A Simple Approach to Truss Deflections

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The determination of deflections for a longspan truss has long been considered a troublesome subject by structural engineers. The most popular approach, other than utilizing established computer programs, is the unit-load method,[1] in which

$$\Delta = \Sigma \frac{SL}{AE} u \qquad (1)$$

where

S = internal force in any member due to actual loads
u = internal force in the same member due to a fictitious unit load at the point where the deflection is sought, acting along the desired direction
L = length of the member
A = cross-sectional area of the member
E = elastic modulus of the member

The simplest way to use Eq. (1) is to compile a table to compute (1) L/EA, (2) S, (3) u, (4) SLu/EA, and (5) the summation of item (4), so that the deflection at the joint and along the direction of the fictitious unit load is determined. It can be seen from this procedure that the use of Eq. (1) is very time consuming because member stresses must be provided before the evaluation of deflections. This paper proposes a new and simple approach to preclude the evaluation of member stresses, so that deflections of longspan trusses of parallel chords can be easily and accurately estimated by hand calculation.

The basic concept in the estimation of vertical deflections for a longspan truss is that the truss is considered as a beam, the deflection of which is composed of two portions: one due to beam bending and the other due to beam shearing. The bending deflection is evaluated by using established deflection formulas of beams as in any structural analysis references, for instance, Ref. 2. The moments of inertia in the formulas are to be determined from chord members. The shear deflection involves web members only. It is known that the shear deflection is not important in a regular beam.[3] However, it is very significant in the simulated beam of a truss. This paper introduces a simple concept of estimating shear deflections; these deflections, together with the bending deflection, give the approximate deflection of a truss.

In order to clarify the determination of truss deflection, the author classifies trusses into four basic types, namely, N-braced (Pratt) type, W-braced (Warren) type, K-braced type, and X-braced type. For each type, the equivalent moment of inertia for bending deflection, the shear deflection due to diagonals, and the shear deflection due to posts are individually discussed and applied. The elastic modulus is assumed to be constant for simplicity's sake.

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N-BRACED TYPE

The diagonals of an N-braced (Pratt) truss are all parallel, as shown in Fig. 1. For this type of truss, the equivalent moment of inertia for the bending deflection is intuitively given by the cross-sectional area of chord members about their averaged neutral axis in the depth. The result is

$$I = \frac{A_t A_b}{A_t + A_b} h^2 \qquad (2)$$

where A_t and A_b are the cross-sectional areas of the upper and lower chords, respectively, and h is the depth of the truss.

The shear deformation, as shown in Fig. 1, consists of two portions: one due to diagonals and the other due to posts. With a shear force V in the panel, the elongation in the diagonal **AD** produces a vertical displacement **DD′** as $Vp/EA_d \sin^2\theta \cos\theta$. At the same time, the displacement of the post **AC** is given by Vh/EA_p. Therefore, the total

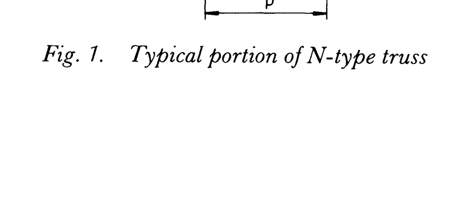

Fig. 1. Typical portion of N-type truss

shear deflection difference between two joints distance L_s apart is given by the proration of L_s to panel length p:

$$\Delta_s = \frac{VL_s}{EA_d}\left(\frac{1}{\sin^2\theta\cos\theta} + \frac{A_d}{A_p}\tan\theta\right) \qquad (3)$$

It is noted that for practical applications the shear force V may not be constant throughout the truss span. When this is the case, V represents an averaged value of the shear force over the distance L_s.

Example 1

Given:
An 8-panel simply supported N-braced truss, as in Fig. 2. The depth is 4⁄3 of the panel length. The cross-sectional area of each chord member is $2A$ and of each web member is A. Two types of loading condition are considered: a vertical load W at joint $\mathbf{L_4}$ and a set of vertical loads W at each lower chord joint. Determine the deflection at $\mathbf{L_4}$ for each loading case.

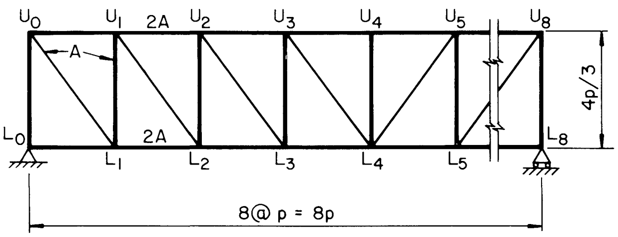

Fig. 2. Example 1

Solution:
The equivalent beam of this truss is considered as a simply supported beam subjected to two types of load: a concentrated load at the middle span and a uniform load W/p over the whole span. The moment of inertia is $I = 16Ap^2/9$. From Eq. (2), the span for the bending deflection is $L = 8p$, the distance for the shear deflection is $L_s = 4p$, the web members are $A_d = A_p = A$, $\tan\theta = 4/3$, and $\sin^2\theta\cos\theta = 0.384$. Then, the deflection of the truss is as follows:

For the case of concentrated load:

Bending deflection:

$$\Delta_b = \frac{WL^3}{48EI} = \frac{W(8p)^3}{48E(16Ap^2/9)} = 6\frac{Wp}{EA}$$

Shear deflection:

From Eq. (3), with $V = 0.5W$:

$$\Delta_b = \frac{0.5W(4p)}{EA}\left(\frac{1}{0.384} + \frac{4}{3}\right) = 7.875\frac{Wp}{EA}$$

Therefore, the approximate total deflection at $\mathbf{L_4}$ is:

$$\Delta = \Delta_b + \Delta_s = 13.875\frac{Wp}{EA}.$$

The result utilizing Eq. (1) is 14.063 Wp/EA. The error in the approximate result is only 1.3%.

For the case of uniform load:

Bending deflection:

$$\Delta_b = \frac{5wL^4}{384EI} = \frac{5(W/p)(8p)^4}{384\,E(16\,Ap^2/9)} = 30\frac{Wp}{EA}$$

Shear deflection at $\mathbf{L_4}$:

With $V = 2W$ averaged over $L_s = 4p$:

$$\Delta_s = 31.5\frac{Wp}{EA}$$

Hence, the approximate total deflection at $\mathbf{L_4}$ is:

$$\Delta = 61.5\frac{Wp}{EA}$$

Compared to the result of 61.875 Wp/EA from Eq. (1), the error in the approximate result is only 0.6%.

W-BRACED TYPE

Figure 3 is a typical portion of a Warren truss, showing the diagonals arranged in a W shape. Since the stress distribution in chord members for this type of truss is similar to that for the N-type, the moment of inertia for the W-type is assumed to be the same as that for the N-type and Eq. (2) can be utilized.

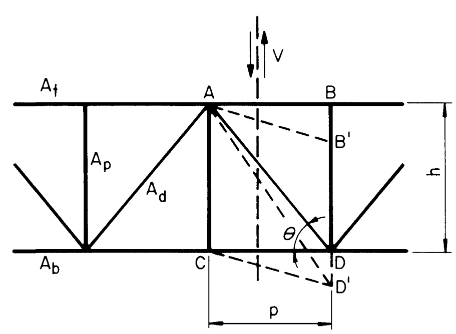

Fig. 3. Typical portion of W-type truss

The shear deformation of a W-braced truss can be seen from Fig. 3. The displacement **DD′** due to diagonals is the same as that for the N-type. However, the deflection due to posts is different. Since post members within L_s do not affect the deflection difference between the ends of L_s, only the posts at both ends of L_s may contribute their elongation Vh/EA_p to the shear deflection. The total shear deflection can then be written as:

$$\Delta_s = \frac{VL_s}{EA_d}\left(\frac{1}{\sin^2\theta\cos\theta} + c\,\frac{h}{L_s}\frac{A_d}{A_p}\right) \qquad (4)$$

where c is a constant which is 0, 1 or 2, depending on whether no post, one post, or two posts at both sides of L_s are under stress.

Example 2

Given:
An 8-panel simply supported W-braced truss as shown in Fig. 4.

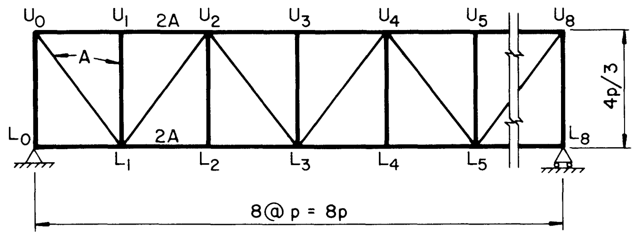

Fig. 4. Example 2

Solution:
The bending deflections are the same as those in Example 1. The only portion of deflection which is different from Example 1 is that due to the posts. For both the concentrated load and the uniform load cases, the stresses in posts $\mathbf{U_0L_0}$ and $\mathbf{U_4L_4}$ are nonvanishing. Hence, $c = 2$ and Eq. (4) gives:

For the case of concentrated load:

$$\Delta_s = \frac{0.5W(4p)}{EA}\left(\frac{1}{0.384} + 2\frac{4/3}{4}\right) = 6.542\,\frac{Wp}{EA}$$

For the case of uniform load:

$$\Delta_s = 26.167\,\frac{Wp}{EA}$$

These shear deflections together with the bending deflections $6Wp/EA$ for the concentrated load and $30Wp/EA$ for the uniform load give the following total deflections at joint $\mathbf{L_4}$:

For the concentrated load case:

$$\Delta_{\text{approx.}} = 12.542\,\frac{Wp}{EA}$$

$$\Delta_{\text{exact}} = 13.396\,\frac{Wp}{EA} \quad (\text{error} = 6.4\%)$$

For the uniform load case:

$$\Delta_{\text{approx.}} = 56.167\,\frac{Wp}{EA}$$

$$\Delta_{\text{exact}} = 57.208\,\frac{Wp}{EA} \quad (\text{error} = 1.8\%)$$

It is noted that when all diagonals in this example are changed across the other corners of every panel (for example, $\mathbf{U_0L_1}$ and $\mathbf{L_1U_2}$ become $\mathbf{L_0U_1}$ and $\mathbf{U_1L_2}$, respectively), the stress at both ends of L_s vanishes and the constant c in Eq. (4) equals zero. Hence, the shear deflections are independent of post members.

For the concentrated load case:

$$\Delta_s = 5.208\,\frac{Wp}{EA}$$

$$\Delta_{\text{approx.}} = 11.208\,\frac{Wp}{EA}$$

and for the uniform load case,

$$\Delta_s = 20.833\,\frac{Wp}{EA}$$

$$\Delta_{\text{approx.}} = 50.833\,\frac{Wp}{EA}$$

The exact solutions for these two loading cases are $11.396Wp/EA$ and $51.208\ Wp/EA$, respectively, and the error in the approximate result is only 1.6% for the concentrated load and 0.7% for the uniform load.

K-BRACED TYPE

K-braced trusses are more complicated than N-type and W-type trusses. Since the result for the N-type is so accurate, as illustrated in Example 1, the bending deflection for the K-type will be estimated with the N-type as a base. First, it has to be recognized that the upper and lower chord forces at any panel of an N-truss are not of equal magnitude, whereas the corresponding chord forces at a panel of a K-truss are equal. From this it follows that if two trusses are of the same size, but one is K-braced and the other is N-braced, the panel moments due to chord forces are different in each truss. Figure 5 illustrates the panel moments for both trusses by solid lines which are step functions. The averaged panel moments are shown by straight dashed lines as a continuous function of the moment at the middle point of a panel. It is seen that the moments for both trusses always have a distance lag of a half-panel. This means that the moment at any point in either truss is equal to the

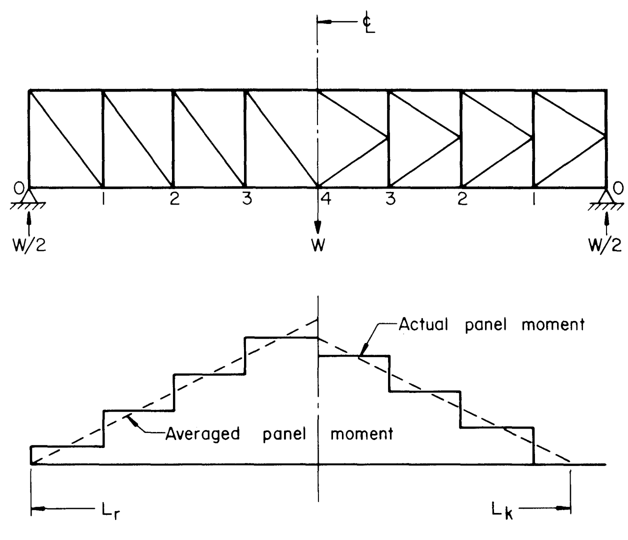

Fig. 5 Diagram of panel moments

moment at a point approximately a half-panel distance away in the other truss. By extending this throughout the trusses, it may be said that the effective span for each type of truss is different. It follows that with the span, L_r, of the equivalent N-type truss as a reference, an artificial span L_k has to be introduced to modify the bending deflection of the K-truss. Since the bending deflection of a beam is always proportional to L^3/I, the modification is to divide the deflection of the N-truss by L_r^3 and to multiply the result by L_k^3. It is also clear that the modification for the K-truss span can be used to modify the moment of inertia instead of the truss span. The modified moment of inertia is thus given as:

$$I = \frac{A_t A_b}{A_t + A_b} h^2 \left(\frac{L_r}{L_k}\right)^3 \qquad (5)$$

The method for determination of L_r and L_k follows.

In most applications, the reference span L_r is taken as the actual span of the truss, because the averaged panel moment can be approximately drawn to the ends of the truss. However, the modified span L_k is dependent upon the pattern of K-bracing. If the K-braces diverge toward midspan in the direction that the absolute value of panel moment is increasing, as shown in Fig. 6(a), the chord forces of the K-type truss are equal to the smaller chord forces of the equivalent N-truss. For this case, the bending deflection of the K-truss is smaller than that of the N-truss. This is equivalent to saying that the modified moment of inertia for the K-truss has to be larger than that for the N-truss and extensibly equivalent to $L_k < L_r$. It follows from the previous discussion that the modified span L_k is taken as the reference span, L_r, minus a half-panel distance at each end, as shown in Fig. 5(b). For instance, in Example 3 (see below) L_r is the actual span $8p$, whereas L_k is $7p$, which is L_r minus $p/2$ at each end of the truss.

On the other hand, if the K-braces diverge away from midspan, in the direction that the absolute value of panel moment is decreasing, as shown in Fig. 6(b), the chord forces of the K-truss are the same as the larger chord forces of the equivalent N-truss. Accordingly, the bending deflection of the K-truss is larger than that of the N-truss and the modified span L_k has to be larger than the reference span L_r. Similarly to Fig. 5(b), L_k is approximately equal to L_r plus a half-panel distance at each end of the truss. This is equivalent to attaching an imagined panel to the ends of the truss as that shown in Fig. 6(b), and L_k is measured to the middle point of that panel. For example, if all bracing in Example 3 were turned aside, L_r would be given by $8p$, whereas L_k would be $8p$ plus $p/2$ for each end of the truss, or $9p$.

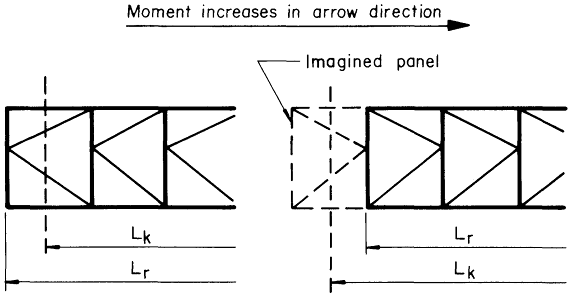

Fig. 6 Measurement of spans L_r and L_k

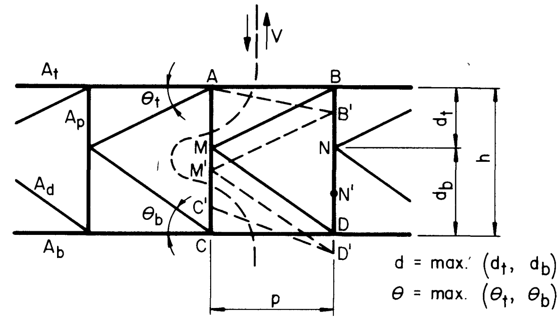

Fig. 7 Typical portion of K-type truss

The shear deformation for a K-truss also consists of two portions, one due to diagonals and the other due to posts. For the portion due to diagonals, either one of the K arms, **MB** and **MD** in Fig. 7, can be utilized to evaluate the shear deflection. However, for a more accurate result, the longer K arm is considered. With this, the first portion of Eq. (3) of the N-type is readily applicable, except that V is replaced by Vd/h, with d as the larger of d_t and d_b, and θ as the larger of θ_t and θ_b. The shear deflection due to posts is determined as follows. The elongation of a post, such as **AC** in Fig. 7, is $Vd(h/d - 2 + 2d/h)/EA_p$. The effect of this elongation on the deflection at any point at distance L_s from the post can be determined by multiplying the post elongation by the ratio L_s/p, where p is the panel length.

Having determined both the shear deflection components, the total shear deflection of a K-truss is:

$$\Delta_s = \frac{VL_s}{EA_d}\frac{d}{h}\left[\frac{1}{\sin^2\theta\cos\theta} + \left(2 - 2\frac{h}{d} + \frac{h^2}{d^2}\right)\frac{A_d}{A_p}\tan\theta\right] \qquad (6)$$

Example 3

Given:

The truss shown in Fig. 8 is K-braced and is subjected to two types of loadings: one a concentrated load at $\mathbf{L_4}$ and the other a set of equal joint loads over the lower chord. Determine the deflection at $\mathbf{L_4}$.

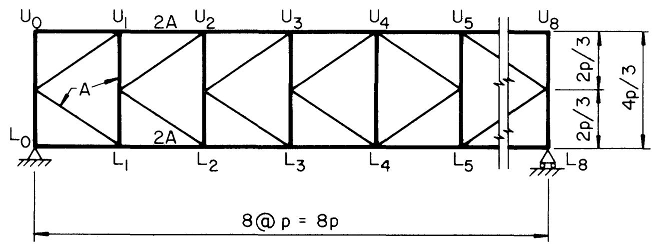

Fig. 8. Example 3

Solution:
Since $L_r = 8p$ and $L_k = 7p$, as discussed previously, the moment of inertia is:

$$I = \frac{A_t A_b}{A_t + A_b} h^2 \left(\frac{L_r}{L_k}\right)^3$$

$$= \frac{(2A)(2A)}{2A + 2A} \left(\frac{4p}{3}\right)^2 \left(\frac{8p}{7p}\right)^3$$

$$= 2.654\ A_p{}^2$$

The bending deflection for the case of concentrated load is:

$$\Delta_b = \frac{WL^3}{48\ EI} = \frac{W(8p)^3}{48\ E(2.654\ A_p{}^2)}$$

$$= 4.019 \frac{Wp}{EA}$$

The shear deflection, with $V = 0.5W$, $L_s = 4p$, $A_d = A_p = A$, $h = 2d$, $\tan\theta = 2/3$, $\sin\theta = 0.5547$, and $\cos\theta = 0.8321$, can be found from Eq. (6):

$$\Delta_s = \frac{(0.5\ w)(4p)}{EA} \times$$

$$\frac{1}{2}\left[\frac{1}{(0.5547)^2(0.8321)} + (2 - 4 + 4)\frac{A}{A}\left(\frac{2}{3}\right)\right]$$

$$= 5.239 \frac{Wp}{EA}$$

Hence, the total deflection at $\mathbf{L_4}$ is $9.258\,Wp/EA$. Compared to the exact solution of $9.010\,Wp/EA$, this result is in error by 2.8%.

For the case of uniform load, the bending deflection is $20.097\,Wp/EA$, the shear deflection is $20.957\,Wp/EA$, and the total deflection is $41.054\,Wp/EA$. Again, compared to the exact solution of $41.166\,Wp/EA$, the error in approximate result is only 0.3%.

Consider, on the other hand, the condition where all diagonals in the same truss are reversed so that they diverge away from midspan, so that the absolute value of panel moment is decreasing in the direction that the K-braces diverge. The reference span L_r is still $8p$, whereas the modified span L_k equals $9p$, as discussed previously. The modified moment of inertia is given by $1.249A_p{}^2$. Upon applying this quantity to the case of concentrated load, the bending deflection is $8.543\,Wp/EA$. This, with the previously determined shear deflection of $5.239\,Wp/EA$, gives an approximate result of $13.782\,Wp/EA$, which is in error by 0.8% compared to the exact solution of $13.677\,Wp/EA$ from Eq. (1).

Also for the truss with diagonals turned aside, if it is subjected to the set of equal joint loads, the bending deflection is $42.704\,Wp/EA$. This, in conjunction with the shear deflection of $20.957\,Wp/EA$, gives a total deflection at $\mathbf{L_4}$ of $63.661\,Wp/Ea$. This result, when compared to the exact solution of $59.332\,Wp/EA$, is in error by 7.3%.

X-BRACED TYPE

An X-braced truss can be considered either as an N-braced truss overlaid by another set of N diagonals, or as a W-braced truss overlaid by another set of W diagonals. The bending deflection may be assumed to be the same as for the N- or W-type. Therefore, the bending deflection of an X-type truss is determined using the equivalent moment of inertia of Eq. (2).

Inspection of shearing stress in an X-type truss shows that the posts do not contribute much to shear deflections. Therefore, this doubly braced truss behaves more like a W-type truss and the shear deflection can be determined from Eq. (4), with A_d replaced by $2A_d$ and $c = 0$:

$$\Delta_s = \frac{VL_s}{2EA_d} \frac{1}{\sin^2\theta\cos\theta} \qquad (7)$$

Example 4

Given:
Determine the deflection at $\mathbf{L_4}$ for the truss of Example 1 (Fig. 2) if it is overlaid by another set of N diagonals, creating an X-type truss.

Solution:
For the case of concentrated load, the bending deflection was found in Example 1 as $6\,Wp/EA$. The shear deflection from Eq. (7) is $2.604\,Wp/EA$. The total deflection, then, is $8.604\,Wp/EA$, which is in error by only 3.2% compared with the computer result, $8.890\,Wp/EA$.

When the truss is subjected to a set of equal joint loads, the bending deflection is $30\,Wp/EA$ and the shear deflection is $10.417\ Wp/EA$. Hence, the total deflection at $\mathbf{L_4}$ is $40.417\,Wp/EA$. Again, this is in error by only 1.1% compared to the computer result, $40.854\,Wp/EA$.

SUMMARY AND CONCLUSIONS

Truss deflection is estimated by considering the truss as a beam. The deflection is composed of two portions. One is the bending deflection, which is determined as for a regular beam with moment of inertia evaluated from chord members. The other portion is the shear deflection, which is evaluated from the deformation of web members.

The most remarkable feature of this approach is that no member stresses are required. The deflection is given by the direct calculation of simple formulas which already involve the properties of material and geometry. The examples given above, together with a good number of other studies, have shown that:

1. This approach requires very little time to attain an accurate result.
2. The results of this approach, compared with the results of either the unit-load method or computer analysis, show an averaged error for all examples of less than 5%.
3. The accuracy of the result depends upon the span-depth ratio. The result is more accurate if the ratio is larger. In general, the error of this approach is less than 5% when the span-depth ratio is larger than 4.

NOMENCLATURE

A = cross-sectional area of a member
A_b = cross-sectional area of lower chord
A_d = cross-sectional area of diagonal
A_p = cross-sectional area of post
A_t = cross-sectional area of upper chord
c = a constant
d = the larger of d_b and d_t
d_b, d_t = lengths of the lower and the upper portions of a K-post, respectively
E = the elastic modulus
I = equivalent moment of inertia of a truss
L = span of bending deflections
L_k, L_r = reference spans
L_s = distance between two joints to measure the difference of shear deflections
p = panel length of a truss
V = a shear force
W = an applied load
Δ = a deflection
Δ_b, Δ_s = bending and shearing deflections, respectively

ACKNOWLEDGMENTS

The author is grateful to his wife, Pin Fun, for her devoted discussion and encouragement throughout the preparation of this paper. He also thanks Mr. R. Onodera for his suggested improvements to the paper and Mr. C. Blue for his work in preparing the illustrations.

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