)

In these equations, the factor C_L accounts for the reduction in column stiffness due to the presence of axial load. As shown in Table 2, the method is very nearly exact, with a maximum error of 1.5% on the conservative side. It will be seen later in this paper that the factor C_L may be ignored in many practical design cases.

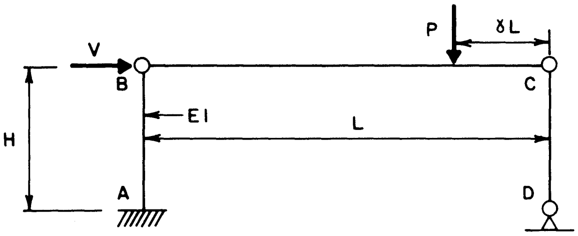


Figure 13

A more general problem in second order analysis is posed by the structure in Fig. 13, with a load in any position defined by the dimension γL. This structure's behavior lies between that of Figs. 8 and 10. The reactions, deformations, and internal forces are shown in Fig. 14 and the moment diagrams for member **AB** are shown in Fig. 15.

Proceeding as before, by the moment area method:

$$\Delta_{pv} = \frac{VH}{P_L} + \frac{P\Delta_{pv}}{P_L} + \frac{\gamma P\Delta_{pv}}{P_L} C_L$$

giving

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P - C_L\gamma P}$$

and

$$A.F. = \frac{1}{1 - \frac{P}{P_L - C_L\gamma P}}$$

It will be seen that deflection and amplification of the whole system are affected by the total load P, but the local stiffness reduction expressed by the term $C_L\gamma P$ is only a function of the axial load γP actually present in member **AB**.

The maximum value of P may be found from

$$P_L - P - C_L\gamma P = 0$$

yielding

$$P_{cr} = \frac{P_L}{1 + \gamma C_L} \tag{41}$$

Table 2

Load	Eq. (31) / Eq. (39)	Eq. (37) / Eq. (40)	Exact C_L	Eq. (15) / Eq. (40)
$6/23\,P_e$	1.004	1.001	0.2038	0.988
$12/23\,P_e$	1.007	1.003	0.2078	0.926
$20/23\,P_e$	1.013	1.011	0.2136	0.547
$0.999\,P_e$	1.015	1.015	0.2159	0.001

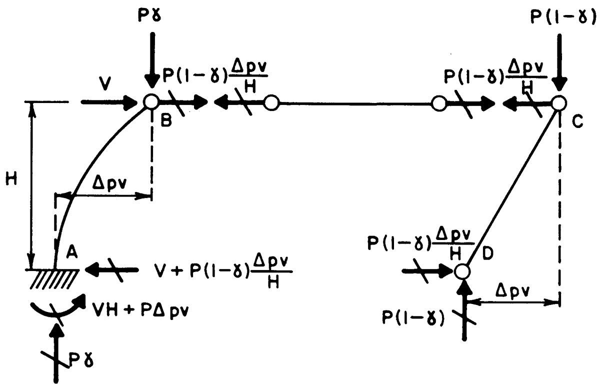


Figure 14

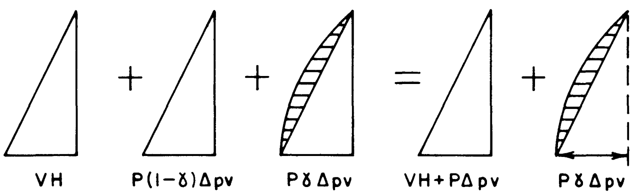


Figure 15

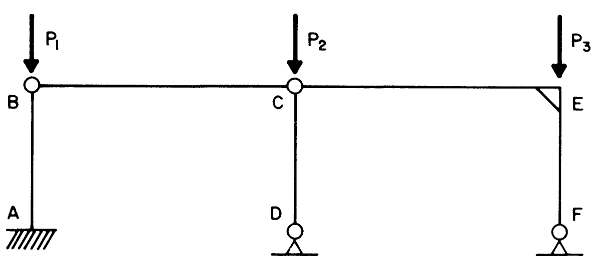


Figure 16

For actual design of member **AB** by the AISC Specification, an effective length factor must be found which reflects the behavior of the whole system **ABCD**. This may be obtained by equating the load in member **AB** to its Euler buckling load expressed in terms of K:

$$\gamma P_{cr} = \frac{\gamma P_L}{(1 + \gamma C_L)} = \left(\frac{\pi}{K}\right)^2 \frac{EI}{H^2}$$

Substituting $\beta(EI/H^2) = P_L$ and solving for K^2:

$$K^2 = \frac{\pi^2}{\beta}\left[\frac{1 + \gamma C_L}{\gamma}\right] \tag{42}$$

For $\gamma = 1$, $K = 2.00$; for $\gamma = 0.5$, $K = 2.70$; for $\gamma = 0.1$, $K = 5.80$. When $\gamma = 0$, K has no meaning, since member **AB** has no load and functions only as a spring.

An even more general case is shown by Fig. 16. Here is a complex frame with two sources of lateral stiffness, namely, member **AB** and member **EF** restrained by member **CE**. The total system stiffness may be identified as ΣP_L, which implies that P_L for member **AB** is added to P_L for member **EF**. (Member **CD** does not resist lateral forces.) All the loads P_1, P_2, and P_3 add up to ΣP.

Equations (31) and (37) may be rewritten for a whole story, such as Fig. 16, as follows:

$$\frac{\Delta_{pv}}{H} = \frac{1}{P_L - P - C_L P}$$

becomes

$$\frac{\Delta_{pv}}{H} = \frac{1}{\Sigma P_L - \Sigma P - \Sigma(C_L P)} \tag{43}$$

and

$$A.F. = \frac{1}{1 - \dfrac{P}{P_L - C_L P}}$$

becomes

$$A.F. = \frac{1}{1 - \dfrac{\Sigma P}{\Sigma P_L - \Sigma(C_L P)}} \tag{44}$$

Equation (44) depends on the relation of loads to stiffness. Therefore, for design at working load levels, a load factor must be introduced to give results at working loads which are proportional to ultimate loads. Thus,

$$A.F. = \frac{1}{1 - \dfrac{\Sigma P_w \times L.F.}{\Sigma P_L - \Sigma(C_L P_w \times L.F.)}} \tag{45}$$

The term $\Sigma(C_L P)$ in these expressions means that the factor C_L for each column, individually, is multiplied by the axial load P in that column. There will be a term $C_L P$ for every column in a story. The factor C_L will of course be zero for columns hinged at both ends.

To find the critical load on any column such as **AB** in Fig. 16, proceed as follows:

The maximum load occurs when:

$$\Sigma P_L - \Sigma P_{cr} - \Sigma(C_L P_{cr}) = 0$$

Rearranging,

$$1 = \frac{\Sigma P_{cr} + \Sigma(C_L P_{cr})}{\Sigma P_L}$$

Multiplying the left side by P_{icr} and the right side by $(\pi/K_i)^2(EI_i/H^2)$, and solving for $(K_i)^2$:

$$(K_i)^2 = \frac{\pi^2 EI_i}{\Sigma P_L H^2}\left[\frac{\Sigma P_{cr} + \Sigma(C_L P_{cr})}{P_{icr}}\right]$$

Since the bracketed term is a ratio which is independent of load factor, an expression for the K-factor for the ith member in any story is:

$$(K_i)^2 = \frac{\pi^2 EI_i}{\Sigma P_L H^2}\left[\frac{\Sigma P + \Sigma(C_L P)}{P_i}\right] \quad (46a)$$

or, more usefully,

$$(K_i)^2 = \frac{I_i}{P_i}\left[\frac{\pi^2 E}{H^2}\right]\left[\frac{\Sigma P + \Sigma(C_L P)}{\Sigma P_L}\right] \quad (46b)$$

which can be expressed in terms of β as:

$$K_i{}^2 = \frac{I_i}{P_i}\pi^2\left[\frac{\Sigma P + \Sigma(C_L P)}{(\Sigma\beta I)}\right] \quad (46c)$$

The reader will note the similarity of Eq. (46c) and Eq. (42) for the simple frame of Fig. 13.

In many practical applications, ΣP_L for a complex frame may be determined directly for the whole story by computer analysis using the basic definition $\Sigma P_L = \Sigma VH/\Delta_{ov}$, where Δ_{ov} is the deflection due to ΣV. If such an analysis includes axial and shear deformations as well as bending deformations, the resulting ΣP_L will be less than $\Sigma\beta(EI/H^2)$, which is based on bending alone. If ΣP_L is determined in this way, the expression for K_i may be written as:

$$K_i{}^2 = \frac{I_i}{P_i}\left[\frac{\pi^2 E}{H^3}\frac{\Delta_{ov}}{\Sigma V}[\Sigma P + \Sigma(C_L P)]\right] \quad (46d)$$

The basic equations for second order frame analysis have now been defined. Although the derivations were based on a simple cantilever column, the analysis and definitions are valid for columns with any end conditions.

Figure 17(a) shows a generalized column, free to sidesway, with end conditions defined by G_A and G_B, which are exactly the same parameters used for the nomograph in Sect. 1.8 of the Commentary to the AISC Specification. The horizontal deflection of such a column is given by:

$$\Delta_{ov} = \frac{VH^3}{\beta EI}$$

from which

$$P_L = \frac{\beta EI}{H^2}$$

From Figs. 17(a) and (c):

$$\theta_A = \frac{M_A G_A H}{6EI} \quad (47)$$

$$\theta_B = \frac{M_B G_B H}{6EI} \quad (48)$$

From Fig. 17(b):

$$VH = M_A + M_B \quad (49)$$

From Figs. 17(b) and (c):

$$\theta_B + \frac{VH^2}{2EI} - \frac{M_A H}{EI} = \theta_A \quad (50)$$

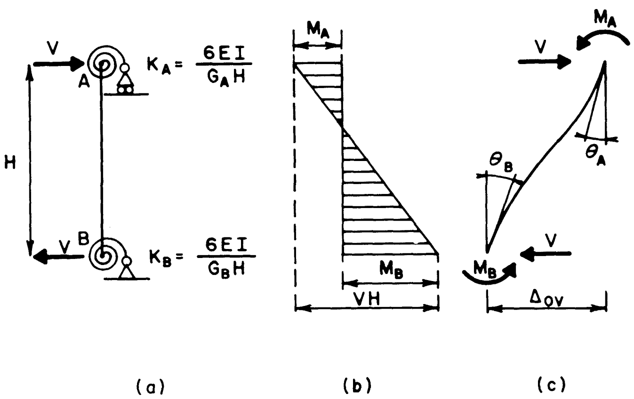


Figure 17

Combining Eq. (50) with Eqs. (47), (48), and (49) yields:

$$\frac{M_A}{M_B} = \frac{G_B + 3}{G_A + 3} \quad (51)$$

$$M_A = \left[\frac{G_B + 3}{G_A + G_B + 6}\right] VH \quad (52)$$

$$M_B = \left[\frac{G_A + 3}{G_A + G_B + 6}\right] VH \quad (53)$$

From Figs. 17(b) and (c):

$$\theta_B H + \frac{VH^3}{3EI} - \frac{M_A H^2}{2EI} = \Delta_{ov} = \frac{VH}{P_L} = \frac{VH^3}{\beta EI} \quad (54)$$

Substituting Eqs. (48), (52), and (53) into Eq. (54):

$$\beta = \frac{6(G_A + G_B) + 36}{2(G_A + G_B) + G_A G_B + 3} \quad (55)$$

The derivation of β is based on simple first order analysis.

The value of K for this general case is given by the well-known expression:[3]

$$\frac{6\frac{\pi}{K}}{\tan\frac{\pi}{K}}(G_A + G_B) - \left(\frac{\pi}{K}\right)^2 G_A G_B + 36 = 0 \quad (56)$$

which is the basis of the nomograph giving K for the sidesway case.

From Eqs. (55) and (56), values of β and K can be computed and C_L can be determined from Eq. (36):

$$C_L = \left[\frac{\beta K^2}{\pi^2} - 1\right] \quad (36)$$

The values of β and C_L have been plotted in Fig. 18.

Having found G_A and G_B at the two ends of a column, to determine C_L from Fig. 18, enter with the larger G-value as G_B at the bottom of the chart and the smaller G-value

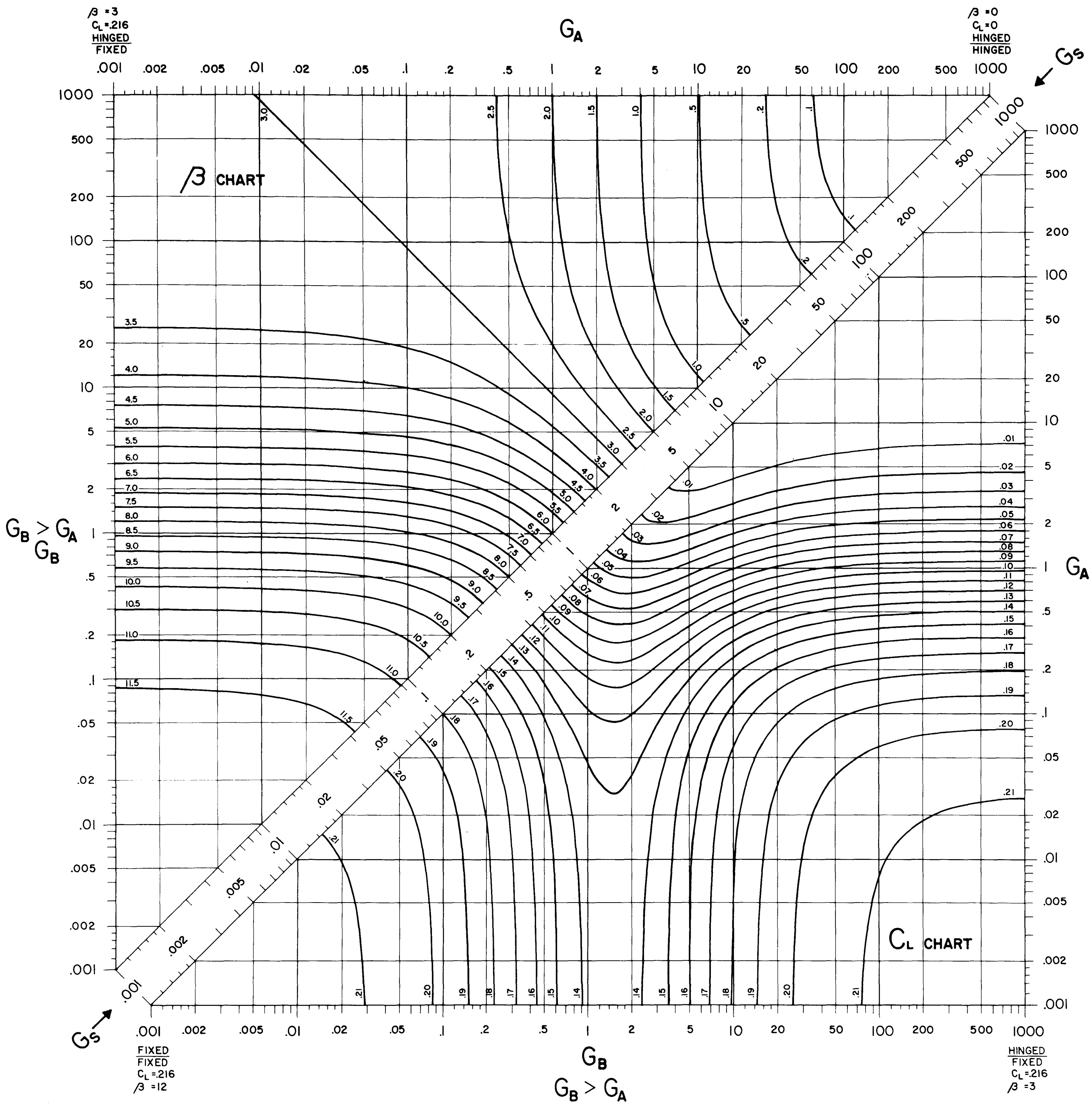


Figure 18

as G_A at the right side of the chart. The point of intersection of a vertical line from G_B and a horizontal line from G_A locates C_L. The value of C_L can be interpolated between the curved lines, which are C_L contours. (See Fig. 25 for a three-dimensional view.)

The value of β may be found in a similar way, starting with the larger G as G_B on the left side and the smaller G as G_A at the top.

When the G's are the same at each end, take $G = G_S$ and enter along the diagonal to find β and C_L. The scales at the sides and on the diagonal are logarithmic.

Since it is most important to determine β accurately, the author recommends the use of Eq. (55), (57), (59), or (61) with the chart as a check. When the G's are equal, or one end is perfectly hinged, Eq. (58) or (60) may be used for C_L. The chart is necessary to find C_L when $G_A \neq G_B$.

Three common cases have simple solutions for β and C_L:

1. When $G_A = G_B = G_S$, the column is symmetrical with a point of inflection at the middle. This is the typical case for a multistory building with regular floor-to-floor heights. In this case:

$$\beta = \frac{12}{(1 + G_S)} \quad \text{(exact)} \tag{57}$$

$$C_L \cong \frac{0.22}{(1 + G_S)^2} \quad \text{(approx.)} \tag{58}$$

The value for C_L, although approximate, is very close to the exact value.

2. When $G_B = \infty$, the column is hinged at one end. For this case:

$$\beta = \frac{3}{\left(1 + \frac{G_A}{2}\right)} \quad \text{(exact)} \tag{59}$$

$$C_L \cong \frac{0.22}{\left(1 + \frac{G_A}{2}\right)^2} \quad \text{(approx.)} \tag{60}$$

3. The value of G for a true hinge is ∞. However, the AISC Specification recommends that a hinge be approximated by $G_B = 10$, which is realistic but means that a point of inflection exists in the column. This results in:

$$\beta = \frac{6G_A + 96}{12G_A + 23} \tag{61}$$

C_L is found from Fig. 18.

The point of inflection is located at

$$\left[\frac{G_A + 3}{G_A + 16}\right] H$$

from the "hinge".

The theory developed above will be best understood by application to practical cases.

DESIGN EXAMPLE 1

Figure 19 shows a problem adapted from the writer's practice. The frame was designed to have slender exterior columns with all lateral rigidity supplied by the central column. Perpendicular bracing is supplied in the plane of the roof and at mid-height of the columns. The loads and reactions are shown on the figure. For simplicity, points **A, B, C, E,** and **F** are initially assumed to be perfect hinges. Point **D** is a rigid connection.

First, column **CD** will be checked by Eq. (1.6-1a) of the AISC Specification, recognizing this as a sidesway case, using the common but *incorrect* assumption that the effective length factor K for column **CD** can be determined from the end conditions and the nomograph.

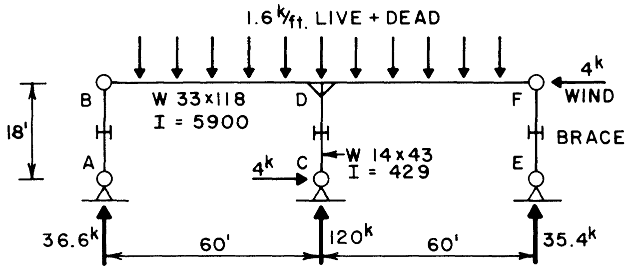


Figure 19

Member **CD** is a W14X43 with its web in the plane of the figure.

For W14X43:

$A = 12.6$ in.^{2}

$I_x = 429$ in.^{4}

$r_x = 5.82$ in.

$r_y = 1.89$ in.

$d/A_f = 3.24$ in.$^{-1}$

$S_x = 62.7$ in.^{3}

$$*G_{TOP} = \frac{429}{18} \times \left(\frac{2 \times 60}{2 \times 5900}\right) = 0.242$$

$G_{BOT} = \infty$

From nomograph: $K = 2.08$

$$\frac{KL_y}{r_y} = \frac{1 \times 9 \times 12}{1.89} = 57.1$$

$$\frac{KL_x}{r_x} = \frac{2.08 \times 18 \times 12}{5.82} = 77.2$$

$F_{ax} = 15.66$ ksi

$F'_{ex} = 25.04$ ksi

Axial load on **CD** $= \frac{5}{8} \times (1.6 \times 60) \times 2$
$= 120$

$$f_a = \frac{0.75 \times 120}{12.6} = 7.14 \text{ ksi}$$

$$f_{bx} = \frac{0.75 \times 4 \times 18 \times 12}{62.7} = 10.33 \text{ ksi}$$

$$F_{bx} = \frac{12 \times 10^3 \times 1.30}{9 \times 12 \times 3.24} = 44.58 \text{ ksi} \qquad \text{[Eq. (1.5-7)]}$$

But, F_{bx} must not exceed 22.0 ksi.

$\therefore$ Use $F_{bx} = 22.0$ ksi

* *The 2 in the numerator is there because BD and DF are hinged at their far ends.*

Equation (1.6-1a):

$$\frac{f_a}{F_{ax}} + \frac{0.85}{\left(1 - \frac{f_a}{F'_{ex}}\right)} \frac{f_{bx}}{F_{bx}} \leq 1$$

$$\frac{7.14}{15.66} + \left[\frac{0.85}{\left(1 - \frac{7.14}{25.04}\right)} \times \frac{10.33}{22.0}\right]$$

$$= 0.456 + (1.189 \times 0.470) = 1.01$$

The W14X43 is satisfactory if a 1% overstress is considered acceptable in meeting the Specification.

The check just made is based on the assumption that column **CD** exists in isolation of the rest of the system. It ignores the fact that **CD** furnishes stability to the "leaning" columns, **AB** and **EF**.

The correct K_x may be found from Eq. (46c). Using the same values, $G_{TOP} = 0.242$, $G_{BOT} = \infty$, in Eq. (59) and Eq. (60):

$$\beta = \frac{3}{\left(1 + \frac{0.242}{2}\right)} = 2.676$$

$$C_L \cong \frac{0.22}{\left(1 + \frac{0.242}{2}\right)^2} = 0.175$$

$$\Sigma P = 1.6 \times 2 \times 60 = 192 \text{ kips}$$

$$\Sigma C_L P = (0 \times 36.6) + (0.175 \times 120) + (0 \times 35.4)$$
$$= 21 \text{ kips}$$

Substituting in Eq. (46c):

$$K_x{}^2 = \frac{429}{120} \pi^2 \left[\frac{192 + 21}{2.676 \times 429}\right] = 6.55$$

$$K_x = 2.56$$

For this example, since only column **CD** provides stability to the system, Eq. (46c) reduces to:

$$K_{CD}{}^2 = \frac{\pi^2}{P_{CD}} \left[\frac{\Sigma P + (C_L P)_{CD}}{\beta_{CD}}\right]$$

In this form it can be seen that K may be correctly obtained with only β and C_L in combination with the loads. If ΣP is taken incorrectly as the 120 kips on column **CD** alone, it is found that $K = 2.08$, which is identical with the nomograph solution. With ΣP taken as the true system load, the correct value is found to be $K = 2.56$.

Using $K = 2.56$:

$$\frac{K_x L_x}{r_x} = \frac{2.56 \times 18 \times 12}{5.82} = 94.9$$

$$F_{ax} = 13.61 \text{ ksi}$$

$$F'_{ex} = 16.57 \text{ ksi}$$

Substituting these new values in Eq. (1.6-1a):

$$\frac{7.14}{13.61} + \left[\frac{0.85}{\left(1 - \frac{7.14}{16.57}\right)} \times \frac{10.33}{22}\right]$$

$$= 0.525 + 1.49 \times 0.470 = 1.23$$

It can now be seen that the column is substantially overstressed! But that is not all. The number 1.49 is the AISC amplification factor, which includes the term $C_m = 0.85$. C_m is a very approximate factor which is only correct for a fixed end column with $f_a \geq 0.84F'_e$. If Eq. (45) is used and the AISC load factor of 23/12 is reduced for wind to $3/4 \times 23/12 = 23/16$, an accurate amplification factor can be determined.

$\beta = 2.676$ (previously calculated)

$$P_L = \beta \frac{EI}{H^2} = \frac{2.676 \times 29{,}000 \times 429}{(18 \times 12)^2} \qquad \text{Eq. (34)}$$

$$= 713 \text{ kips}$$

$C_L = 0.175$ (previously calculated)

$$C_L P \times L.F. = 0.175 \times 120 \times (23/16) = 30 \text{ kips}$$

$$\Sigma P \times L.F. = 192 \times (23/16) = 276 \text{ kips}$$

Substituting in Eq. (45):

$$A.F. = \frac{1}{1 - \left(\frac{276}{713 - 30}\right)} = 1.68$$

Using this value, AISC Eq. (1.6-1a) yields:

$$0.525 + (1.68 \times 0.470) = 1.31$$

To get the correct amplification factor, the AISC formula would require that $C_m = 0.95$ for this case.

The author believes that the load factor in the AISC amplification method is unnecessarily high at 23/12 and should be 5/3, which is the same value used for bending and tension. (Both of these are reduced by 3/4 for wind loads.) Two changes to the sidesway amplification procedure are presently under consideration by the AISC Specification Advisory Committee. One is the change of load factor to 5/3 and the other is to make $C_m = 1$ for the sidesway case. These changes offset each other somewhat. A third change under consideration is to offer designers the ability to use any rational method for elastic second order analysis, with a load factor of 5/3 for gravity and 5/4 for gravity plus wind. If these changes are adopted, the author's procedure, Eq. (45), would yield:

$$A.F. = \frac{1}{1 - \left(\frac{192 \times 5/4}{713 - 0.175 \times 120 \times 5/4}\right)} = 1.53$$

or, ignoring the $\Sigma C_L P$ term,

$$A.F. = \frac{1}{1 - \dfrac{192 \times 5/4}{713}} = 1.51$$

The present AISC Eq. (1.6-1a), with $C_m = 1$ and the Load Factor reduced to 5/4, would yield:

$$A.F. = \frac{1}{1 - \left(\dfrac{7.14 \times \dfrac{16}{23} \times \dfrac{5}{4}}{16.57}\right)} = 1.60$$

which is clearly conservative.

Using the value of $A.F.$ from Eq. (45) in AISC Eq. (1.6-1a):

$$0.525 + (1.53 \times 0.470) = 1.24$$

which would be the author's interpretation of the 1969 AISC Specification, including the proposed change of load factor. Note that this last result, 1.24, is very close to the value 1.23 obtained above using the present Specification with the correct value for K of 2.56. This is because the unconservative coefficient C_m is offset by the conservative load factor and the inherently conservative term $1/[1 - (f_a/F'_e)]$.

All of the foregoing analysis is very conservative in treating joint **C** as a pure hinge. A more reasonable design would take G for joint **C** as 10. Figure 19 will be reanalyzed with this assumption, using the author's method and a load factor of 5/4:

$G_{TOP} = 0.242$ (as before)

$G_{BOT} = 10$

From Eq. (61):

$$\beta = \frac{(6 \times 0.242) + 96}{(12 \times 0.242) + 23} = 3.76$$

$C_L = 0.14$ (from Fig. 18)

$$K^2{}_{CD} = \frac{\pi^2}{P_{CD}} \left[\frac{\Sigma P + (C_L P)_{CD}}{\beta_{CD}}\right]$$

$$= \frac{\pi^2}{120} \left[\frac{192 + (0.14 \times 120)}{3.76}\right]$$

$K_{CD} = 2.14$

With this value for K:

$$\frac{K_x L_x}{r_x} = \frac{2.14 \times 18 \times 12}{5.82} = 79.3$$

$F_a = 15.43$

$$P_L = \frac{3.76 \times 429 \times 29{,}000}{(18 \times 12)^2} = 1{,}003 \text{ kips}$$

$C_L P = 0.14 \times 120 = 16.8$ kips

The assumption of $G = 10$ at point **C** means that the column has a point of inflection, reducing the maximum bending moment. From Eq. (52), the maximum bending moment is:

$$\left[\frac{10 + 3}{0.242 + 10 + 6}\right] VH = 0.800\ VH$$

From Eq. (45):

$$A.F. = \frac{1}{1 - \left(\dfrac{192 \times 5/4}{1{,}003 - (16.8 \times 5/4)}\right)} = 1.32$$

or, without the $C_L P$ correction,

$$A.F. = \frac{1}{\left(\dfrac{192 \times 5/4}{1{,}003}\right)} = 1.31$$

Substituting these new values in AISC Eq. (1.6-1a) and using the author's amplification factor:

$$\frac{7.14}{15.43} + 1.32 \left(\frac{0.800 \times 10.33}{22.0}\right)$$

$$= 0.463 + (1.32 \times 0.376) = 0.960$$

Using the current AISC amplification procedure and the correct $K = 2.14$:

$F'_{ex} = 23.75$

$$A.F.\ (\text{AISC}) = \frac{0.85}{\left(1 - \dfrac{8.27}{23.75}\right)} = 1.30$$

and

$$0.463 + (1.30 \times 0.376) = 0.953 \qquad \text{Eq. (1.6-1a)}$$

By either method, the column is acceptable for strength with $G = 10$.

A complete design should include a check of deflection or drift at working loads. The first order drift ratio is easily obtained from

$$\frac{\Delta_{ov}}{H} = \frac{\Sigma V}{\Sigma P_L} = \frac{4}{1{,}003} = 0.00399$$

There are no universally accepted drift criteria, but many designers would consider 0.004 acceptable for an industrial structure.

The true drift ratio at working loads is, however, substantially higher. It may be found accurately from Eq. (31), or very closely from Eq. (12). Using Eq. (31):

$$\frac{\Delta_{pv}}{H} = \frac{4}{1{,}003 - 192 - (0.14 \times 120)}$$

$$= 0.00504$$

If this is thought to be excessive, the designer might consider studying the footing for column **CD**. If larger stiffness and strength at joint **C** can be justified, the structure might still be acceptable.

In this example it has been shown how the author's practical method can be applied in practice. For those designers who wish to apply the method within the present AISC Specification, the author suggests the use of Eq. (46b), (46c), or (46d) to find the correct K for each member. Using these values of K for finding F_a and F'_e will give very reasonable results.

DESIGN EXAMPLE 2

In order to emphasize the method of finding correct K-value, let us analyze the structure shown in Fig. 20. This structure is taken from Fig. 15.18 of the *Guide to Stability Design Criteria for Metal Structures*[4].

Equation (46c) is used to find the K-values, and Eqs. (59) and (60) are used for β and C_L. The work is summarized in Table 3. Simplifying Eq. (46c) after substituting the appropriate β and C_L values:

$$K_i{}^2 = \frac{I_i}{P_i} \times \pi^2 \left[\frac{\Sigma P + \Sigma(C_L P)}{\Sigma \beta I} \right]$$

$$= \frac{I_i}{P_i} \times \pi^2 \left[\frac{210 + 41.4}{1{,}357} \right] = \frac{I_i}{P_i} \times 1.83$$

The tabulated values of K are slightly low because axial strains are not accounted for in this frame. ΣP_L is assumed as:

$$\Sigma P_L = 1{,}357 \times \frac{29{,}000}{(12 \times 12)^2} = 1{,}898 \text{ kips}$$

The author performed an accurate first order calculation of Δ_{ov} from a load of 210 kips applied horizontally at joint **2**, using the STRESS program which accounts for axial deformations. The result was $\Delta_{ov} = 16.04$ in. This yields a more accurate value of ΣP_L:

$$\Sigma P_L = \frac{210 \times 144}{16.04} = 1{,}885 \text{ kips}$$

Using this result and Eq. (46b):

$$K_i{}^2 = \frac{I_i}{P_i} \left[\frac{\pi^2 \times 29{,}000}{(144)^2} \right] \left[\frac{210 + 41.4}{1{,}885} \right]$$

$$= \frac{I_i}{P_i} \times 1.84$$

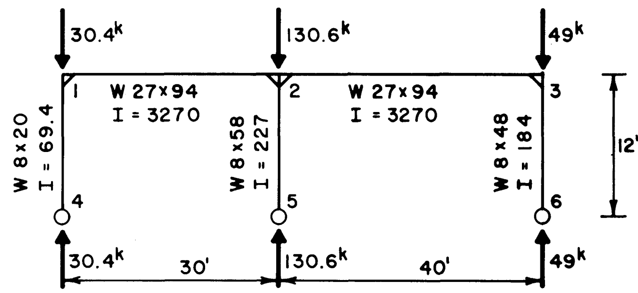


Figure 20

These values are listed as K' in Table 3. They are identical with the "exact computer" values reported in Ref. 4. It is obvious that the values obtained from Eq. (46c) based on β only are entirely adequate for design purposes.

An underlying assumption in this example is that elastic behavior can be assumed all the way to failure. The axial stresses in the three columns at design loads are:

Column **1-4**: $\dfrac{30.4}{5.89} = 5.16$ ksi

Column **2-5**: $\dfrac{130.6}{17.1} = 7.64$ ksi

Column **3-6**: $\dfrac{49}{14.1} = 3.48$ ksi

If the loads on all columns are increased at a uniform rate, column **2-5** will be the first to reach a level of stress which will cause a loss of stiffness. The true reduced stiffness of column **2-5** will be a function of τEI, where τ depends on the axial stress. As the stiffness drops, the K's will change. The K of column **2-5** will decrease while the others increase.

The maximum load for the frame of Fig. 20, assuming all loads are increased by the same load factor, can be obtained by the following procedure:

Step 1: Assume a load factor.

Step 2: Calculate the axial stress f_a in each column.

Table 3. Summary of Elastic Stability Analysis, Design Example 2

Column	P (kips)	I (in.^{4})	G_A	G_B	β	βI (in.^{4})	C_L	$C_L P$ (kips)	K	K'
1-4	30.4	69.4	0.0531	∞	2.92	203	0.209	6.35	2.04	2.05
2-5	130.6	227.0	0.0992	∞	2.86	649	0.200	26.08	1.78	1.79
3-6	49.0	184.0	0.188	∞	2.74	505	0.184	9.01	2.62	2.63
$\Sigma P = 210$						$\Sigma\beta I = 1{,}357$	$\Sigma C_L P = 41.4$			

Step 3: Find τ for each column from

$$\tau = 4\left(1 - \frac{f_a}{F_y}\right)\frac{f_a}{F_y}$$

Step 4: Find G based on τI, and calculate $\beta(\tau EI/H^2)$ and $C_L P$ for each column.

Step 5: Check if

$$\Sigma\left[\beta\frac{\tau EI}{H^2}\right] < \Sigma P + \Sigma(C_L P)$$

Step 6: If not, increase the load factor until

$$\Sigma\left[\beta\frac{\tau EI}{H^2}\right] = \Sigma P + \Sigma(C_L P)$$

If such an analysis is performed, the maximum load factor for Fig. 20 is 4.49. The resulting *inelastic* K's can be found from Eq. (46c) modified to the following inelastic form:

$$K_i^2 = \frac{\tau I_i}{P_i}\pi^2\left[\frac{\Sigma P + \Sigma(C_L P)}{\Sigma(\tau\beta I)}\right] \quad (46e)$$

The analysis is summarized in Table 4. For the whole frame,

$$\Sigma P_L = \Sigma(\tau\beta I)\frac{E}{H^2}$$

$$= 813 \times \frac{29{,}000}{144^2} = 1{,}137 \text{ kips}$$

$$\Sigma P + \Sigma(C_L P) = 943 + 193 = 1{,}136 \text{ kips}$$

Since $\Sigma P_L \cong \Sigma P + \Sigma(C_L P)$, the maximum load factor must be 4.49. Substituting in Eq. (46e):

$$K_i^2 = \frac{\tau I_i}{P_i} \times \pi^2\left[\frac{943 + 193}{813}\right]$$

$$= \frac{\tau I_i}{P_i} \times 13.8$$

The individual K's are found from this last expression and are listed in Table 4. The large differences from the elastic K values in Table 3 should be noted. For column **2-5**, K decreases from 1.78 to 0.98, while K for column **3-6** increases from 2.62 to 3.39.

This rather complex calculation was performed to show that the elastic K's of Table 3 are very conservative. With elastic K's, column **2-5** limits the frame's capacity to a load factor of 3.98, or 836 kips. The inelastic analysis shows that the frame becomes unstable at 943 kips, which is 13% greater. (Both of those loads would be slightly lower if axial deformations are accounted for.)

It has been shown how the method of second order analysis can be used for inelastic problems as well as for the elastic cases from which it is derived. The next example will again demonstrate the use of the inelastic approach on a practical design problem.

DESIGN EXAMPLE 3

The structure in Fig. 21 is a market shed, designed deliberately to have slender columns. Column **CD** is on an open side and is slender for architectural reasons. Column **AB** was chosen to limit drift to a maximum of 1 in., a liberal criteria but acceptable for a market shed. The problem is to check the hypothetical designer's computations.

The designer treated joints **A, C,** and **D** as perfect hinges and joint **B** as rigid. He reasoned that since the structure

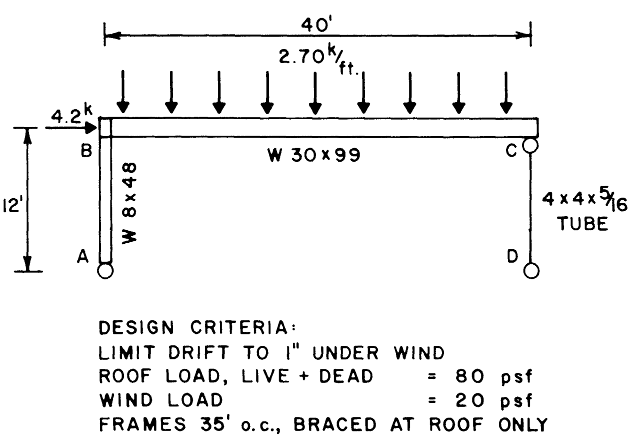


Figure 21

Table 4. Summary of Inelastic Stability Analysis with $L.F.$ = 4.49, Design Example 2

Column	$4.49P$ (kips)	f_a (ksi)	τ	τI (in.^{4})	G_A	G_B	β	$\tau\beta I$ (in.^{4})	C_L	$4.49C_LP$ (kips)	K
1-4	136.5	23.2	0.917	64.0	0.049	∞	2.93	186.5	0.206	28.2	2.54
2-5	586.4	34.3	0.181	41.0	0.018	∞	2.97	122.0	0.212	124.5	0.98
3-6	220.0	15.6	1.000	184.0	0.188	∞	2.74	504.7	0.182	40.1	3.39
	ΣP = 943							$\Sigma(\tau\beta I)$ = 813	$\Sigma(C_LP)$ = 193		

was three-hinged, it was statically determinate and therefore no gravity moment would exist in column **AB.** The design moment at joint **B** is therefore only from wind, and equals 4.2 kips × 12 ft = 50.4 kip-ft.

Because the designer was rusty in drift analysis, he analyzed his design on a computer using the STRESS program. For gravity alone, he was pleased to find no moments in column **AB,** which confirmed his initial analysis. Under the 4.2-kip wind load, he found that the structure deflected 0.90 in., which met his criteria. His check of the column **AB** follows:

W8×48:

I_x = 184 in.^{4}; S_x = 43.2 in.^{3}; A = 14.1 in.^{2};
r_x = 3.61 in.; r_y = 2.08 in.

W30×99:

I_x = 4,000 in.^{4}; S_x = 270 in.^{3}

$$G_{TOP} = \frac{184}{12} \times \left(\frac{2 \times 40}{4{,}000}\right) = 0.307$$

$$G_{BOT} = \infty$$

$$K_x = 2.10 \text{ (from nomograph)}$$

$$\frac{K_xL_x}{r_x} = \frac{2.10 \times 144}{3.61} = 83.77$$

$$\frac{K_yL_y}{r_y} = \frac{1 \times 144}{2.08} = 69.23$$

$$F_{ax} = 14.93 \text{ ksi}$$

$$F'_{ex} = 21.28 \text{ ksi}$$

$$F_b = 22.0 \text{ ksi}$$

The actual stresses used for combined gravity and wind are:

$$f_a = \frac{(40 \times 1/2 \times 2.7) + (50.4/40)}{14.1} = 3.92 \text{ ksi}$$

$$f_b = \frac{4.2 \times 12 \times 12}{43.2} = 14.00 \text{ ksi}$$

Substituting in AISC Eq. (1.6-1a), the designer found:

$$\frac{3.92 \times 0.75}{14.93} + \left[\frac{0.85}{\left(1 - \frac{3.92 \times 0.75}{21.28}\right)} \times \frac{14.00 \times 0.75}{22.0}\right]$$
$$= 0.197 + (0.986 \times 0.477) = 0.668 < 1.0$$

At this point he felt very confident, since strength was no problem and drift control had governed the design. He filed his computations for checking.

From the earlier discussion in Design Example 1, it should immediately be apparent that an error was made by the designer in determining the K-factor for column **AB.** K was taken directly from the nomograph without considering the load on "leaning" column **CD.** Less obvious, however, is the designer's failure to recognize the second order gravity moment in column **AB.**

In Part 1,[1] the subject of sidesway under gravity load was discussed in detail. The reader is referred to the equations developed there. To check the structure shown in Fig. 21, use Eq. (25) from Part 1:

$$(V_{P1})_w = \left[\left(\frac{\Delta_{po}}{H}\right)_w \times \Sigma P_w \times L.F.\right] \qquad (25)$$

The meaning of this expression can be seen from Fig. 22. If the structure sidesways under gravity load a distance $(\Delta_{po})_w$, equilibrium requires a fictitious lateral reaction of $(\Delta_{po}/H)_w \times \Sigma P$. For design at working load level, a load factor is needed to account for the non-linearity of behavior at ultimate load. The final force $(V_{P1})_w$ from Eq. (25) can be used like a real horizontal load. Its effect must be amplified by the amplification factor $A.F.$ When wind also acts, the force $)V_{P1})_w$ may be added directly to the wind force.

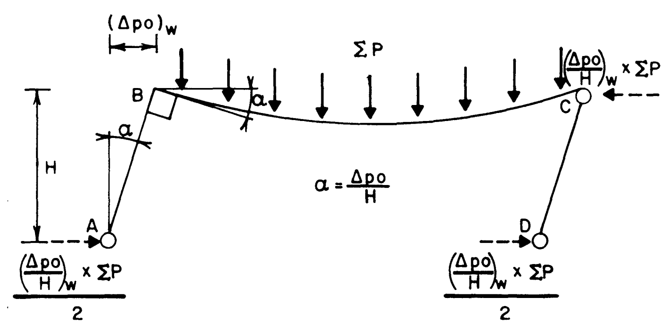


Figure 22

If the hypothetical designer had not thrown out his gravity calculations from the STRESS program, he would have noticed a horizontal deflection $(\Delta_{po})_w$ of 1.29 in. This may be calculated by observing that the slope α of column **AB** must match the end slope α of beam **BC,** which has a simple bending moment diagram with a maximum moment of $wL^2/8$. From the area of one-half the M/EI diagram for **BC:**

$$\alpha = \frac{2}{3} \times \frac{2.7 \times (40)^2 \times 12 \times 40 \times 12 \times 1/2}{8 \times 29{,}000 \times 4{,}000}$$
$$= 0.00894 = \left(\frac{\Delta_{po}}{H}\right)_w$$

From Eq. (25):

$$(V_{P1})_w = 0.00894 \times 2.7 \times 40 \times 5/4 = 1.21 \text{ kips}$$

The load factor of 5/4 is used to combine with wind. The total design lateral load is 1.21 + 4.2 = 5.41 kips.

Now, check the design using a second order analysis. Using the same G_{TOP} = 0.307 and $G_{BOT} = \infty$, Eqs. (59) and (60) give:

$$\beta = \frac{3}{1 + \frac{0.307}{2}} = 2.601$$

$$C_L = \frac{0.22}{\left(1 + \frac{0.307}{2}\right)^2} = 0.165$$

Equation (46c) may be simplified in this case to:

$$K^2 = \pi^2 \left(\frac{2 + C_L}{\beta}\right) = \pi^2 \left(\frac{2 + 0.165}{2.601}\right)$$

$$K = 2.87$$

$$\frac{K_x L_x}{r_x} = 114.3$$

$$F_{ax} = 11.08 \text{ ksi}$$

The axial stress is:

$$f_a = \frac{(40 \times 1/2 \times 2.7) - [(5.41 \times 12)/40]^*}{14.1} = 3.71 \text{ ksi}$$

The amplification factor can be computed by Eq. (45) with a load factor of 5/4.

$$P_L = \beta \frac{EI}{H^2} = 2.601 \times \left(\frac{29{,}000 \times 184}{144^2}\right) = 669 \text{ kips}$$

$$A.F. = \frac{1}{1 - \left(\frac{40 \times 2.7 \times 5/4}{669 - (0.165 \times 20 \times 2.7 \times 5/4)}\right)}$$

$$= 1.258 \text{ (1.253 without } C_L P)$$

The bending stress is:

$$f_b = \frac{5.41 \times 12 \times 12}{43.2} = 18.03 \text{ ksi}$$

Using a modified AISC Eq. (1.6-1a):

$$\frac{f_a}{F_a} + A.F. \left(\frac{f_b}{F_b}\right) \leq 1$$

$$\frac{0.75 \times 3.71}{11.08} + 1.258 \left(\frac{0.75 \times 18.03}{22.0}\right) = 1.025$$

If the preceding computations were carried out using the AISC stability load factor of 23/16 for wind force, Eq. (25) gives:

$$(V_{P1})_w = 1.21 \times 23/16 \times 4/5 = 1.39 \text{ kips}$$

Adding the wind load, the design lateral load becomes:

$$1.39 + 4.2 = 5.59 \text{ kips}$$

The axial stress becomes:

$$f_a = \frac{(40 \times 1/2 \times 2.7) - [(5.59 \times 12)/40]}{14.1} = 3.71 \text{ ksi}$$

* *The negative sign is used since wind acts to the right to combine with sidesway.*

and the bending stress becomes:

$$f_b = \frac{5.59 \times 12 \times 12}{43.2} = 18.63 \text{ ksi}$$

Using the correct K-factor of 2.87, for $K_x L_x / r_x = 114.3$, $F'_e = 11.43$ ksi. Substituting in AISC Eq. (1.6-1a):

$$\frac{0.75 \times 3.71}{11.08} + \left[\frac{0.85}{\left(1 - \frac{0.75 \times 3.71}{11.43}\right)} \times \frac{0.75 \times 18.63}{22.0}\right]$$

$$= 0.251 + (1.124 \times 0.635) = 0.965 < 1.0$$

The designer could certainly argue that this is defensible. However, he has been saved by the C_m-factor of 0.85. Without it, the amplification factor would be 1.124/0.85 = 1.322** and AISC Eq. (1.6-1a) would yield:

$$0.251 + (1.322 \times 0.635) = 1.091 > 1.0$$

The working load deflection of the structure by first order analysis is:

$$\text{Gravity } (\Delta_{po})_w = 0.00894 \times 144 = 1.29 \text{ in.}$$

$$\text{Wind } (\Delta_{ov})_w = \frac{4.2 \times 144}{669} = \frac{0.90 \text{ in.}}{2.19 \text{ in.}}$$

$$(A.F.)_w = \frac{1}{1 - \left(\frac{40 \times 2.7}{669}\right)} = 1.19$$

The true second order working load deflection is 2.19 × 1.19 = 2.61 in. Of course, the designer might have found 2.19 in. if he had added his computer solutions for wind and gravity.

Since the column is slightly overstressed and the deflection is excessive, the author believes that the structure should be redesigned.

DESIGN EXAMPLE 4

Figure 23 shows a framing plan and elevation of a 30-story apartment building designed with the staggered truss system. In this system, story deep trusses span the 60-ft width of the building. Since the trusses fully brace the columns in the N-S direction, the columns are turned with their webs in the E-W direction. The columns and spandrels together make a portal frame for wind loads in the E-W direction. The columns and spandrels between the 3rd and 4th floors are to be designed. See Fig. 23(c).

The axial gravity load on an interior column just below the 4th floor comes from 28 levels (including the roof):

$$P = 28 \times 25 \times 30 \times 0.125 = 2{,}625 \text{ kips}$$

Since the effective length for weak axis bending is 9 ft, a trial W14 column can be picked from the column tables in Ref. 2. The trial selection is W14X455. Stability and

** *The value from Eq. (45) with a load factor of 23/16 is 1.310.*

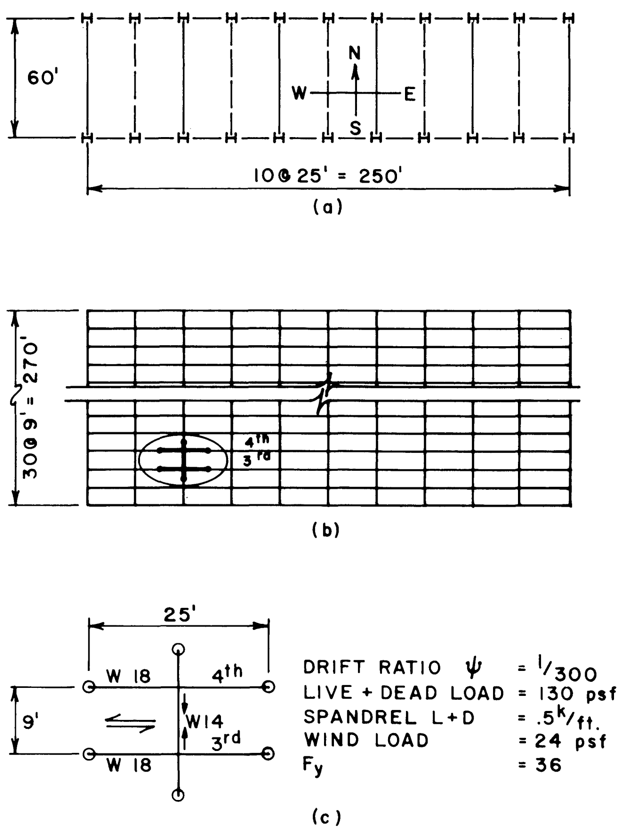


Figure 23

strength in the E-W plane will be checked after choosing a spandrel beam.

The spandrel must be adequate for strength, but must also be stiff enough to limit first order drift to 1/300 at design wind load.

The wind shear in each column between the 3rd and 4th floors is:

$$V = 9 \times 27.5 \times 60 \times \frac{1}{2} \times 0.024 \times \frac{1}{10} = 17.8 \text{ kips}$$

The wind moment in the columns is:

$$17.8 \times 9 \times \frac{1}{2} = 80.2 \text{ kip-ft}$$

which is also the average wind moment in the 3rd and 4th floor spandrels. The spandrel has a gravity moment of:

$$0.5 \times (25)^2 \times 1/12 = 26.0 \text{ kip-ft}$$

For strength, the section modulus required for wind-plus-gravity moments in the spandrel, assuming full lateral bracing, is:

$$\frac{0.75(80.2 + 26.0) \times 12}{24} = 39.8 \text{ in.}^3$$

This could be provided by a W14X30. However, a spandrel will also be selected for drift control.

$$\frac{\Delta_{ov}}{H} = \frac{V}{P_L}$$

$$P_L = \frac{17.8}{1/300} = 5{,}346 \text{ kips}$$

$$I \text{ of W14X455} = 7{,}220 \text{ in.}^4$$

From Eq. (34):

$$\beta = \frac{P_L H^2}{EI} = \frac{5{,}346 \times (9 \times 12)^2}{29{,}000 \times 7{,}220} = 0.298$$

From Eq. (57), for symmetrical G's:

$$G_S = \left(\frac{12}{\beta}\right) - 1 = \left(\frac{12}{0.298}\right) - 1 = 39.3$$

Since $G_S = \dfrac{I_c}{H} \times \dfrac{L}{I_g}$:

$$I_g = \frac{I_c L}{G_S H} = \frac{7{,}220 \times 25}{39.3 \times 9} = 510 \text{ in.}^4$$

From the AISC Manual,[2] the most economical member with this moment of inertia is a W18X35, with $I_x = 513$ in.^{4} and $S_x = 57.9$ in.^{3}.

With a spandrel selected, the stability of the E-W frame can now be checked. To do this, it must be recognized that the columns will be inelastic at the full gravity load times the load factor. A procedure first suggested by Yura[5,6] will be followed:

Step 1: Find axial working stress f_a in the column:

$$f_a = \frac{2{,}625}{134} = 19.59 \text{ ksi}$$

Step 2: Find the KL/r for f_a:

From Table 1-36, Ref. 2:

For $f_a = 19.59$, $KL/r = 34.86$

Step 3: Find the load factor $L.F.$ from KL/r and C_c:

From denominator of Eq. (1.5-1), Ref. 2:

With $C_c = 126.1$ and $KL/r = 34.86$, $L.F. = 1.768$

Step 4: Find the ultimate stress $f'_a = f_a \times L.F.$:

$$f'_a = 1.768 \times 19.59 = 34.63 \text{ ksi}$$

Step 5: Find:

$$\tau = 4\left(1 - \frac{f'_a}{F_y}\right)\frac{f'_a}{F_y}$$

$$= 4\left(1 - \frac{34.63}{36}\right)\frac{34.63}{36} = 0.147$$

Step 6: Find:

$$G_S = \frac{\tau I_c L}{H I_g}$$

$$= \frac{0.147 \times 7{,}220 \times 25}{9 \times 513} = 5.73$$

Step 7: Find K from the nomograph or from Eq. (36):

$$\beta = \frac{12}{1 + 5.73} = 1.783 \qquad \text{Eq. (58)}$$

$$C_L \cong \frac{0.22}{(1 + 5.73)^2} = 0.005 \qquad \text{Eq. (59)}$$

Rearranging Eq. (36):

$$K = \sqrt{\frac{\pi^2(1 + C_L)}{\beta}}$$

$$= \sqrt{\frac{\pi^2(1.005)}{1.783}} = 2.36$$

Step 8: Find K_xL_x/r_x; if less than required in Step 2, column is satisfactory. If not, increase spandrel stiffness.

$$\frac{K_xL_x}{r_x} = \frac{2.36 \times 9 \times 12}{7.35} = 34.66$$

Since the value in Step 8 is less than the minimum value required in Step 2, the column is satisfactory and the allowable stress $F_a \geq 19.59$ ksi.

To check the column by AISC Eq. (1.6-1a), the elastic K should be used in determining the stress F'_e.

$$\text{Elastic } G_S = 7{,}220 \times 25/9 \times 513 = 39.1$$

$$\beta = 0.299; \quad C_L = 0.0001; \quad K = 5.74$$

$$K_xL_x/r_x = 5.74 \times 9 \times 12/7.35 = 84.4$$

$$F'_e = 20.97 \text{ ksi}$$

Checking the column for wind plus gravity, the wind stress in bending is:

$$f_b = 80.2 \times 12/758 = 1.27 \text{ ksi}$$

and the allowable bending stress is 24 ksi. Substituting in AISC Eq. (1.6-1a):

$$\frac{0.75 \times 19.59}{19.59} + \left[\frac{0.85}{\left(1 - \frac{0.75 \times 19.59}{20.97}\right)} \times \frac{1.27 \times 0.75}{24}\right]$$

$$= 0.750 + (2.84 \times 0.040) = 0.863 < 1.0$$

The column is satisfactory and according to usual practice the design is complete, since the strength of both column and spandrel have been checked and the frame stiffness has been controlled.

Unfortunately, usual practice has traditionally overlooked the amplification of girder moments. In the column just checked, the AISC amplification procedure caused an increase in moment of 2.84 times. Logic requires a similar increase in the girder moment at the same joint.

Use Eq. (45) to compute the amplification factor with a load factor of 5/4. The C_LP term may be ignored since, at 0.0001, C_L is trivial. For each column:

$$P_L = \frac{\beta EI}{H^2} = \frac{0.299 \times 29{,}000 \times 7{,}220}{(9 \times 12)^2} = 5{,}373 \text{ kips}$$

$$A.F. = \frac{1}{\left(1 - \frac{2{,}625 \times 5/4}{5{,}373}\right)} = 2.57$$

The true girder moment for design should be the gravity moment plus amplified wind moment:

$$0.75[(2.57 \times 80.2) + 26.0] = 174 \text{ kip-ft}$$

Even at working load level the amplification factor is 1.96, giving a working stress of:

$$f_b = \frac{[(1.96 \times 80.2) + 26.0]12}{57.9} = 38.0 \text{ ksi}$$

This means that a plastic hinge begins to form at working loads. Since the plastic capacity of a W18X35 is 200 kip-ft, the non-proportional load factor for wind is:

$$L.F. = \frac{200 - 26.0}{1.96 \times 80.2} = 1.11$$

and the proportional load factor, obtained by iteration, is 1.05. These are load factors against collapse, since the stiffness is halved after formation of plastic hinges at one end of the spandrels. At half-stiffness, the amplification factor equals ∞!

Clearly the girder size must increase. A W18X40 is also unsatisfactory. For a W18X45, P_L becomes 7,325, $A.F.$ becomes 1.81, and the actual bending stress $f_b = 26.0$ ksi, which is satisfactory for wind.

Using the W18X45, the first order drift ratio at working load can be found from:

$$\frac{\Delta_{ov}}{H} = \frac{V}{P_L} = \frac{17.8}{7{,}325} = 0.00243$$

But the true second order drift ratio, remembering that C_LP is small, is:

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P} = \frac{17.8}{7{,}325 - 2{,}625} = 0.00379$$

Since this still exceeds the original criteria, it is necessary to use a W18X50, with $P_L = 8{,}283$, and

$$\frac{\Delta_{pv}}{H} = \frac{17.8}{8{,}283 - 2{,}625} = 0.00315$$

The final size is substantially greater than the W18X35 determined by traditional procedures.

This example demonstrates several important points:

1. Column design under the 1969 AISC Specification is conservative. The AISC amplification factor of 2.84 is greater than 2.57 because of the generous AISC load factor.
2. The design of girders must also account for second order moment amplification. In this example, which is a practical case, ignoring the second order effects is courting danger.
3. Control of drift does not necessarily prevent large second order effects.

DRIFT CONTROL AND AMPLIFICATION

The previous example demonstrated that design for drift does not insure stability. The dimensions of the stability problem can be derived from Fig. 24. This shows a regular rectangular building of length L in the direction of the wind. Drawn within the building is a prism of space which is 1 ft^2 on the windward face and L long. The average wind force in lbs/sq ft is w, and ψ_1 is the *designed* first order drift ratio. The P_L for the prism can be found from

$$P_L = \frac{VH}{\Delta_{ov}}$$

Substituting,

$$P_L = \frac{wH}{\psi_1 H} = \frac{w}{\psi_1} \text{ psf}$$

The weight of the prism is the average density of the building, γ , in lbs/cu ft, times the length L. For the whole prism, $P = \gamma L$ psf. Using Eq. (15), the amplification factor is:

$$A.F. = \frac{1}{1 - \dfrac{P}{P_L}} = \frac{1}{1 - \dfrac{\gamma L \psi_1}{w}}$$

or, with a load factor of 5/4,

$$A.F. = \frac{1}{1 - \left(\dfrac{\gamma L \psi_1}{w} \times \dfrac{5}{4}\right)} \tag{62}$$

The true second order drift, ψ_2 , can be obtained at working loads from Eq. (12):

$$\frac{\Delta_{pv}}{H} = \psi_2 = \frac{V}{P_L - P}$$

$$\psi_2 = \frac{w}{\dfrac{w}{\psi_1} - \gamma L} \tag{63}$$

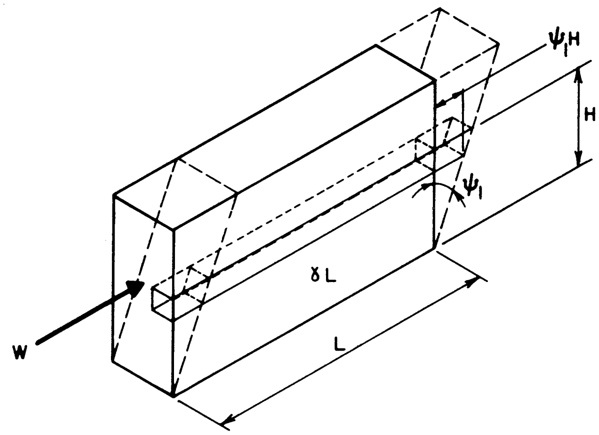
If the designer wishes to limit the true second order drift, ψ_2 , the first order drift, ψ_1 , must be limited to:

$$\psi_1 = \frac{w}{\dfrac{w}{\psi_2} + \gamma L} \tag{64}$$

These ideas could have been applied from the beginning to Example 4 and greatly simplified the design procedure. From $L = 250$ ft, $w = 24$ psf, and $\gamma = 125/9 = 13.9$ pcf. To achieve a true drift ratio of 1/300 at working loads, use Eq. (64):

$$\psi_1 = \frac{24}{\dfrac{24}{1/300} + (13.9 \times 250)} = 0.00225$$

If the structure is designed by usual procedures to limit the drift ratio to 0.00225, the amplification factor for strength from Eq. (62) will be:



W = WIND, psf
γ = BUILDING DENSITY, pcf
ψ_1 = DESIGNED DRIFT RATIO

Figure 24

$$A.F. = \frac{1}{1 - \left(\dfrac{13.9 \times 250 \times 0.00225}{24} \times \dfrac{5}{4}\right)}$$

$$= 1.69$$

If the design were carried out using a modified wind force in the E-W direction of $1.69 \times 24 = 40.5$ psf, the drift ratio ψ_1 could be adjusted to $1.69 \times 0.00225 = 0.0038$ for use with the increased wind. In this way the design could proceed in the usual manner with first order computations. The amplification term for sidesway moment in AISC Eq. (1.6-1a) would then be superfluous.

This discussion has assumed that C_L and/or its effect is small. The author recommends that the $\Sigma(C_L P)$ term be omitted in calculating the amplification factor whenever $A.F. \leq 1.5$. The error, when compared to exact elastic solutions, will never exceed 4%. C_L should, however, always be used in determining K-factors, unless C_L is less than 0.01, which will be true when $G_S > 4$ for symmetrically restrained columns.

SUMMARY

This paper has developed a method for analyzing one-story rigid frames and multistory frames which are reasonably regular from floor to floor. Three essential frame problems were dealt with.

1. The amplification of the effects of lateral forces on a story may be found from Eq. (45):

$$A.F. = \frac{1}{1 - \dfrac{\Sigma P_w \times L.F.}{\Sigma P_L - \Sigma(C_L P_w \times L.F.)}} \tag{45}$$

Whenever the result is less than 1.5, the simpler Eq. (18) from Part 1 is recommended:

$$A.F. = \frac{1}{1 - \dfrac{\Sigma P_w \times L.F.}{\Sigma P_L}} \tag{18}$$

A special form of this equation was developed in Design Example 4 for design of regular high-rise buildings:

$$A.F. = \frac{1}{1 - \dfrac{\gamma L \psi_1 \times L.F.}{w}} \tag{62}$$

2. The true drift ratio or rotational displacement of a single-story structure, neglecting axial stresses, can be found from Eq. (43):

$$\text{Drift} = \frac{\Delta_{pv}}{H} = \frac{\Sigma V_w}{\Sigma P_L - \Sigma P_w - \Sigma(C_L P)_w} \tag{43}$$

This may be simplified to Eq. (16) from Part 1 when $A.F.$ is less than 1.5 or C_L's are small:

$$\text{Drift} = \frac{\Delta_{pv}}{H} = \frac{\Sigma V_w}{\Sigma P_L - \Sigma P_w} \tag{16}$$

A special form of this last equation, for regular high-rise buildings, gives the true second order drift:

$$\psi_2 = \frac{w}{\dfrac{w}{\psi_1} - \gamma L} \tag{63}$$

In this case, ψ_1, the first order drift, should include the effect of axial deformations for tall, narrow buildings.

3. Accurate elastic K-factors for any column in an irregular single story structure may be found from Eq. (46c):

$$K_i^2 = \frac{I_i}{P_i} \times \pi^2 \left[\frac{\Sigma P + \Sigma(C_L P)}{\Sigma(\beta I)}\right] \tag{46c}$$

The use of this equation may be extended to find *inelastic* K-factors, as shown in Design Example 2, by:

$$K_i^2 = \frac{\tau I_i}{P_i} \times \pi^2 \left[\frac{\Sigma P + \Sigma(C_L P)}{\Sigma(\tau \beta I)}\right]$$

with the β's and C_L's based on τI for each column. The C_L-factor is essential for accurate results in these K-equations. β and C_L may be readily found from Eqs. (57) through (60) for symmetrical or singly pin ended columns. Figure 18 may be used to find C_L in irregular cases.*

The reader may wonder how the designation C_L originated. In the author's view, C_L is a "clarification" factor. It has been shown to have a physical meaning. More importantly, the author hopes that C_L will clear up some of the mystery surrounding the subject of frame stability.

* *The author has found that Eqs. (36), (55), and (56) may be readily programmed for a 224 step card programmable pocket calculator.*

ACKNOWLEDGMENTS

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Extraordinary thanks are also due to D. Fraser Sinclair, an engineer in the writer's office, who has a special flair for computer graphics. He suggested the construction of Fig. 18 and programmed it for an automatic plotter. As a bonus he prepared Fig. 25, which is a three-dimensional view of the C_L landscape.

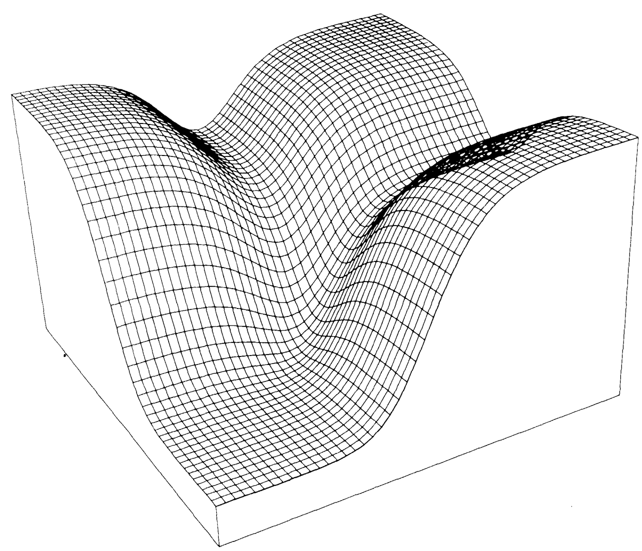

Figure 25

APPENDIX A

NOMENCLATURE

A	=	Area of a member
$A.F.$	=	Amplification factor. The ratio of second order internal forces to first order internal forces resulting from lateral loads
C_L	=	Stiffness reduction factor for a column
E	=	Modulus of elasticity
G_A	=	Joint stiffness factor
G_B	=	Joint stiffness factor
G_S	=	Joint stiffness factor
H	=	Height of a column or story
I	=	Moment of inertia
K	=	Effective length factor
L	=	Length of a girder
$L.F.$	=	Load factor, taken as 5/3 for gravity alone or 5/4 for wind plus gravity

P = Vertical load on a column
P_{cr} = Elastic buckling load
P_e = Euler load $= \dfrac{\pi^2 EI}{(KH)^2}$
P_L = Force which produces unit rotational displacement (unit "drift") of a member or subsystem
V = Horizontal load
V_{P1} = $\dfrac{\Delta_{po}}{H} \times \Sigma P$
w = A subscript meaning working load; also, wind force in pounds per square foot
α = A factor variously defined in the text
β = Column first order stiffness factor
Δ_{ov} = First order deflection from V or ΣV acting alone
Δ_{po} = First order lateral deflection from P or ΣP acting alone
Δ_{pp} = Second order lateral deflection, from P or ΣP acting alone
Δ_{pv} = Second order lateral deflection from both V and P or ΣV and ΣP acting together
γ = Load position factor; also, building density in pounds per cubic foot
ψ_1 = First order drift ratio
ψ_2 = Second order drift ratio
ΣP = Algebraic total of the vertical loads on a story
ΣP_L = Total of P_L's for a story
ΣP_w = ΣP at working load level
ΣV = Total of all horizontal loads on a story
ΣV_w = ΣV at working load level
τ = Ratio of tangent modulus to E

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