A Practical Method of Second Order Analysis

Part 1—Pin Jointed Systems

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MOST DESIGNERS were trained in methods of structural analysis which assume that the change in shape of a structural system due to loading does not significantly affect the magnitude of the internal forces of the system. Except for consideration of local effects such as buckling of individual columns or beams, knowing the absolute stiffness of members in relation to loads has generally been unnecessary for determining the internal forces of a system.

In 1961 the curtain was lifted on a new era by a revised AISC Specification which included the expression

$$\frac{C_m}{\left(1 - \frac{f_a}{F'_e}\right)}$$

in the bending term of its interaction equation. This expression may be written in terms of forces as

$$\frac{C_m}{\left(1 - \frac{P_w \times L.F.}{P_e}\right)}$$

It has the two essential characteristics of second order analysis. The ratio P_w / P_e measures the relation of the actual working load to the absolute stiffness of an element which is part of a system. P_e depends on EI_c / H of a column together with EI_g / L of the girders at each end. The second characteristic feature of the expression is the inclusion of a load factor, *L.F.* This is necessary because the relationship between load and stiffness at ultimate strength is needed as a measure of ultimate forces. The ratio changes with load, and must be measured at either increased load or reduced stiffness to give results at working load levels which are proportional to ultimate loads.

Since 1961 there has been growing discussion of the problem of second order analysis under various names such as $P\Delta$ effect, moment amplification, and non-linear analysis. The purpose of this paper is to clarify this problem and to develop a practical but accurate method of second order analysis suitable for design office use.

Specifically, in this paper, "second order analysis" means a determination of internal forces in a structural system taking into account the actual deformations of the structure, provided that the deformations are small. If the actual deformations of a member do not exceed 1% of its length at working loads, this restriction will be met. The method developed here is general, but will be limited in this paper to building structures of structural steel. However, it is applicable to all elastic materials and to materials whose inelasticity is a straightforward function of load or time. A system of notation has been developed as part of the method and is summarized in Appendix A.

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DEFINITIONS

A central idea used throughout this and following Parts of this paper is a measure of absolute structural stiffness which is defined in units of force. This is the parameter P_L, defined as the force applied at the end of a member or subsystem that will produce a unit rotational displacement of the member or subsystem. Rotational displacement is defined as relative displacement of the ends of an initially straight member perpendicular to its axis divided by the length of the member. Using a simple structure as illustration, the cantilever column of Fig. 1(a), of height H and

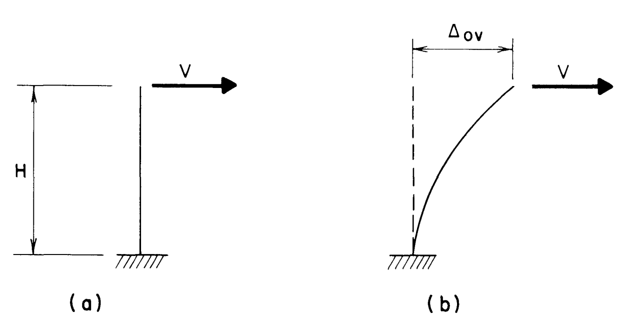

Figure 1

loaded with force V, deforms as shown in Fig. 1(b) to develop the deflection Δ_{ov}. The subscript ov indicates that the axial load is zero and V is the only load applied. Unit rotational displacement is reached when $\Delta_{ov}/H = 1$. Therefore, the value of V which makes $\Delta_{ov}/H = 1$ is P_L. It follows that

$$P_L = \frac{VH}{\Delta_{ov}} \tag{1}$$

In general, for Fig. 1, the deflection is

$$\Delta_{ov} = \frac{VH^3}{3EI} + \frac{VH}{A_v G} \tag{2}$$

where I is the moment of inertia, A_v is the shear area, and E and G are the moduli of elasticity and rigidity, respectively. Rewriting Eq. (2),

$$\frac{VH}{\Delta_{ov}} = \frac{1}{\dfrac{H^2}{3EI} + \dfrac{1}{A_v G}} = P_L \tag{3}$$

For many structures the shear term $1/A_vG$ is small in relation to $H^2/3EI$, and Eq. (3) may therefore be simplified to

$$P_L = \frac{3EI}{H^2} \tag{4}$$

in effect ignoring local shear strains.

Hereafter, local shear strains in the derivations are ignored unless otherwise noted. They may easily be included where necessary, since the definition of P_L is entirely general.

PIN JOINTED SYSTEMS

To develop the method of second order analysis, a simple pin jointed structure as shown in Fig. 2(a) will be analyzed. This structure must resist both the vertical load P and the horizontal load V. Since column **DE** has pinned ends, it cannot resist the load V, and therefore its stability is dependent upon brace member **AB.** The subsystem **ABC,** Fig. 2(b), provides the resistance to lateral load and is the stability subsystem of the whole structure.

The value of P_L for subsystem **ABC** will now be determined. Using the method of virtual work, the deflection of point **B** horizontally in the direction of a unit force of 1 kip is given by

$$\frac{\left(\dfrac{H}{W}\right)^2 H}{A_H E} + \frac{\left(\dfrac{D}{W}\right)^2 D}{A_D E} = \Delta_{ov},\ V = 1 \tag{5}$$

Letting $\lambda = DA_H/A_DH$ and substituting, Eq. (5) may be rewritten as

$$\frac{H}{A_H E}\left[\left(\frac{H}{W}\right)^2 + \lambda\left(\frac{D}{W}\right)^2\right] = \Delta_{ov} \tag{6}$$

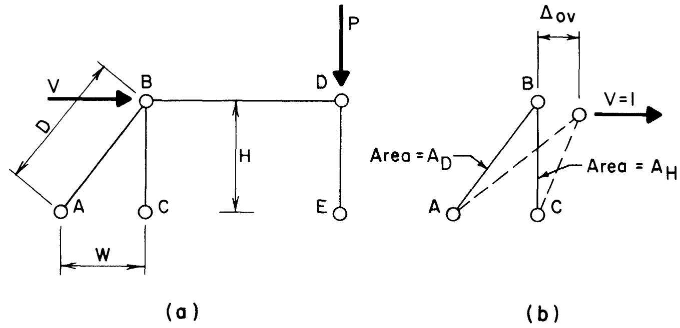

Figure 2

and, since $D^2 = H^2 + W^2$,

$$\frac{H}{A_H E}\left[(1+\lambda)\left(\frac{H}{W}\right)^2 + \lambda\right] = \Delta_{ov} \tag{7}$$

Finally, from Eq. (1), for $V = 1$,

$$P_L = \frac{A_H E}{\left[(1+\lambda)\left(\dfrac{H}{W}\right)^2 + \lambda\right]} \tag{8}$$

Next, the structure in Fig. 2 will be analyzed for combined loads V and P. Figure 3(a) shows the deflected shape under both loads P and V. The structure has deflected horizontally a distance Δ_{pv} and as a result has developed a horizontal reaction at **E** of $P(\Delta_{pv}/H)$. Figure 3(b) shows the forces acting on system **ABC.** Rewriting Eq. (1) as

$$\frac{\Delta_{ov}}{H} = \frac{V}{P_L} \tag{9}$$

and applying it to Fig. 3(b):

$$\frac{\Delta_{pv}}{H} = \frac{V + P\dfrac{\Delta_{pv}}{H}}{P_L} \tag{10}$$

Rearranging Eq. (10):

$$\frac{\Delta_{pv}}{H} P_L - P\frac{\Delta_{pv}}{H} = V \tag{11}$$

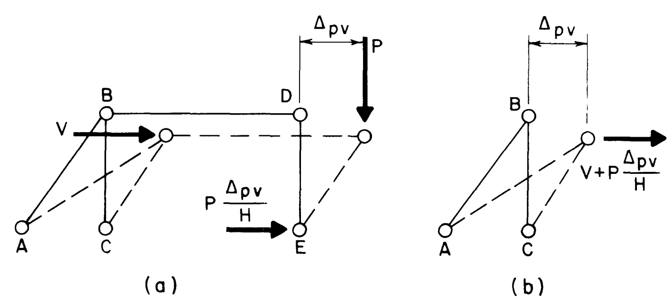

Figure 3

Solving Eq. (11) for Δ_{pv}/H:

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P} \tag{12}$$

Equation (12) is of considerable importance, since it shows how to calculate the true rotational displacement (or "drift") of the structure in Fig. 2. If it is compared to Eq. (9), which calculates the rotational displacement under load V only, it is seen that the stiffness P_L is reduced by the presence of load P. As P is increased, holding V constant, the rotational displacement Δ_{pv}/H will increase. This is the second order effect.

Equation (12) shows that as P approaches the magnitude of P_L, the rotational displacement Δ_{pv}/H will grow extremely large. When $P = P_L$, the rotational displacement will become infinite at any value of V. This means that the structure becomes unstable when $P = P_L$ and, therefore, that P_L is the maximum vertical load that the structure can support. If the load at which the structure becomes unstable is P_{cr}, it follows that

$$P_{cr} = P_L \tag{13}$$

If only force V acts, it can be determined from Fig. 2(b) that the force in member **BC** equals VH/W. From Fig. 3(b) it can be seen that the total horizontal force applied at joint **B** increases to $V + P(\Delta_{pv}/H)$ when both V and P act. The force in member **BC** will become

$$\frac{\left(V + P\frac{\Delta_{pv}}{H}\right)H}{W}$$

Dividing this force by the force resulting from V only,

$$\frac{\left(V + P\frac{\Delta_{pv}}{H}\right)H}{VH} = 1 + \frac{P\Delta_{pv}}{VH} \tag{14}$$

The expression $1 + (P\Delta_{pv}/VH)$, which is the ratio of the second order force based on both P and V to the first order force based on V only, will be called the amplification factor $A.F.$ Substituting Eq. (12) into Eq. (14),

$$A.F. = 1 + \frac{PV}{V(P_L - P)}$$

which reduces to

$$A.F. = \frac{1}{1 - \frac{P}{P_L}} \tag{15}$$

Three important points now have been established. For a structure such as that shown in Fig. 2, the rotational displacement is

$$\frac{\Delta_{pv}}{H} = \frac{V}{P_L - P} \tag{12}$$

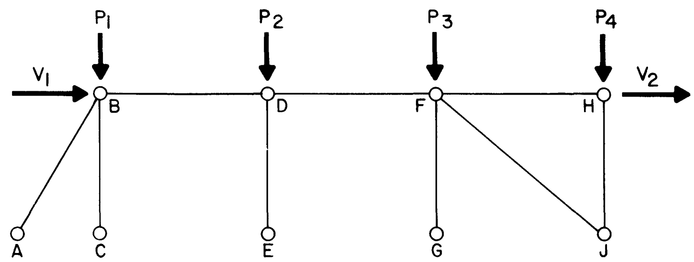

Figure 4

the maximum value of P is

$$P_{cr} = P_L \tag{13}$$

and the ratio of second order forces to first order forces is

$$A.F. = \frac{1}{1 - \frac{P}{P_L}} \tag{15}$$

To design the structure of Fig. 2 with a working load of P_w, Eqs. (13) and (15) must be revised to include a load factor $L.F.$:

$$P_w \leq \frac{P_L}{L.F.} \tag{13a}$$

$$A.F. = \frac{1}{1 - \frac{P_w \times L.F.}{P_L}} \tag{15a}$$

where P_L would be defined by Eq. (8).

The principles developed above apply equally well to the structure shown in Fig. 4, which has four loads, P_1 through P_4, and two sources of lateral stiffness, namely subsystems **ABC** and **GFJ**. Expanding Eqs. (12), (13a), and (15a),

$$\left(\frac{\Delta_{pv}}{H}\right)_w = \frac{\Sigma V_w}{\Sigma P_L - \Sigma P_w} \tag{16}$$

$$\Sigma P_w = \frac{\Sigma P_L}{L.F.} \tag{17}$$

$$A.F. = \frac{1}{1 - \frac{\Sigma P_w \times L.F.}{\Sigma P_L}} \tag{18}$$

In these equations, the term Σ denotes summation of all sources of stiffness in ΣP_L and summation of all horizontal and vertical loads in ΣV_w and ΣP_w. These equations are perfectly general and are exact when applied to a pin jointed structure such as that shown in Fig. 4. The subscript w denotes working load.

Linking elements such as **BD, DF,** and **FH** are normally treated as rigid in most building structures. In this paper members which are part of a floor or roof diaphragm will be assumed to be axially rigid unless otherwise stated.

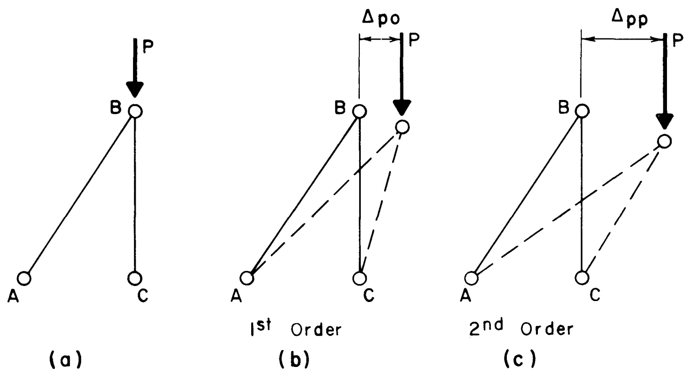

Figure 5

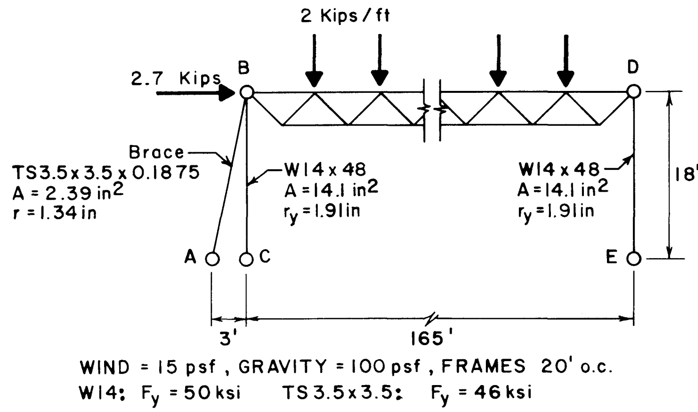

Figure 6

PIN-JOINTED SYSTEMS WITH SIDESWAY

A special case of second order analysis applies to structures which have sidesway under gravity loads only. Lateral deflection will occur to some degree on unsymmetrical structures and on unsymmetrically loaded symmetrical structures. Figure 5(a) shows a simple structure **ABC** loaded with force P. Ordinary analysis would show that the force in member **BC** would be exactly equal to P and that there would be no force in member **AB.** But even ordinary analysis would show a horizontal deflection Δ_{po} of point **B.** Since **BC** obviously shortens, some rotational displacement Δ_{po}/H of member **BC** must take place for compatibility with the length of member **AB** and the shortened length of **BC.** The lateral displacement Δ_{po} may be found by the method of virtual work:

$$\Delta_{po} = \frac{P\left(\frac{H}{W}\right)H}{A_H E}$$

or

$$\frac{\Delta_{po}}{H} = \frac{P\left(\frac{H}{W}\right)}{A_H E} \qquad (19)$$

The force P acting on the displaced member **AB** will produce horizontal reactions at the ends of the member which will produce further rotational displacement to a final equilibrium rotation of Δ_{pp}/H. (The subscript po in Δ_{po} signifies first order deflection. The subscript pp in Δ_{pp} signifies that the compounding second order effect of P acting on the deformed structure is included.)

In the equilibrium state the horizontal force at point **B** is $P(\Delta_{pp}/H)$ and the part of the rotation produced by $P(\Delta_{pp}/H)$ is, from Eq. (9),

$$\frac{P(\Delta_{pp}/H)}{P_L}$$

The total rotational displacement is

$$\frac{\Delta_{pp}}{H} = \frac{\Delta_{po}}{H} + \frac{P(\Delta_{pp}/H)}{P_L} \qquad (20)$$

Rearranging,

$$\frac{\Delta_{pp}}{H}\left[1 - \frac{P}{P_L}\right] = \frac{\Delta_{po}}{H}$$

$$\frac{\Delta_{pp}}{H} = \frac{\Delta_{po}}{H}\left[\frac{1}{1 - \frac{P}{P_L}}\right]$$

Multiplying both sides by P,

$$P\frac{\Delta_{pp}}{H} = \frac{\Delta_{po}}{H}\left[\frac{P}{1 - \frac{P}{P_L}}\right] = V_{P2} \qquad (21)$$

where V_{P2} is defined as the induced second order shear force resulting from first order sidesway Δ_{po}/H under load P. If the structure **ABC** were used as part of a larger structure, such as that shown in Fig. 6, with multiple "leaning" columns braced by **ABC**, the final form of Eq. (21) becomes

$$V_{P2} = \frac{\Delta_{po}}{H}\left[\frac{\Sigma P}{1 - \frac{\Sigma P}{\Sigma P_L}}\right] \qquad (22)$$

or, with load factors applied to working loads,

$$(V_{P2})_w = \left(\frac{\Delta_{po}}{H}\right)_w\left[\frac{\Sigma P_w \times L.F.}{1 - \frac{\Sigma P_w \times L.F.}{\Sigma P_L}}\right] \qquad (23)$$

This may also be written as

$$(V_{P2})_w = \left[\left(\frac{\Delta_{po}}{H}\right)_w \times \Sigma P_w \times L.F.\right] \times A.F. \quad (24)$$

where $A.F.$ is given by Eq. (18). The bracketed term in Eq. (24) is the first order shear force, $(V_{P1})_w$:

$$(V_{P1})_w = \left[\left(\frac{\Delta_{po}}{H}\right)_w \times \Sigma P_w \times L.F.\right] \qquad (25)$$

Equation (23) may be used to find the effective shear $(V_{P2})_w$ from ΣP_w acting on a structure with any initial deformation Δ_i/H. An example would be a structure built out-of-plumb by an amount Δ_i/H. In this case,

$$(V_{P2})_w = \frac{\Delta_i}{H}\left[\frac{\Sigma P_w \times L.F.}{1 - \dfrac{\Sigma P_w \times L.F.}{\Sigma P_L}}\right] \tag{26}$$

Numerical Example of a Pin Jointed System—Figure 6 shows a long span roof structure consisting of 165-ft long trusses at 20 ft o.c., carrying 100 psf total load. The structure is 18 ft high and must resist a wind load of 15 psf. The column loads are 1/2 × 165 ft × 2 kips/ft = 165 kips. W14×48 is the chosen column size in steel with F_y = 50 ksi. A brace is provided at one side; for architectural reasons it is made from a square tube 3½ × 3½ × 3/16. This brace is intended to act in both compression and tension. The allowable compressive load on this member [AISC Formula (1.5-2)] is 13.4 kips × 4/3 = 17.8 kips for wind. The ultimate load is 13.4 × 23/12 = 25.6 kips. The brace is also connected to a foundation capable of resisting an ultimate tension of 25.6 kips. A first order analysis of the structure will be made initially.

The structure is geometrically stable and statically determinate. Therefore the gravity load does not produce a force in member **AB.** The wind load in **AB** is

$$2.7 \times \left(\frac{\sqrt{18^2 + 3^2}}{3}\right) = 16.4 \text{ kips} < 17.8 \text{ kips}$$

Therefore, member **AB** is acceptable for strength.

To complete the analysis a responsible designer would check lateral drift under the wind load. Using Eqs. (8) and (9) [Eq. (9) is a first order equation] with the same dimensional nomenclature as in Fig. 2,

$$W = 3.00 \text{ ft} \qquad H = 18.00 \text{ ft}$$

$$D = \sqrt{18^2 + 3^2} = 18.25 \text{ ft}$$

$$A_H = 14.1 \text{ in.}^2 \qquad A_D = 2.39 \text{ in.}^2$$

$$\lambda = \frac{18.25 \times 14.1}{2.39 \times 18} = 5.98$$

$$P_L = \frac{A_H E}{\left[(1+\lambda)\left(\dfrac{H}{W}\right)^2 + \lambda\right]} \tag{8}$$

$$= \frac{14.1 \times 29{,}000}{\left[6.98 \times \left(\dfrac{18}{3}\right)^2 + 5.98\right]} = 1{,}589 \text{ kips}$$

$$\frac{\Delta_{ov}}{H} = \frac{V}{P_L} \tag{9}$$

$$= \frac{2.7}{1589} = 0.00170$$

The drift ratio of 0.00170 would be considered conservative by most designers. The first order horizontal deflection is 0.00170 × 18 × 12 = 0.367 in., which is modest for a height of 18 ft.

Now, consider the methods of second order analysis that have been developed. Equations (16), (17), and (18) will be used. The total working load is

$$\Sigma P_w = 165 \times 2 = 330 \text{ kips}$$

The second order wind drift ratio in the presence of ΣV = 2.7 kips and ΣP_w = 330 is

$$\left(\frac{\Delta_{pv}}{H}\right)_w = \frac{\Sigma V_w}{\Sigma P_L - \Sigma P_w} \tag{16}$$

$$= \frac{2.7}{1589 - 330} = 0.00214$$

which is still moderate.

Rearranging Eq. (17),

$$L.F. = \frac{\Sigma P_L}{\Sigma P_w} = \frac{1589}{330} = 4.82$$

The load factor against buckling is entirely adequate.

For Eq. (18), a wind load factor of 1.3 will be used, in accordance with Sect. 2.1 of the AISC Specification.

$$A.F. = \frac{1}{1 - \left(\dfrac{330 \times 1.3}{1589}\right)} = 1.370$$

The true second order wind force for design is 1.370 × 16.4 = 22.50 kips. Since 22.50 > 17.8, member **AB** appears to be unsatisfactory in compression. However, a subsequent check for sidesway under gravity load alone will show that the compressive strength of **AB** is actually adequate.

Before the analysis can be considered complete, the gravity loaded structure must be examined for sidesway, since it is unsymmetrical. Applying Eq. (19) to **BC:**

$$\left(\frac{\Delta_{po}}{H}\right)_w = \frac{165 \times \dfrac{18}{3}}{14.1 \times 29{,}000} = 0.00242^*$$

* *It is possible for temperature change to also produce second order effects, even in statically determinate structures. In the structure of Fig. 6, even if member BD is rigid (i.e., is part of a roof diaphragm with a very large effective area), if the temperature rises 50°F and the coefficient of expansion is 0.00065/100°, member DE will develop an initial rotational displacement without load of*

$$\frac{0.5 \times .00065 \times 165 \times 12}{216} = 0.00298.$$

Since both BC and DE have the same loads, P_w, and the same rotational displacement of $(\Delta_{po})_w$ = 0.00242, the weighted average is

$$\left[\frac{2 \times 0.00242 + 0.00298}{2}\right] = \left(\frac{\Delta_{po}}{H}\right)_{w+t} = 0.00391$$

Selection of the appropriate load factor for this situation is left to the reader.

This drift ratio from sidesway under gravity load alone exceeds that from wind. The resulting second order force $(V_{P2})_w$ on subsystem **ABC** is given by Eq. (23), using a load factor of 1.3 (to combine with wind).

$$(V_{P2})_w = 0.00242 \left[\frac{330 \times 1.3}{1 - \left(\frac{330 \times 1.3}{1589} \right)} \right] = 1.42 \text{ kips}$$

Force $(V_{P2})_w$ will produce a tension force in member **AB** of

$$1.42 \times \left(\frac{\sqrt{18^2 + 3^2}}{3} \right) = 8.65 \text{ kips}$$

This tension force adds to the wind force, giving:

$$22.50 + 8.65 = 31.15 \text{ kips (tension)}$$

$$-22.50 + 8.65 = 13.85 \text{ kips (compression)}$$

Since 31.15 kips is a design load, the ultimate uplift capacity of the foundation, previously computed as 25.6 kips, must be increased.

The sidesway force will also increase member forces for gravity alone. This increase should be calculated with a higher load factor, such as 1.7. Recomputing,

$$(V_{P2})_w = 0.00242 \left[\frac{330 \times 1.70}{1 - \frac{330 \times 1.70}{1589}} \right] = 2.10 \text{ kips}$$

This force will produce 2.10 × (18/3) = 12.6 kips in member **BC**. Adding this to the first order load of 165 kips in **BC** gives the second order load of 177.6 kips. The AISC capacity of the W14×48 is 165 kips. However, the failure load is (23/12) × 165 = 316 kips. Therefore, the member load factor on the second order load is 316/177.6 = 1.78.*

The example of Fig. 6 illustrates the important aspects of second order analysis. It has been seen that the tension force in the brace is increased from 16.4 kips to 31.2 kips when sidesway and wind forces are combined with factored gravity loads. It is important to recognize that this large increase occurs in spite of the fact that the structure has satisfactory stiffness, measured by the drift ratio at working wind load, and a large margin of safety against buckling. Of course, buckling in the pure sense of bifurcation does not occur in this structure, since lateral deformation occurs from the beginning of gravity loading. In fact, the supposed load factor of 4.82, determined from Eq. (17), is not valid unless the tension brace is infinitely strong. If the foundation for member **AB** can only develop an ultimate uplift of 25.6 kips, the structure will fail at 1.81 times gravity load alone instead of 4.82 times. (Member **BC** fails at 1.77 times gravity.)

* *The maximum system load factor is 1.77, obtained by iterating the previous procedure.*

Table 1. Summary of Analyses of Fig. 6

Function	First Order Analysis	Second Order Analysis
Design wind force in AB	±16.4 kips	±22.5 kips
Design sidesway force in AB	0	+8.7 kips
Design tension in AB (wind)	+16.4 kips	+31.2 kips
Design compression in AB (wind)	−16.4 kips	−13.8 kips
Design compression in BC (gravity)	−165.0 kips	−177.6 kips
Drift at working wind	0.00170 ft	0.00214 ft
Drift from sidesway at working gravity	0.00242 ft	0.00306 ft
Total drift at working loads	0.00412 ft	0.00520 ft
A.F. at working load	. . .	1.26
A.F. at 1.3 × working load	. . .	1.37

GENERAL DESIGN PROCEDURE FOR SINGLE STORY PIN JOINTED STRUCTURES

The previous example has shown how to apply the principles of second order analysis to a simple structure. For the general case of structures such as those illustrated in Figs. 4 and 6, proceed as follows:

Step 1:
Perform a first order analysis for lateral loads ΣV_w acting alone. Find the rotational displacement $(\Delta_{ov}/H)_w$ for the whole story. This analysis can be done using hand methods or with a first order computer program, such as STRESS.

If STRESS were used to analyze the structure in Fig. 6, the results for V_w = 2.70 kips would be:

Force in **AB** = 16.42 kips
Force in **BC** = 16.20 kips
Horizontal displacement of **B** = 0.367 in.

The rotational displacement is

$$\frac{0.367}{18 \times 12} = 0.00170$$

Step 2:
Using $(\Delta_{ov}/H)_w$ and ΣV from Step 1, find ΣP_L:

$$\Sigma P_L = \frac{\Sigma V_w}{\left(\frac{\Delta_{ov}}{H} \right)_w} = \frac{2.70}{0.00170} = 1589 \text{ kips}$$

Determine the amplification factor *A.F.* from Eq. (18):

$$A.F. = \frac{1}{1 - \frac{\Sigma P_w \times L.F.}{\Sigma P_L}} = \frac{1}{1 - \left(\frac{330 \times 1.3}{1589} \right)} = 1.370$$

Step 3:
Perform a first order analysis for gravity loads, including determination of rotational displacement $(\Delta_{po}/H)_w$, if any.

Using STRESS for Fig. 6, the results for $\Sigma P_w = 330$ kips would be:

Force in **AB** = 0

Force in **BC** = 165 kips

Horizontal displacement of **B** = 0.523 in.

The rotational displacement of **B** is

$$\frac{0.523}{18 \times 12} = 0.00242$$

Compute the first order force:

$$(V_{P1})_w = \left[\left(\frac{\Delta_{po}}{H}\right)_w \times \Sigma P_w \times L.F.\right] \qquad (25)$$

$$= 0.00242 \times 330 \times 1.30$$

$$= 1.04 \text{ kips}$$

Step 4:
Add $(V_{P1})_w$ from Step 3 to the external lateral load ΣV_w from Step 1 and multiply the sum by the amplification factor *A.F.* from Step 2.

For the sample case, $V_w = \pm 2.70$ kips and $(V_{P1})_w = 1.04$ kips.

$$(+2.70 + 1.04)\ 1.370 = 5.12 \text{ kips}$$

$$(-2.70 + 1.04)\ 1.370 = -2.27 \text{ kips}$$

The plus sign indicates that the load acts to the right. These forces may now be treated as equivalent first order forces.

Step 5:
Modify the results from Step 1 by the ratio of the equivalent first order forces from Step 4 to the value of V_w used in Step 1. [If there is no sidesway force, $(V_{P1})_w$, this ratio will simply be the amplification factor *A.F.*] Combine these results with the results of Step 3 for member design.

Max. tension in **AB** for wind plus sidesway:

$$\frac{5.12}{2.70} \times 16.4 = 31.2 \text{ kips}$$

Max. compression in **AB** for wind minus sidesway:

$$\frac{-2.27}{2.70} \times 16.4 = -13.8 \text{ kips}$$

Max. compression in **BC** for wind plus sidesway plus gravity:

$$\left(\frac{5.12}{2.70} \times 16.2\right) + 165 = 195.7 \text{ kips}$$

Step 6:
If the exact horizontal deflection at working load is desired, using the first order deflections from Steps 1 and 3, find:

$$(\Delta_{pv})_w = \frac{(\Delta_{ov} + \Delta_{pv})_w}{\left(1 - \dfrac{\Sigma P_w}{\Sigma P_L}\right)} \qquad (27)$$

$$= \frac{0.367 + 0.523}{\left(1 - \dfrac{330}{1589}\right)}$$

$$= 1.123 \text{ in.}$$

If it can be determined by inspection that the structure will not develop sidesway under vertical gravity loading, Steps 3 and 4 may be omitted. The amplification factor from Step 2 may be directly multiplied by the results of Step 1 to get final second order results. In actual practice this will be true for most structures.

When structures are symmetrical, the designer may wish to use a minimum out-of-plumb ratio, Δ_i/H, such as 0.002 in Step 3.

Summary of Steps for Second Order Analysis

Step 1: Find forces and $(\Delta_{ov}/H)_w$ from first order analysis for ΣV_w.

Step 2: Find:

$$\Sigma P_L = \frac{\Sigma V_w}{\left(\dfrac{\Delta_{ov}}{H}\right)_w}$$

$$A.F. = \frac{1}{1 - \dfrac{\Sigma P_w \times L.F.}{\Sigma P_L}}$$

Step 3: Find:

$$\left(\frac{\Delta_{po}}{H}\right)_w$$

$$(V_{P1})_w = \left(\frac{\Delta_{po}}{H}\right)_w \times (\Sigma P_w \times L.F.)$$

from first order analysis for ΣP_w.

Step 4: Add algebraically ΣV_w and $(V_{P1})_w$ and multiply result by *A.F.*

Step 5: Multiply forces from Step 1 by the ratio of the results from Step 4 to ΣV_w

$$\frac{[\Sigma V_w + (V_{P1})_w]A.F.}{\Sigma V_w}$$

to get final forces from lateral loads.

Step 6: Find exact displacement at working load:

$$\left(\frac{\Delta_{pv}}{H}\right)_w = \frac{\left[\frac{\Delta_{ov}}{H} + \frac{\Delta_{po}}{H}\right]_w}{\left(1 - \frac{\Sigma P_w}{\Sigma P_L}\right)}$$

Steps 1 through 5 can be further condensed to a single equation as follows:

$$\Sigma V_{Design} = \frac{\left[\Sigma V_w \pm \left(\frac{\Delta_{po}}{H}\right)_w \Sigma P_w \times L.F.\right]}{\left[1 - \frac{\Sigma P_w \times L.F.}{\Sigma V_w}\left(\frac{\Delta_{ov}}{H}\right)_w\right]} \quad (28)$$

where

ΣV_w = Applied working lateral load on a story

ΣP_w = Total working gravity load on a story

$\left(\frac{\Delta_{po}}{H}\right)_w$ = First order rotational displacement (drift) from ΣP_w acting alone

$\left(\frac{\Delta_{ov}}{H}\right)_w$ = First order rotational displacement (drift) from ΣV_w acting alone

For gravity alone, set $\Sigma V_w = 0$ in the numerator of Eq. (28) and use a load factor for gravity alone.

LIMITATION

All of the previous discussion has assumed perfectly elastic behavior. This is appropriate because the members of the structure in Fig. 6 fail by buckling at stresses below $0.5F_y$. On the other hand, when steel members are axially loaded to a level sufficiently high to cause local yielding by the combination of applied and residual stresses, the stiffness of the section will be reduced. In such cases members should be analyzed with a reduced effective stiffness based on the tangent modulus of elasticity calculated at maximum second order load. This procedure, which is necessarily iterative for second order analysis, will be discussed in detail in a continuation of this paper, to be published in the AISC *Engineering Journal* in the near future.

CONCLUSION

The method discussed here was developed for pin jointed structures without bending moments. It gives exact results within the limitations of small deflection theory.

In a continuation of this paper, to be published in the AISC *Engineering Journal* in the near future, the method will be extended to include frames with bending moments. The procedures for moment resisting frames will be essentially the same as the six steps just outlined, with the addition of an allowance for stiffness reduction in members having both axial loads and bending moments. Applications to single story and multistory structures will be shown.

APPENDIX A

NOMENCLATURE

A = Area of a member

A_v = Shear area of a member

$A.F.$ = Amplification factor. The ratio of second order internal forces to first order internal forces resulting from lateral loads

E = Modulus of elasticity

G = Modulus of rigidity (shear modulus of elasticity)

H = Height of a column or story

I = Moment of inertia

K = Effective length factor

L = Length of a girder

$L.F.$ = Load factor, taken as 1.7 for gravity alone or 1.3 for wind plus gravity

P = Vertical load on a column

P_{cr} = Elastic buckling load

P_e = Euler load = $\frac{\pi^2 EI}{(KH)^2}$

P_L = Force which produces unit rotational displacement (unit "drift") of a member or subsystem

V = Horizontal load

V_{P1} = $\frac{\Delta_{po}}{H} \times \sum P$

V_{P2} = $\frac{\Delta_{pp}}{H} \times \sum P$

w = A subscript meaning working load

Δ_i = Initial deflection from vertical without load

Δ_{ov} = First order deflection from V or ΣV acting alone

Δ_{po} = First order lateral deflection from P or ΣP acting alone

Δ_{pp} = Second order lateral deflection, from P or ΣP acting alone

Δ_{pv} = Second order lateral deflection from both V and P or ΣV and ΣP acting together

λ = A factor defined in the text

ΣP = Algebraic total of the vertical loads on a story

ΣP_L = Total of P_L's for a story

ΣP_w = ΣP at working load level

ΣV = Total of all horizontal loads on a story

ΣV_w = ΣV at working load level