On Inelastic Column Buckling

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IN 1971 Yura[1] presented the concept of the inelastic K-factor, which provides a more accurate estimate of the effective length of a column that buckles at a stress level beyond the proportional limit. For the most simple applications, the essence of the idea is that the end restraint factor, G, as required for use of the alignment charts or equations for evaluation of K, should be modified as follows:

$$G_{inelastic} = \frac{E_T}{E} G_{elastic} \tag{1}$$

where

E_T = slope of the column stress-strain curve
E = initial value of E_T; Young's modulus

The problem is how to determine the proper value of E_T/E. Disque[2] presented a procedure for design, suggested by Yura, which can be summarized as follows:

(a) For specified axial load, assume column size and evaluate $f_a = P/A$.
(b) Determine Kl/r for which $f_a = F_a$, which is the allowable stress according to AISC Specification.
(c) Determine F'_e for that particular Kl/r.
(d) Then $E_T/E = f_a/F'_e$.

It was recognized that the above equation for E_T/E is not theoretically correct, since F_a and F'_e are based on different factors of safety, but the error was considered, quite properly, to be small.

The purpose of this paper is to present a procedure for the evaluation of E_T/E which retains all the simplicity of Disque's method, but eliminates that small error. The slight differences in E_T/E values would hardly be worth any additional efforts, but if a correct formulation gives more precise results consistent with the AISC Specification and can be achieved at no extra cost, why not use it?

DERIVATION OF EQUATION

In the development of the AISC column formulas, it has been assumed that the average stress-strain curve (as determined from a stub column test) becomes nonlinear at $f = 0.5F_y$, due primarily to residual stresses in the cross-section. That is, for $f \leq 0.5F_y$, the slope of the stress-strain curve is equal to E; for $f > 0.5F_y$, the slope is $E_T < E$ until, at $f = F_y$, $E_T = 0$. In what follows, an equation for E_T/E is assumed as a function of compressive stress f. Then, it is shown that the given expression for E_T/E, when used in conjunction with the tangent modulus definition of critical load, gives the AISC equation for column critical stress. This then serves as justification of the validity of the assumed form for E_T/E.

We begin with the relationships

$$E_T = 1 \qquad (\text{for } f \leq 0.5F_y) \tag{2a}$$

$$E_T = 4\frac{f}{F_y}\left(1 - \frac{f}{F_y}\right) \qquad (\text{for } 0.5F_y < f < F_y) \tag{2b}$$

which are implied by Eq. (2.11) of Ref. 3. The magnitude of critical stress for inelastic buckling is:

$$f_{cr} = \frac{\pi^2 E_T}{(Kl/r)^2} = \frac{\pi^2 E}{(Kl/r)^2}\frac{E_T}{E} \tag{3}$$

Recognizing that $f \geq 0.5F_y$ by assumption of inelastic buckling, and substituting Eq. (2b) into Eq. (3),

$$f_{cr} = \frac{\pi^2 E}{(Kl/r)^2}\left[4\frac{f_{cr}}{F_y}\left(1 - \frac{f_{cr}}{F_y}\right)\right]$$

This can be reduced to

$$f_{cr} = \left[1 - \frac{1}{2}\left(\frac{Kl/r}{C_c}\right)^2\right]F_y \tag{4}$$

where $C_c = \sqrt{2\pi^2 E/F_y}$. Equation (4) is recognized as the critical stress relationship when $Kl/r < C_c$ according to the AISC Specification. In other words, the AISC equation for allowable stress, Eq. (1.5-1) in the Specification, can be interpreted as essentially specifying the tangent modulus variation shown in Eq. (2).

Define $\alpha = Kl/r/C_c$. Then from Eq. (4),

$$\frac{f_{cr}}{F_y} = 1 - 0.5\alpha^2 \tag{5}$$

If Eq. (5) is substituted into Eq. (2), the tangent modulus at the inelastic buckling stress is:

$$\frac{E_T}{E} = \alpha^2(2 - \alpha^2) \qquad (\text{for } 0 < \alpha \leq 1) \tag{6}$$

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Equation (6) can also be derived very simply by noting that

$$\frac{E_T}{E} = \frac{f_{cr}}{F_e} = \frac{(SF)_1 f_a}{(SF)_2 F'_e} = \frac{F_y[1 - 0.5\alpha^2]}{\dfrac{\pi^2 E}{(Kl/r)^2}} = \alpha^2(2 - \alpha^2)$$

DESIGN PROCEDURE

The recommended procedure for evaluation of the inelastic K-value is as follows for the most simple applications:

1. For known value of axial load P, select a trial column size.
2. Determine Kl/r such that $F_a = f_a = P/A$. This is the critical Kl/r value for the assumed column and the given axial load.
   If $Kl/r \geq C_c$, then $E_T/E = 1$ and no adjustment is necessary.
   If $Kl/r < C_c$, evaluate $\alpha = Kl/r/C_c$.
3. Then, $E_T/E = \alpha^2(2 - \alpha^2)$. This value of E_T/E is used to determine $G_{inelastic}$ according to Eq. (1) above.
4. Determine K from alignment charts or equations, using $G_{inelastic}$ as appropriate.
5. If the resulting Kl/r is less than or equal to the critical Kl/r from step 2, then the trial column is satisfactory. The allowable load is guaranteed to equal or exceed the applied load.
6. If Kl/r from step 4 is greater than Kl/r from step 2, the trial column size is no good. The allowable load will be less than the actual load.

It is the contention of this paper that the evaluation of E_T/E in step 3 above is no more difficult than evaluating f_a/F'_e, and the result does not suffer the small inaccuracy associated with the approximation that $f_a/F'_e = f_{cr}/F_e$. Furthermore, it would be very simple to prepare tables which give values of E_T/E for different values of $f = P/A$.

DESIGN EXAMPLE 1

Given:
Yura's[1] Design Example 1, with W16×40 girders (I_x = 517 in.^{4}, length = 24 ft) at both ends of a 12-ft column which must support 750 kips. Use A36 steel.

Solution:
For the column, try W14×127:

$A = 37.3$ in.^{2}; $I_x = 1480$ in.^{4}; $r_x = 6.29$ in.

$$G_{elastic} = \frac{2(1480/12)}{2(517/24)} = 5.72$$

$$f_a = \frac{750}{37.3} = 20.11 \text{ ksi}; \quad \frac{Kl}{r} = 27.5$$

$$\alpha = 27.5/126.1 = 0.218$$

$$E_T/E = (0.218)^2[2 - (0.218)^2] = 0.093$$

$$G_{inelastic} = (0.093)(5.72) = 0.53$$

From alignment chart, $K = 1.16$

$$\frac{Kl}{r} = \frac{(1.16)(12)(12)}{6.29} = 26.6 < 27.5$$

∴ **W14×127 is satisfactory.**

DESIGN EXAMPLE 2

Given:
Compute the allowable concentric axial compressive load for a W14×228 column, 12 ft long. At the upper end there are two W16×88 girders 40 ft long. The column is pinned at the lower end. The column steel has a yield stress of 50 ksi. Assume that buckling is governed by major axis stiffness of the column.

Solution:

For W16×88: $I_x = 1220$ in.^{4}

For W14×228: $A = 67.1$ in.^{2}
$I_x = 2940$ in.^{4}
$r_x = 6.62$ in.

At bottom of column: $G_B = 10$

At top of column:

$$G_{elastic} = \frac{2(2940/12)}{2(1220/40)} = 8.03$$

Assume elastic buckling:

$K_x = 2.85$ (from alignment chart)

$$\frac{Kl}{r} = \frac{(2.85)(12)(12)}{6.62} = 62 \; (< C_c = 107.0)$$

Therefore, buckling will be inelastic.

For $Kl/r = 62$:

$$\alpha = 62/107 = 0.579$$

$$E_T/E = (0.579)^2\,[2 - (0.579)^2] = 0.558$$

$$G_{inelastic} = 0.558(8.03) = 4.48$$

$$K = 2.5$$

$$\frac{Kl}{r} = \frac{2.5(144)}{6.62} = 54.4$$

For $\dfrac{Kl}{r} = 52$:

$$\alpha = 52/107 = 0.486$$

$$E_T/E = (0.486)^2[2 - (0.486)^2] = 0.417$$

$$G_{inelastic} = 0.417(8.03) = 3.35$$

$$K = 2.35$$

$$\frac{Kl}{r} = \frac{2.35(144)}{6.62} = 51.1 \text{ (close enough)}$$

Say $\frac{Kl}{r} = 51$:

$$F_a = 24.2 \text{ ksi}$$

$$P_{allow} = (24.2)(67.1) = 1624 \text{ kips}$$

REFERENCES

1. *Yura, Joseph A.* The Effective Length of Columns in Unbraced Frames *AISC Engineering Journal, April 1971, pp. 37–42.*
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3. *Johnston, Bruce G., ed.* Guide to Design Criteria for Metal Compression Members *Second Edition, Column Research Council, John Wiley and Sons, 1966, p. 26.*