Effective Length of Columns with Semi-Rigid Connections

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THE EFFECTIVE length of a column depends on the boundary conditions at the ends of its unbraced length. In a framed structure, the boundary conditions depend on the stiffness of the beams framed to the column. Design guides available for determining the effective length are provided for cases where beams are rigidly framed to the column and have their far ends rigidly framed to another column, fixed, or hinged. This paper will present a solution for the case where beams are framed to columns using semi-rigid connections.

Frequently, in the design of frames for tall buildings, it is found that beams must be selected based on their stiffness to resist frame drift, rather than on the strength needed to resist wind forces. In this case the actual bending moments required to be resisted at the ends of beams may be substantially smaller than the moment capacity of the beams. It is economically attractive to consider the use of semi-rigid connections adequate for the actual bending moment. However, the semi-rigid connection may have a lower stiffness than an equivalent portion of the beam, since its strength is weaker than the beam.

This paper will present a set-up for a general solution for the stiffness of beams with semi-rigid connections. A solution will be made for a simplified case and available information for one type of semi-rigid connection will be used to illustrate the effect on the effective length of a column.

EFFECTIVE LENGTH USING ALIGNMENT CHART

In Ref. 1, an alignment chart is presented which gives an effective length factor in terms of a dimensionless ratio G calculated at each end joint of the column:

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$$G = \frac{\Sigma(I_c/L_c)}{\Sigma(I_g/L_g)} \tag{1}$$

where

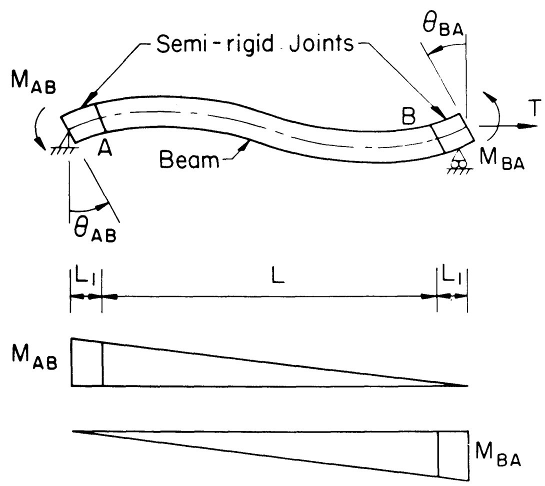
I_c = moment of inertia of column corresponding to plane in which buckling is being considered
L_c = corresponding unbraced length of column
I_g = moment of inertia of beam or girder corresponding to plane of buckling
L_g = corresponding unbraced length of girder

Also, Σ indicates a summation for all members rigidly connected to the joint and lying in the plane in which buckling is being considered. The girder stiffness I_g/L_g should be multiplied by a factor to correct for end conditions different from those assumed in development of the alignment chart. A special case not covered in the references on the use of the alignment chart is a member connected to the column by a semi-rigid connection. The following article will present a method for determining the factor by which the I_g/L_g term must be multiplied to allow for semi-rigid connections.

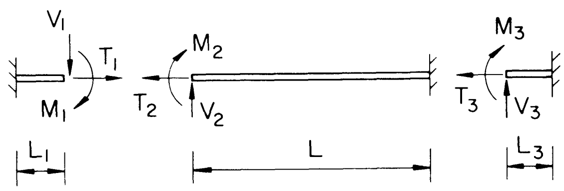
FORMULATION OF PROBLEM

In order to formulate a solution, the end force and deformation relationships for a structure comprised of a beam and two semi-rigid connections will be solved by the matrix force method.[2]

The structure to be assumed is shown in Fig. 1(a). The beam member, assumed to be prismatic, is connected at each end to a semi-rigid joint assumed to be a short extension of the beam. The object of the problem is to obtain the moments at each end of the beam in terms of the corresponding end rotations. In Fig. 1(b), the structure is broken into three basic elements, representing the beam and the two joints.



(a) Structure assumed



(b) Three basic elements

Figure 1

Formulated as a cantilever beam, the basic element flexibilities may each be represented by the matrix equation

$$\{x\} = [f]\{SR\} \tag{2}$$

which expands for one member to

$$\begin{vmatrix} e \\ \delta \\ \phi \end{vmatrix} = \begin{vmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{vmatrix} \begin{vmatrix} T \\ V \\ M \end{vmatrix} \tag{3}$$

where

x = vector of basic element deformations
e = elongation component
δ = deflection component
ϕ = rotation component
SR = vector of basic element stress resultants
T = thrust component
V = shear component
M = moment component
f = matrix of flexibility coefficients relating deformations to stress resultants

The flexibility coefficients f_{ij} will differ for every kind of semi-rigid connection and values are not generally available. However, values determined from tests or special derivations could be substituted in the equations given.

The flexibility coefficients for a prismatic beam are derived by strength of materials as (neglecting shear strains)

$$\begin{vmatrix} e \\ \delta \\ \phi \end{vmatrix} = \begin{vmatrix} \frac{L}{AE} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{vmatrix} \begin{vmatrix} T \\ V \\ M \end{vmatrix} \tag{4}$$

The unassembled structure flexibility matrix for the whole structure is given by the expansion of Eq. (2):

$$\begin{vmatrix} e_1 \\ \delta_1 \\ \phi_1 \\ e_2 \\ \delta_2 \\ \phi_2 \\ e_3 \\ \delta_3 \\ \phi_3 \end{vmatrix} = \begin{vmatrix} f_{11}{}^A & 0 & 0 & & & & & & \\ 0 & f_{22}{}^A & f_{23}{}^A & & & & & & \\ 0 & f_{32}{}^A & f_{33}{}^A & & & & & & \\ & & & \frac{L}{AE} & 0 & 0 & & & \\ & & & 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} & & & \\ & & & 0 & \frac{L^2}{2EI} & \frac{L}{EI} & & & \\ & & & & & & f_{11}{}^B & 0 & 0 \\ & & & & & & 0 & f_{22}{}^B & f_{23}{}^B \\ & & & & & & 0 & f_{32}{}^B & f_{33}{}^B \end{vmatrix} \begin{vmatrix} T_1 \\ V_1 \\ M_1 \\ T_2 \\ V_2 \\ M_2 \\ T_3 \\ V_3 \\ M_3 \end{vmatrix} \tag{5}$$

where

L = length of basic element
E = Young's modulus of elasticity
A = cross-sectional area of basic element
I = moment of inertia of basic element

A set of equilibrium equations can be used to express all the basic element stress resultants in terms of the loads applied to the structure:

$$\{SR\} = [B]\ \{W\}$$

$$\begin{vmatrix} T_1 \\ V_1 \\ M_1 \\ T_2 \\ V_2 \\ M_2 \\ T_3 \\ V_3 \\ M_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \frac{1}{L_g} & \frac{1}{L_g} & 0 \\ \frac{L + L_3}{L_g} & \frac{-L_1}{L_g} & 0 \\ 0 & 0 & 1 \\ \frac{1}{L_g} & \frac{1}{L_g} & 0 \\ \frac{-(L + L_3)}{L_g} & \frac{L_1}{L_g} & 0 \\ 0 & 0 & 1 \\ \frac{1}{L_g} & \frac{1}{L_g} & 0 \\ \frac{-L_3}{L_g} & \frac{L_1 + L}{L_g} & 0 \end{vmatrix} \begin{vmatrix} M_{AB} \\ M_{BA} \\ T \end{vmatrix} \tag{6}$$

where

B = matrix of influence coefficients for stress resultants

W = vector of external node loads

$L_g = L_1 + L + L_3$

The B and the f matrix are all that are needed to formulate the assembled structure flexibility matrix giving the node displacements in terms of the node loads:

$$\{X\} = [B]^T [f][B]\{W\} = [F]\{W\} \tag{7}$$

where

X = vector of node displacements corresponding to node loads

F = assembled structure flexibility matrix

The desired stiffness matrix of the original structure may be obtained by inverting the flexibility matrix:

$$\{W\} = [F]^{-1}\{X\} = [K]\{X\} \tag{8}$$

The form of the final stiffness matrix is a three-by-three array useful in a typical structural analysis:

$$\begin{vmatrix} M_{AB} \\ M_{BA} \\ T \end{vmatrix} = \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} \begin{vmatrix} \theta_{AB} \\ \theta_{BA} \\ \Delta \end{vmatrix} \tag{9}$$

where

θ = rotation of node corresponding to end moment

Δ = total elongation of member

A general solution for stiffness of a beam with semi-rigid connections has been formulated. This solution would make it possible to include the behavior of joints that would either be long enough in length or have another configuration to permit shear and axial type deflections. Information that is available on semi-rigid connections is usually limited to joints that are compact in length, so that only moment-rotation characteristics are considered. This constitutes a special case which allows some simplification in the solution of problems.

SOLUTION OF SPECIAL CASE

To obtain a solution for a special case it will be assumed that a prismatic beam of properties L, E, I, A is connected to its columns by two semi-rigid connections of essentially zero length and having a known moment-rotation characteristic. The moment-rotation characteristic of the joints is

$$\phi = ZM \tag{10}$$

where

Z = rotation of the joint for a unit value of moment

When effects of axial deformation are neglected, Eqs. (5) and (6) simplify to the following:

$$\{X\} = [F]\,\{SR\}$$

$$\begin{vmatrix} \delta_1 \\ \phi_1 \\ \delta_2 \\ \phi_2 \\ \delta_3 \\ \phi_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & & & & \\ 0 & Z_A & & & & \\ & & \frac{L^3}{3EI} & \frac{L^2}{2EI} & & \\ & & \frac{L^2}{2EI} & \frac{L}{EI} & & \\ & & & & 0 & 0 \\ & & & & 0 & Z_B \end{vmatrix} \begin{vmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{vmatrix} \tag{11}$$

$$\{SR\} = [B]\,\{W\}$$

$$\begin{vmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{vmatrix} = \begin{vmatrix} \frac{1}{L} & \frac{1}{L} \\ 1 & 0 \\ \frac{1}{L} & \frac{1}{L} \\ -1 & 0 \\ \frac{1}{L} & \frac{1}{L} \\ 0 & 1 \end{vmatrix} \begin{vmatrix} M_{AB} \\ M_{BA} \end{vmatrix} \tag{12}$$

By performing the operations of Eqs. (7) and (8) the following expression is obtained for Eq. (9):

$$\begin{vmatrix} M_{AB} \\ M_{BA} \end{vmatrix} = \begin{vmatrix} \frac{6EI}{DL} \end{vmatrix} \begin{vmatrix} 2(K_B + 1) & +1 \\ +1 & 2(K_A + 1) \end{vmatrix} \begin{vmatrix} \theta_{AB} \\ \theta_{BA} \end{vmatrix} \tag{13}$$

where

$K_A = 3Z_A\,(EI/L)$

$K_B = 3Z_B\,(EI/L)$

$D = 4(K_A + 1)(K_B + 1) - 1$

Equation (13) may be solved for absolute stiffnesses useful in various structural problems by substituting proper boundary conditions and determining the moment corresponding to a unit rotation.

1. For use in rigid frame analysis:

$$\theta_{AB} = 1; \qquad \theta_{BA} = 0$$

$$K_{AB} = M_{AB} = 4E\left[\frac{3(2K_B + 1)}{4(K_A + 1)(K_B + 1) - 1}\right]\frac{I}{L} \tag{14}$$

2. For use with alignment charts (free to sidesway):

$$\theta_{AB} = 1; \qquad \theta_{BA} = \theta_{AB}$$

$$K_{AB} = M_{AB} = 6E\left[\frac{2(K_B + 1) + 1}{4(K_A + 1)(K_B + 1) - 1}\right]\frac{I}{L} \tag{15}$$

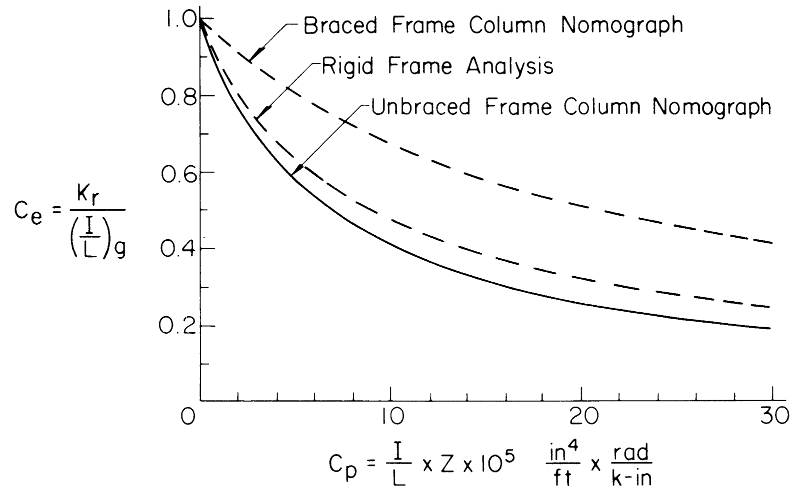


Fig. 2. Chart for determination of K_r

3. For use with alignment charts (sidesway prevented):

$$\theta_{AB} = 1; \qquad \theta_{BA} = -\theta_{AB}$$

$$K_{AB} = M_{AB} = 6E\left[\frac{2(K_B+1)-1}{4(K_A+1)(K_B+1)-1}\right]\frac{I}{L} \quad (16)$$

where

K_{AB} = absolute stiffness of member AB for joint rotations at **A**

M_{AB} = bending moment at **A** in member **AB**

Equations (14), (15), and (16) may be converted into relative stiffness equations in order to conserve computational effort involving the modulus of elasticity. This is achieved by dividing by E and an appropriate constant. For a further special case, K_A is assumed to equal K_B.

1. For use in rigid frame analysis:
   Divide Eq. (14) by $4E$:

$$K_r = \left[\frac{3}{4(K_B+1)-1(K_B+1)}\right]\frac{I}{L} \quad (17)$$

2. For use with alignment charts (frames free to sidesway):
   Divide Eq. (15) by $6E$:

$$K_r = \left[\frac{1}{2K_B+1}\right]\frac{I}{L} \quad (18)$$

3. For use with alignment charts (sidesway prevented):
   Divide Eq. (16) by $2E$:

$$K_r = \left[\frac{3}{2K_B+3}\right]\frac{I}{L} \quad (19)$$

where

K_r = relative stiffness of the beam with two identical semi-rigid connections

The relative stiffnesses obtained may each be used in an appropriate type of structural analysis or with the correct alignment chart for effective column lengths.

PREPARATION OF DESIGN CHART

DeFalco and Marino developed an effective chart for the determination of the relative stiffnesses of beams with semi-rigid connections to be used in determining effective lengths of columns.[3] Their chart was based on an equation equivalent to Eq. (17) derived by the author in Ref. 4. At that time both the author and Messrs. DeFalco and Marino were unaware that the alignment charts for columns in frames free to sway were based on a different boundary condition for end rotation of the beams than is used for ordinary structural analysis.[1] In this paper a revision of DeFalco and Marino's chart will be presented.

Figure 2 is a chart for the determination of K_r containing plots of Eqs. (17), (18), and (19). In this chart

C_p = flexibility index of beam and connections

C_e = efficiency coefficient for the beam and semi-rigid connection assembly.

It should be noted that the values of C_p and C_e in Fig. 2 are based on the beam length in feet. The efficiency coefficient C_e is the ratio of effective beam stiffness with semi-rigid connections to effective beam stiffness with fully rigid connections.

For use with other unit systems (i.e., metric) C_p may be determined in consistent units from

$$C_p = \frac{4K_B}{E} \times 10^5$$

resulting in units for C_p of rotation units divided by the units of E.

The three curves for relative stiffness are plotted in Fig. 2. It will be noted that the curve for effective beam stiffness to be used in the alignment charts for columns in frames subject to sidesway is the lowest curve. The curve for beam stiffness in rigid frame analysis is the same as was given by DeFalco and Marino for sway frames. Therefore, their result was slightly unconservative. However, for the two examples they presented, the error was too small to make a difference in the effective length factor selected from the nomographs, and was thus negligible.

SAMPLE RESULTS

To test the theory, a study was made of three varieties of semi-rigid connections. The general makeup of the three types is shown in Fig. 3. They consist of Type A, a bolted web angle connection; Type B, a bolted top angle and unstiffened seat connection; and Type C, a connection with both flange and web angles bolted. The flexibility coefficients Z have been derived for each of the three types and theoretical formulas are presented by Lothers.[4] The theoretical curves agree well with the initial slopes of experimental moment-rotation curves for bolted connections obtained by Rathbun.[5]

Connection Strength—In order that the connections studied would be matched with appropriate beam sizes, the connections were selected for the beams needed in design examples of a multi-story frame.[6] Connections were needed for beams ranging from W12X22 with a length of 12 ft to W27X84 with a length of 28 ft. Connection strengths derived by Lothers are based on the allowable bending stresses of the angles.[5] It would also be necessary to check fastener capacities, which was not done in this study because the other results were of primary importance.

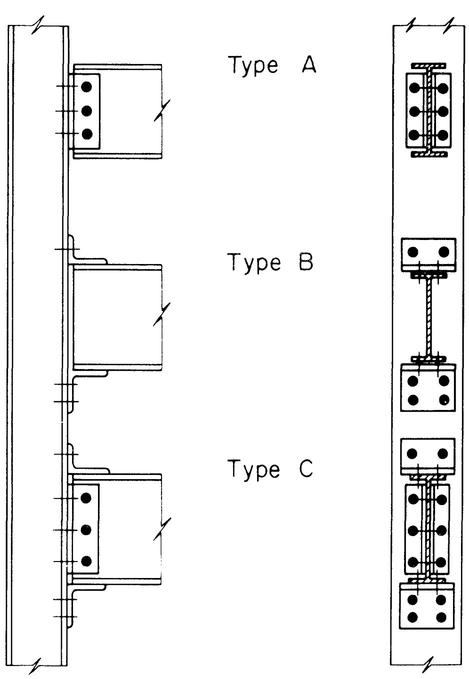


Fig. 3. Three types of semi-rigid connections

It was found that Type C connections could be selected to match the needed moments in all the members. Even larger beams could have been accommodated with heavier Type C connections.

Type B connections with the heaviest available angle sections could resist the expected moments in all the beams of this study. However, for beams larger than those in this study, Type B would not be sufficient.

Type A connections are ordinarily used only as flexible shear connections. However, they do have some moment capacity, which would have been adequate for some of the smaller beams in the study.

Beam Stiffness—The primary purpose of this study was to obtain effective stiffnesses of beams with semi-rigid connections as they relate to effective column length alignment charts for sidesway frames. Table 1 presents the range of relative beam stiffnesses obtained in the study for sidesway frames, braced frames, and rigid frame analysis.

For the sidesway frames alignment chart, connection Type C provides relative stiffnesses from 0.75 to 0.85 of a rigidly connected beam. Connection Type B can achieve high stiffnesses of $0.85I/L$, but it can become much more flexible when thinner angles are used. Connection Type A exhibits much less contribution to beam stiffness, because it does not engage the strength of the beam flanges.

Influence on Effective Column Length—The influence of the effective beam stiffnesses determined in the preceding paragraph on the effective column length in a sidesway frame can now be determined. It will be assumed that one story of a column will be restrained by an identical combination of beams at both the top and bottom joints.

Table 1. Range of Relative Stiffnesses from Sampling of Beams with Semi-Rigid Connections

1. SIDESWAY FRAMES (Use In Effective Length Alignment Chart)

Connection	Range of Relative Stiffness
Type C	$0.75\,I/L$ to $0.85\,I/L$
Type B	$0.48\,I/L$ to $0.86\,I/L$
Type A	$0.10\,I/L$ to $0.38\,I/L$

2. BRACED FRAMES (Use in Effective Length Alignment Chart)

Connection	Range of Relative Stiffness
Type C	$0.90\,I/L$ to $0.94\,I/L$
Type B	$0.73\,I/L$ to $0.95\,I/L$
Type A	$0.24\,I/L$ to $0.64\,I/L$

3. RIGID FRAME ANALYSIS

Connection	Range of Relative Stiffness
Type C	$0.80\,I/L$ to $0.88\,I/L$
Type B	$0.54\,I/L$ to $0.89\,I/L$
Type A	$0.14\,I/L$ to $0.44\,I/L$

Therefore, the parameter G for entering the alignment charts will be identical at top and bottom joints. Table 2 lists effective length factors considering the maximum and minimum effective beam stiffnesses previously derived for each of the three types of connections. For each case a result is provided assuming very stiff restraining members (G = 1.0) and very flexible restraining members (G = 10). Since the relationship between G and the column K is a smooth curve between these extremes, the results will be indicative of the entire range.

With the stiffest Type C connection and beam members which would result in a G value of 1.0 in fully rigid construction, the resulting G is 1/0.85 or 1.18, resulting in a K value of 1.37. With the weakest Type C connection and the same beams, G would increase to 1.33 but K would only increase to 1.41, a difference actually undetectable at the usual plotted scale of the alignment charts. With more flexible beam members yielding a G of 10, the same range of Type C connections would result in column K of 3.2 to 3.3, again a difference undetectable on the alignment charts. Obviously a designer could be quite comfortable with using $0.75I/L$ of the beam in calculating G for all preliminary stages of design involving Type C connections, without having to calculate Z values of the connections until the final design is checked.

With Type B connections, the assumption of the weakest type of connection for preliminary design would probably be conservative, but would not be as accurate as is possible for the Type C connection.

The results for Type A connections show why that type has only limited possibilities for application in frames subject to sidesway.

Reliability of Joint Flexibility Factors—Because of the non-linear character of the experimental moment-rotation curve of semi-rigid connections, care should be exercised in the use of the theoretical connection flexibility Z. Figure 4 has a shape typical of all experimental curves for semi-rigid connections of the types covered in this study. The

Table 2. Influence of Semi-Rigid Connection Stiffness on Effective Length Factor for Sidesway Columns

Effective Beam Stiffness	Stiff Restraints		Flexible Restraints	
	G	K	G	K
Fully Rigid Conn.	1.0	1.33	10.0	3.0
Type C Conn.:				
Max 0.85 I/L	1.18	1.37	11.8	3.2
Min 0.75 I/L	1.33	1.41	13.3	3.3
Type B Conn.:				
Max 0.86 I/L	1.16	1.36	11.6	3.2
Min 0.48 I/L	2.08	1.60	20.8	4.2
Type A Conn.:				
Max 0.38 I/L	2.63	1.74	26.3	4.6
Min 0.10 I/L	10.0	3.00	100.	9.0

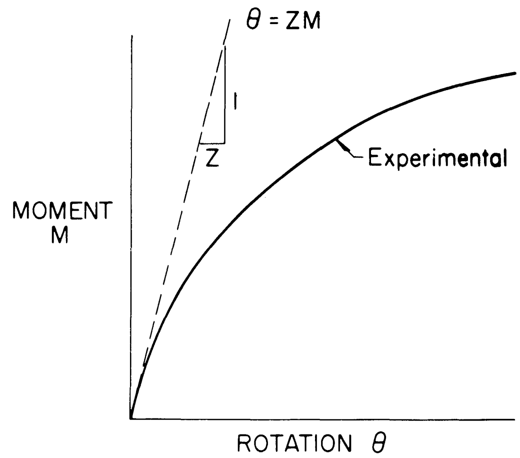


Fig. 4. Typical moment rotation curve of semi-rigid connection

slope of the moment-rotation curve decreases after a very small increase in moment and no part of the curve has a substantial straight line elastic portion. Therefore, the theoretical value of Z is accurate only for the initial value where moment first exceeds zero.

The theoretical value of Z is quite appropriate to use in bifurcation type buckling solutions where no moment exists at the joints. It is the initial instantaneous tendency to resist deformation which enters such buckling solutions.

However, in frame analysis solutions the theoretical Z values should be used with caution. Here the moments at joints would be substantial and the instantaneous slope of the moment-rotation curve would be smaller and the effective value of Z larger than the theoretical value. A linear frame analysis would not be valid.

Frye and Morris have fitted a series of empirical non-linear flexibility formulas to the test results for seven types of semi-rigid connections including those referred to here as Type A and Type B.[7] The flexibility formulas are used in an iterative non-linear analysis procedure for solving structures with the seven particular connection types. Such a procedure should be followed when load-deflection behavior of a structure is desired.

SUMMARY AND CONCLUSIONS

A brief theoretical study was conducted to determine the effect of semi-rigid beam-to-column connections on the effective length of columns in framed steel structures. Material for the study was gathered from available references on connections and structures. The results of the study are summarized as follows:

General Solution for Beam Stiffness—A general solution for the stiffness of a beam element with semi-rigid connections has been formulated.

It was reduced to a special case of semi-rigid connections having rotational flexibility only.

Absolute and relative stiffness equations were derived for beams used in rigid frame analysis and in alignment charts for effective column length.

A design chart for relative stiffness of beams with semi-rigid connections was presented. The relative stiffnesses may be used for three cases:

1. Rigid frame analysis.
2. Effective length of columns in frames free to sidesway.
3. Effective length of columns in frames prevented from sidesway.

Connection Influence on Beam Stiffness—To illustrate use of the theory, a study was made considering three types of semi-rigid connection using bolted or riveted web angles and top and bottom flange angles.

The results of the study indicate that a beam connected by the types of semi-rigid connections considered would have an effective stiffness ranging from 0.10 to 0.85 of the relative stiffness of a fully rigidly connected beam as used in the alignment charts for columns in frames subject to sidesway. Relative stiffness efficiencies of 0.75 to 0.85 can be assured by selecting the stiffer connection types and using stiffer angles. The connection type with the highest stiffness of the three studies was Type C with both web angles and top and bottom flange angles.

Good relative beam stiffnesses can be provided by Type B connections having only top and bottom flange angles, but the angles must not be too thin.

Connection Influence on Effective Column Length—The influence on effective column length was determined for examples of the relative beam stiffnesses obtained.

It was found that the G nomograph parameter for members with Type C connections was from 18 to 33 percent greater than G for the same members rigidly framed, but that the resulting difference in the effective length factor for the columns was too small to be detected in the usual alignment charts.

It is suggested that a convenient design approximation is to assume that the effective I/L of beams with Type C connections is $0.75I/L$ in calculating G for entering the alignment charts for sidesway frames.

It was concluded that the use of the theoretical values of joint flexibility factors Z in bifurcation type buckling solutions is appropriate. A bifurcation problem determines the initial tendency to buckle and the value Z is a linear approximation of the initial tendency to buckle.

It is recommended that careful consideration be taken before using the values of Z in analyses of rigid frame behavior under increasing load, because of the severe nonlinearity of typical semi-rigid connection behavior. Instead, a non-linear analysis such as given in Ref. 7 should be used.

The general beam stiffness equation derived would be suitable for use with other semi-rigid connections, provided they had linear characteristics of load vs. shear deformation, axial deformation, and rotation.

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