Approximate Analytical Model for Multistory Frames

WILLIAM J. LeMESSURIER, ROBERT J. McNAMARA, AND J. C. SCRIVENER

In THE PRELIMINARY design of tall, slender buildings, the member forces and displacements produced by the action of lateral loads on the structure are of prime importance in proportioning the members. For all but the simplest of structures, the analysis of the structure is a highly indeterminate and complex problem, and for its accurate study, in particular to find the effects of varying girder, column or bracing stiffnesses, electronic computers must be used. Complete analysis of the structural frame by computer programs is presently available.[1,2,3] However, the preparation of input data and cost of computation time for investigating alternate framing and bracing schemes may not be justified within the budget of a given project. A method of analytically modeling multistory building frames to reduce the scale of the problem is presented herein. The modeling of large structures into smaller ones enables the designer to analyze very large structures efficiently and economically on a small computer.

MODELING TECHNIQUE

Assumptions and Limitations—In deriving the equivalent model stiffnesses as outlined in Appendix A, it is assumed that the points of inflection in the prototype and model girders and columns occur at mid-span and mid-height, respectively. This assumption is reasonable for many tall frames provided the story heights and member stiffnesses of the prototype do not change appreciably within any story of the model. Further, it is necessary that the lateral loading be the same at each story in the modeled region.

Because of the above assumptions and of the desire to retain simple modeling procedures, discrepancies between prototype and model deflection profiles do occur. A detailed study of the effects of the above approximations in model vs. prototype deflection profiles is presented in Ref. 4.

Further discrepancies occur as follows:

(a) When the number of prototype stories modeled into one model story vary throughout the height of the modeled structures

(b) At the interface of modeled and unmodeled parts of the structure

(c) At the interface of braced with unbraced portions of the structure.

These discrepancies, although they may be large locally, do not detract, in general, from the overall equivalence of prototype and model deflection profiles.

Unbraced Frames—Consider a typical multistory frame as shown in Fig. 1a. This prototype frame is to be modeled as the frame shown in Fig. 1b. The modeling is accomplished by keeping the overall frame geometry fixed and lumping q stories ($q = 4$ in the example of Fig. 1) of the prototype into one model story.

To achieve the same lateral deflections of joints in model and prototype, it is necessary to provide model section properties as follows:

$$I_{mc} = q \sum_{i=1}^{q} (I_{pc})_i$$

$$A_{mc} = \frac{1}{q} \sum_{i=1}^{q} (A_{pc})_i$$

$$I_{mg} = \sum_{i=1}^{q} (I_{pg})_i$$

$$A_{mg} = \sum_{i=1}^{q} (A_{pg})_i$$

where

q = Number of stories modeled into one

I_{mc}, I_{mg} = Moments of inertia of the model column and model girder, respectively

A_{mc}, A_{mg} = Cross-sectional areas of the model column and girder, respectively

$(I_{pc})_i$, $(I_{pg})_i$ = Moments of inertia of the ith story of the q stories modeled into one for the prototype column and girder, respectively

---

William J. LeMessurier is President, LeMessurier Associates, Inc. Consulting Structural Engineers, Cambridge, Massachusetts.

Robert J. McNamara is Associate, LeMessurier Associates, Inc. Consulting Structural Engineers, Cambridge, Massachusetts.

J. C. Scrivener is Reader in Civil Engineering, Univ. of Canterbury, Christchurch, New Zealand, formerly Visiting Assoc. Prof. of Civil Engineering, Massachuseets Institute of Technology, Cambridge, Massachusetts.

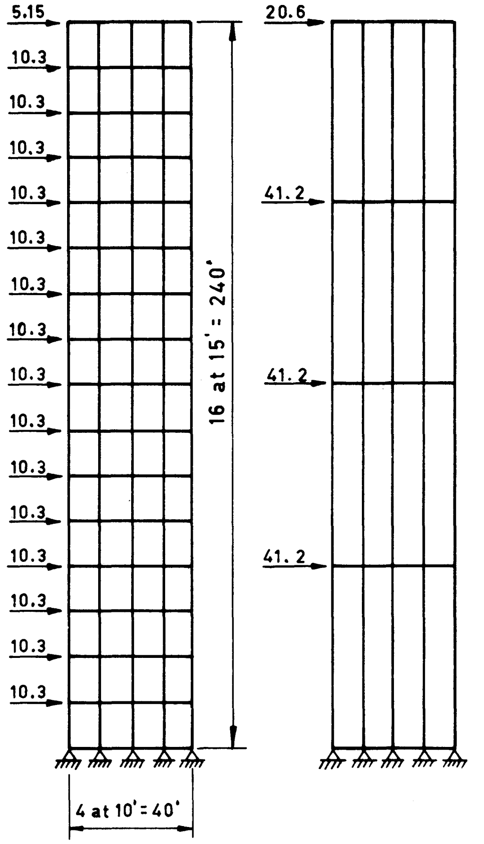


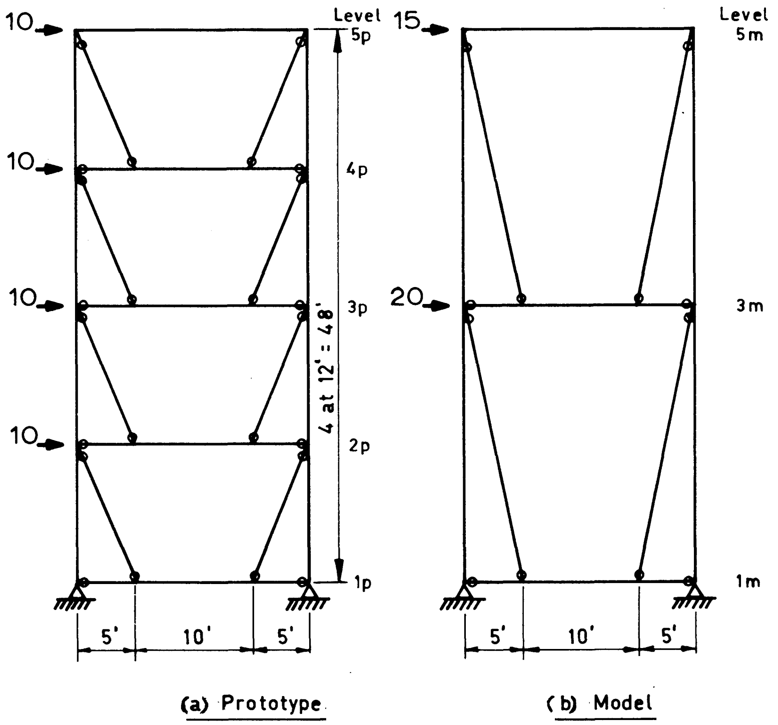
Fig. 1a-Prototype. **Fig. 1b-Model.**

Fig. 1. Unbraced prototype and model frames—Example 1

$(A_{pc})_i$, $(A_{pg})_i$ = Cross-sectional area of the ith story of the q stories modeled into one for the prototype column and girder, respectively

The model column occupies the equivalent position of the prototype columns, whereas the model girder is in the equivalent mean position of the prototype girders. When q is an even integer, the two formulas for the girders are interpreted so that the furthermost girders from the central girder being modeled will have one half of the I (or A) taken in the summation of q. For instance, with the frame of Fig. 1(a), the 1st floor model girder section properties will be obtained from the summation of the 3rd, 4th and 5th together with one half of the 2nd and 6th prototype girder properties.

Braced Frames—Consider the knee-braced frame of Fig. 2a being modeled by the frame of Fig. 2b. The braces are pin-ended and the girders are pinned to the columns. The model column and girder section properties may be found from the prototype section properties using the unbraced frame formulas above. However, because of the geometric connection of the braces to the girders, the prototype girders are "lumped down" to obtain the model girders. For instance, in the frame of Fig. 2, prototype girders $4p$ and $3p$ are modeled as model girder $3m$, prototype girders $2p$ and $1p$ are modeled as model girder $1m$, and prototype girder $5p$ retains its section properties in model girder $5m$.



(a) Prototype **(b) Model**

PROTOTYPE SECTION PROPERTIES.

All Girders & Columns

$I = 25.0 \text{ in}^4$

$A = 7.0 \text{ in}^2$

Braces

$I = 1.0 \text{ in}^4$

$A = 0.5 \text{ in}^2$

Fig. 2 Knee-braced prototype and model frames—Example 3

For K, X or knee-braced frames with pin-ended braces,

$$A_{mb} = \frac{1}{q^2}\left(\frac{L_{mb}}{L_{pb}}\right)^3 \sum_{i=1}^{q} (A_{pb})_i$$

where

A_{mb} = Cross-sectional area of the model brace

$(A_{pb})_i$ = Cross-sectional area of the prototype brace at the ith story of the q stories being modeled

L_{mb}, L_{pb} = Lengths of the model and prototype braces, respectively.

As the braces are pin-ended and not loaded along their length, the model brace moments of inertia may be chosen arbitrarily.

Lateral Loading—The lateral loads applied to the model are the loads at the model junctions which produce the same overturning moments at each level of the model as are produced by the loads on the prototype structure. This results in

$$P_m = \sum_{i=1}^{q} (P_p)_i$$

where P_m represents the model lateral load, and $(P_p)_i$ represents the prototype lateral load at the ith story of the q stories modeled into one. P_m is applied at the centroid of the $(P_p)_i$, which corresponds with the position of the model girder.

Internal Moments and Forces—The model internal moments may be used to obtain approximate prototype internal moments at equivalent positions of the frame. Thus,

$$M_{pg} = \frac{M_{mg}}{q}$$

$$M_{pc} = \frac{M_{mc}}{q}$$

where M_{pg}, M_{mg}, (M_{pc}, M_{mc}) are the internal moments in girder (column) at the same joint of prototype and model, respectively.

Further, from the girder and column axial forces at the model joints, the prototype girder and column axial forces may be estimated by:

$$N_{pg} = \frac{N_{mg}}{q}$$

$$N_{pc} = N_{mc}$$

where N_{pg}, N_{mg}, (N_{pc}, N_{mc}) are the internal axial forces in girder (column) at the same joint of prototype and model, respectively.

These formulas must be used with extreme care, as when discrepancy conditions (a), (b) or (c) apply, the errors involved may be very large. Further, the estimation of prototype moments is not particularly accurate in the uppermost and lowermost levels where the assumptions noted in Appendix A are not satisfied. Because of the change of geometry involved in the modeling of braced frames, an estimation of prototype internal brace forces can be approximated using the horizontal component of the model brace force to compute the prototype force. Additional studies of various model and prototype comparisons, using the above formulas and comparisons using modeling formulas of more complex nature, are presented in Ref. 4.

EXAMPLES

The computations of all the following examples were conducted on an IBM 1130 using STRESS (7).

Example 1—Figure 1a shows a 16-story, 4-bay frame with constant story height and uniform lateral loads. A preliminary design produced member properties varying at the different levels. The same frame with X-braces included in one of the internal bays over the complete height of the frame was also analyzed. The actual structures, both braced and unbraced, were modeled three ways, lumping 2, 4 and 8 stories, respectively, into one. Figure 1b shows the model with the 4-story lump. In Fig. 3, the deflected profile of the proto-

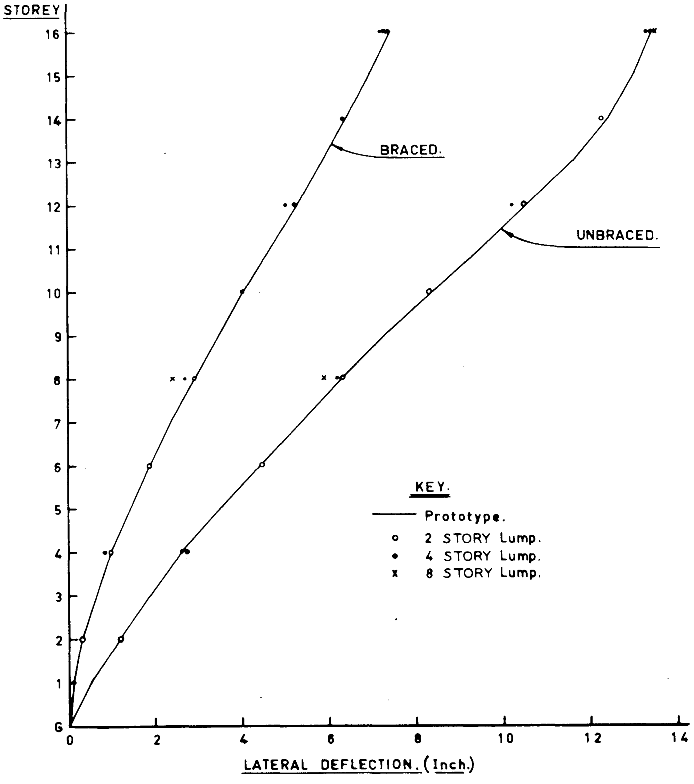


Fig. 3. Comparison of deflected profiles of prototype and models of Example 1

type frame is indicated and the plotted points represent the lateral deflection of the various floors obtained from the model analysis. These results indicate that errors introduced by modeling, even with the simplified model with 8 stories lumped into one, are very small for both the braced and unbraced situations. All of the model results are slightly unconservative.

The relative computation times of the braced frames were:

Actual 16 story structure: 10 units
Model with 2 stories lumped into 1: 4 units
Model with 4 stories lumped into 1: 2 units
Model with 8 stories lumped into 1: 1 unit

Figure 4 compares the internal girder shears in the prototype and models.

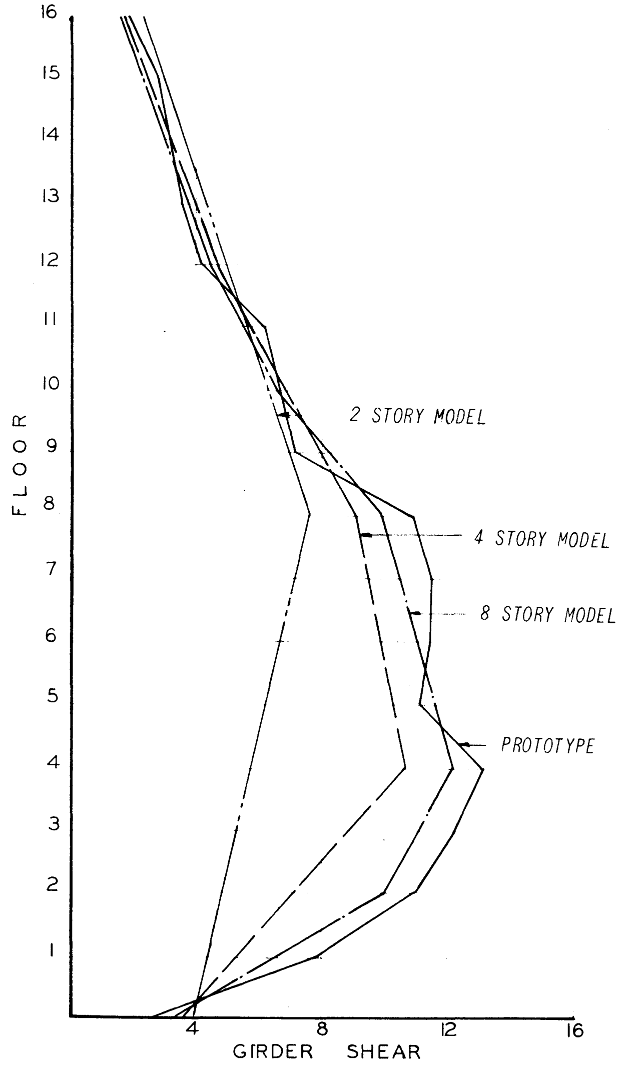


Fig. 4. Braced frame girder shear

Example 2—This example of a 39-story frame gives the results of the preliminary analysis of an office building. Lateral loads are resisted by the interaction of braced and unbraced bents. Figure 5a shows one of the four internal bents of the building with K-bracing up to the 27th level. These frame girders are non-prismatic. The remaining four bents are unbraced rigid frames with the two bents immediately outside the braced internal bents having non-prismatic girders and the two exterior bents having prismatic girders.

For the preliminary analysis, the model of an internal bent (Model 1) shown in Fig. 5b was analyzed first, using an assumed proportion of the total wind load upon it. Model 1 was then further simplified by lumping additional stories together. This simplified model (Model 2) is shown in Fig. 5c. The deflection profile (Fig. 7) indicates that this further modeling produced substantially equivalent deflections. Model 2 was then linked with similar models of the other two unbraced bents, to represent the complete structure as shown in Model 3 (Fig. 6). On the basis of the results of the linked bent analysis the distribution of total wind load to each type of bent was determined. It is at this stage that the full benefit of the modeling technique is apparent, as the three interconnected frames representing the entire structure were small enough to analyze on a small computer.

Using the proportion of wind load to be carried by each type of bent obtained from the linked bent, each prototype bent was analyzed. The accuracy of the load

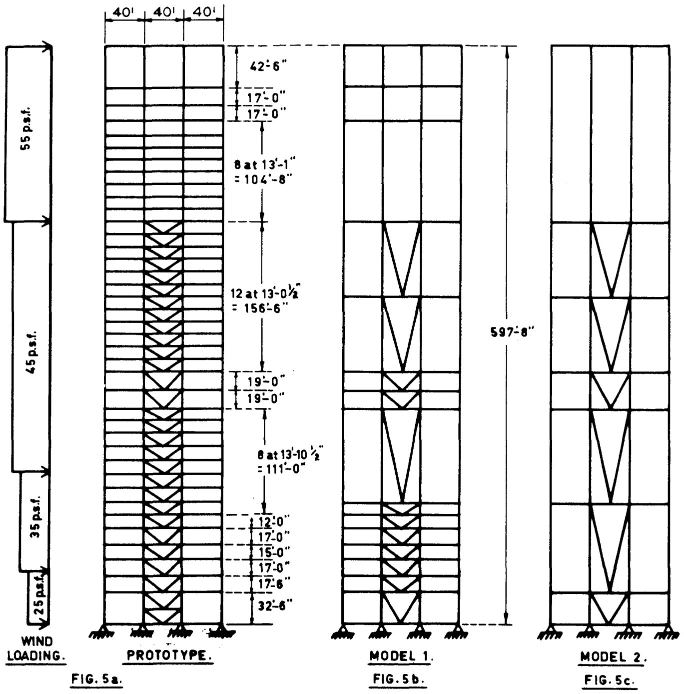


Fig. 5. 39-story frame and models

distribution was substantiated by the very similar deflections of each bent at each level. The deflected profiles are indicated in Fig. 7. The deflections are greater as the initial estimate of the proportion of wind load on the internal bent (Models 1 and 2) was lower than that obtained from the analysis.

Example 3—Figure 2 shows a 4-story knee braced frame which was modeled with 2 stories lumped into one. Comparing the lateral deflections of the top and mid-height of the prototype and model gives:

	Prototype deflection	Model deflection
Mid-height	12.7 in.	12.5 in.
Top	18.4 in.	18.6 in.

It is interesting that the model deflections are within 2% of the prototype deflections.

CONCLUSIONS

The results of Example 1, a tall regular frame, in both the X-braced and unbraced frames with the same lateral loading at each level, indicate that the models exhibit a very close approximation of prototype behavior.

The structure of Example 2 has considerable variation in the inter-story heights, member properties, and loads at each level. These conditions violate the assumptions used in deriving the model properties, but model deflections closely approximate those of the prototype. This example also shows that the modeling technique is capable of handling the transition from a K-braced portion to an unbraced portion within the one frame.

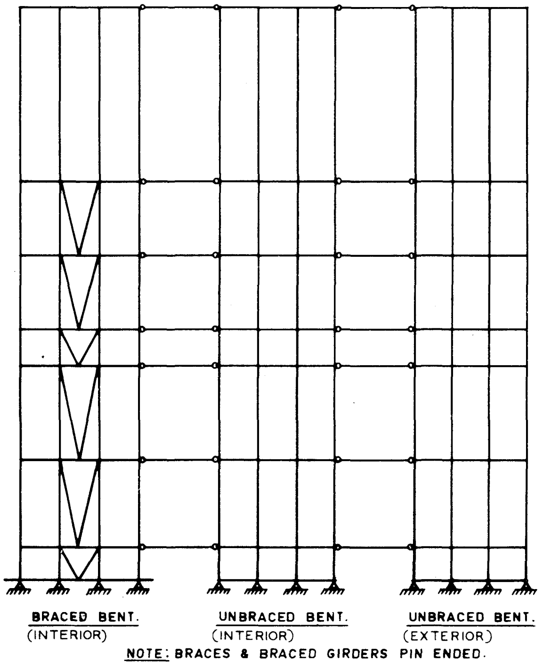


Fig. 6. Linked bents, 39-story frame—Model 3

The advantages of the modeling technique for preliminary analysis and design are listed below:

1. Less input data required and accordingly a simpler task to prepare and check the data.
2. Smaller structure to analyze, needing less computer time and often enabling the preliminary analysis to be conducted on a small computer easily accessible to the designer.

As a consequence of advantages 1 and 2, above, it becomes more economic to analyze large frames, to investigate the interaction between frames of building, and to vary member stiffnesses, frame geometry or bracing systems in order to produce an optimum design, all within the accuracy required for a preliminary design.

Three-dimensional structures, such as tube and tube-in-tube type structures composed of closely spaced columns around the perimeter interconnected with stiff spandrels, can be conveniently analyzed on a small computer with the use of the modeling techniques presented herein.

It should be emphasized that the modeling techniques presented are primarily intended for preliminary analysis, and, as in all approximate methods of analysis, must be applied carefully and with the skilled judgement of an analyst.

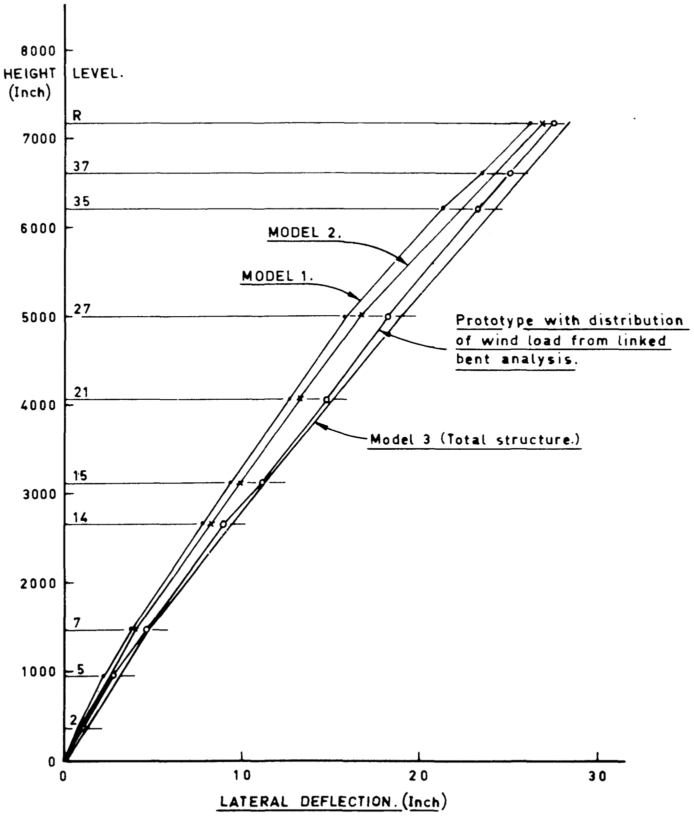


Fig. 7. Comparison of deflected profiles of prototype and models

NOMENCLATURE

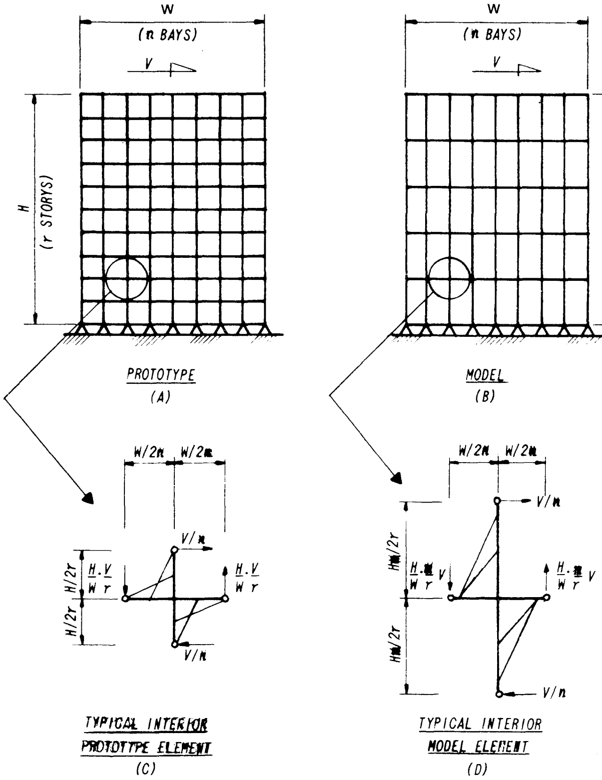
I_{mc} = Moment of inertia of model column
I_{mg} = Moment of inertia of model girder
I_{pg} = Moment of inertia of prototype girder
I_{pc} = Moment of inertia of prototype column
A_{mc} = Area of model column
A_{mg} = Area of model girder
A_{mb} = Area of model brace
A_{pc} = Area of prototype column
A_{pg} = Area of prototype girder
A_{pb} = Area of prototype brace
q = Number of stories modeled into one
L_{mb} = Length of model brace
L_{pb} = Length of prototype brace
P_m = Model load
P_p = Prototype load
M_{pg} = Moment in prototype girder
M_{mg} = Moment in model girder
M_{mc} = Moment in model column
M_{pc} = Moment in prototype column
N_{pg} = Axial force in prototype girder
N_{mg} = Axial force in model girder
N_{pc} = Axial force in prototype column
N_{mc} = Axial force in model column
m = Model ratio

APPENDIX A

The basic concept in deriving the model properties is to equate the deflections in each system. By using the principle of virtual work, the deflections of the model and prototype can be expressed in terms of the respective member properties. By equating the deflections the equivalent model properties can be obtained from the prototype. Extensive mathematical formulations can be developed for these equivalences. However, since the method is approximate, simple methods will suffice for the derivation of model properties.

Assumptions—The following assumptions are made regarding the loading and member sizes in the prototype and the behavior under load of both prototype and model over any modelled region:

1. The lateral loading at each level of the prototype is constant.
2. The story heights of the prototype are constant.
3. The prototype girder section properties are constant.
4. The prototype column section properties are constant.
5. Bending moment diagrams in both prototype and model are such that the points of contraflexure are at the midpoints of the girders and columns.



NOTES: Ipg, Ipc, Img, Imc = MOMENT OF INERTIA OF PROTOTYPE GIRDER AND COLUMN, AND MODEL GIRDER AND COLUMN RESPECTIVELY
m = MODEL RATIO

Figure 8

Unbraced Frames—Consider the prototype and model frames shown in Figs. 8a and 8b. Upon examination of the prototype elements shown in Fig. 8c and the equivalent model element in Fig. 8d, the following equation can be written for the element bending contribution to the overall deflection in each system:

$$\Delta \text{ prototype} = \frac{V}{12E}\left[\frac{H^3}{nr^3 I_{pc}} + \frac{H^2 W}{r^2 n^2 I_{pg}}\right]$$

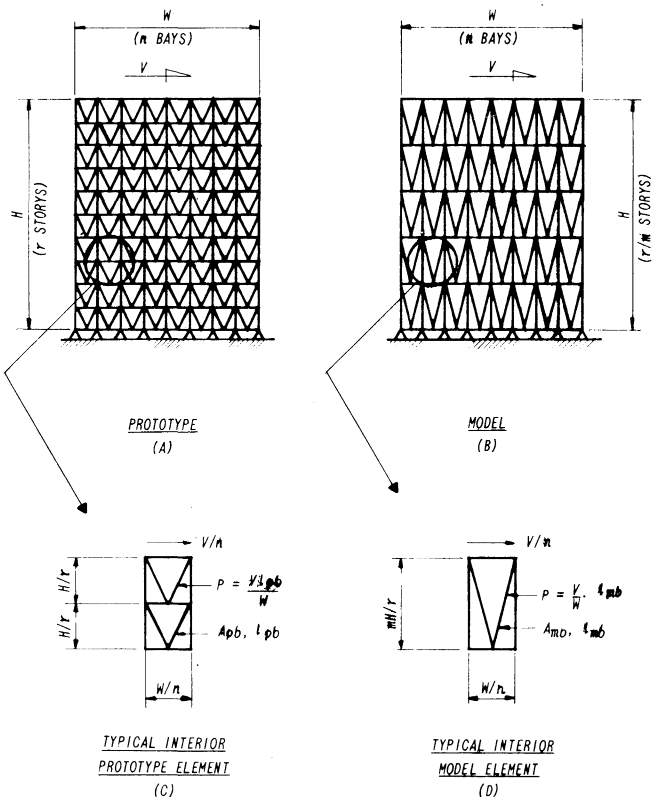
$$\Delta \text{ model} = \frac{V}{12E}\left[\frac{H^3 m^3}{nr^3 I_{mc}} + \frac{H^2 W m^2}{r^2 n^2 I_{mg}}\right]$$

By equating the expressions for the deflection and noting that the model replaces m prototype elements, the equivalent model properties are given by:

$$I_{mc} = m^2 I_{pc}$$

$$I_{mg} = m I_{pg}$$

These expressions approximate the panel (shear or wracking) distortion in the frame. As the model did not change the prototype column geometry, the distribution of axial shortening of the column is unchanged and the actual prototype column areas are used for the model column areas.



NOTE: A_{pb}, A_{mb}, l_{pb}, l_{mb} = AREA AND LENGTH OF PROTOTYPE AND MODEL BRACE, RESPECTIVELY

Figure 9

Braced Frames—By examining the prototype and model frames shown in Figs. 9a and 9b and examining the prototype and model elements as above the element contribution to the overall deflection in each system is:

$$\Delta \text{ prototype} = \frac{2VL_{pb}^3}{W^2EA_{pb}}$$

$$\Delta \text{ model} = \frac{2VL_{mb}^3}{W^2EA_{mb}}$$

Equating, and noting that the model element replaces m prototype elements,

$$A_{mb} = \frac{A_{pb}}{m}\left(\frac{L_{mb}}{L_{pb}}\right)^3$$

Similarly, the horizontal brace model area (A_{mn}) is given by:

$$A_{mp} = \frac{A_{ph}}{m}$$

where A_{ph} is the area of the horizontal prototype member.

As in the case of unbraced frames, the contribution of the column shortening is unchanged in each system and the actual prototype column area is used for the model column area.

REFERENCES

1. *Clough, R. W., E. L. Wilson, and I. P. King* Large Capacity Multistory Frame Analysis Program *Journal of the Structural Division, ASCE Vol. 89, No. ST4, Proc. Paper 3592, August 1963, pp. 179–204.*
2. *Clough, R. W., I. P. King, and E. L. Wilson,* Structural Analysis of Multistory Buildings *Journal of the Structural Division, ASCE Vol. 90, No. ST3, Proc. Paper 3925, June 1964, pp. 19–34.*
3. *Fenves, S. J., et al* STRESS: A Users Manual *MIT Press, 1964.*
4. *Smith, I. R.* Analytical Modeling of Multistory Frames *Master of Engineering Report, Dept. of Civil Engineering, University of Canterbury, Christchurch, New Zealand.*