Dynamic Analysis of Multistory Buildings

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ANALYSIS AND DESIGN of common types of buildings to resist earthquake forces is done by the method described in the Uniform Building Code. Approximate formulas for natural periods of fundamental mode of buildings are available and are used in the static method of aseismic design of Uniform Building Code. However, some building codes in the seismic regions require dynamic analysis of tall buildings and other important structures. The dynamic analysis of multistory structures is laborious and time consuming because of the many degrees of freedom involved. Modern electronic computers have made possible the dynamic analysis of multi-degree freedom systems which was prohibitive by hand calculations. However, when the number of degrees of freedom is large, the dynamic analysis requires the use of digital computers of large capacities and the computer time used can be very expensive. In the modal method of dynamic analysis, the natural periods and mode shapes for various modes of vibrations are determined by computer analysis. The objective in this paper is to develop simplified methods for determining natural periods of vibrations of multistory buildings so that expensive computer time is kept to a minimum. Since the aseismic design by dynamic analysis require a few preliminary trials before final satisfactory design is achieved, the simplified methods can save laborious computer calculations in the preliminary design. The simplified methods discussed in this paper are developed using the concept of an elastic shear wave equation in solid uniform bars. A numerical example is presented to compare the simplified methods with the more rigorous analysis.

METHODS BASED ON SHEAR WAVE EQUATION

Consider first the case of vibrations of tall buildings assuming no rotation and translation of the foundation. A tall building with fixed base is similar to a cantilever bar as regards its vibration characteristics. The horizontal deflection of the building during vibrations consists of shear-type deformations, flexural deformations, and deformations due to joint rotations. The theory of vibrations of uniform prismatic shear and flexural beams can be used to develop simplified formulas for natural periods of multistory buildings.

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SHEAR BUILDINGS

Consider first the vibrations of a uniform cantilever shear beam (Fig. 1). Let

L	=	length of cantilever beam
G	=	modulus of rigidity of material of shear beam
T_1, T_2, T_3	=	natural periods in first, second and third modes of vibrations
a	=	velocity of shear wave in the beam
γ	=	unit weight of material of beam
λ_1, λ_2, λ_3	=	wave lengths of elastic shear wave in first three modes

The cantilever shear beam has infinite degrees of freedom. In the dynamic analysis of buildings, only first, second, and third modes are of importance, which are considered here.

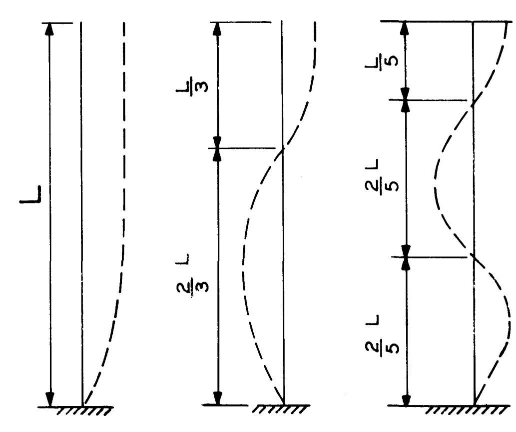

Fig. 1 Transverse vibrations of a cantilever shear beam; shapes of first three natural modes of vibration

From the theory of propagation of an elastic shear wave in long bars, the velocity of the shear wave is given by

$$a = \sqrt{\frac{Gg}{\gamma}} \tag{1}$$

The vibrations of the cantilever beam constitute a stationary wave. The natural periods of vibration of the shear beam in the first three modes are given by:

$$T_1 = \frac{\lambda_1}{a} \tag{2}$$

$$T_2 = \frac{\lambda_2}{a} \tag{3}$$

$$T_3 = \frac{\lambda_3}{a} \tag{4}$$

For vibrations in the i^{th} mode, the wave length is given by

$$\lambda_i = \frac{4L}{(2i - 1)} \tag{5}$$

The natural periods for the first three modes are therefore given by

$$T_1 = 4\frac{L}{a} \tag{6}$$

$$T_2 = \frac{4}{3}\frac{L}{a} \tag{7}$$

$$T_3 = \frac{4}{5}\frac{L}{a} \tag{8}$$

Since L/a is the time taken by the shear wave to travel from base to the top of the cantilever, Eqs. (6), (7), and (8) can be used to determine the natural periods of multistory buildings.

The building mass may be assumed to be concentrated at each floor and roof level. Now consider a multistory shear building (Fig. 2). For simplicity, first assume all stories of equal height and with the same stiffness and equal masses. Let

n = number of stories
h = height of each story
m = mass at each floor level and roof
L = total height of the building
k = stiffness of each story

When the number of stories of the building is large, the periods and mode shapes of the building are the same as an equivalent uniform catnilever bar having the same uniform rigidity as the building.

The expression for the velocity of shear wave along the height of the building can be deduced in terms of story mass, height, and stiffness.

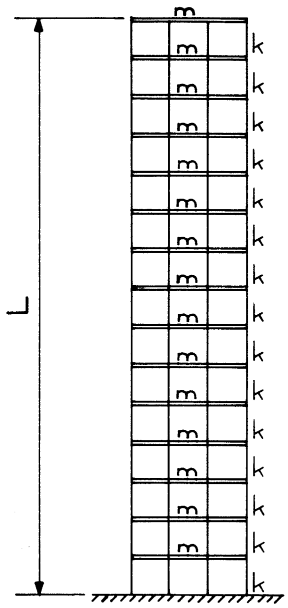

Fig. 2. Fixed base multistory shear building

Let

P = story shear
A = cross-sectional area of the building in plan
Δ = relative story deflection due to shear
γ_e = unit weight of a dynamically equivalent solid uniform bar having the same vibrational characteristics as the building
G_e = modulus of rigidity or shear modulus of material of a solid uniform bar which is dynamically equivalent to the shear building
a_e = velocity of shear wave in the building

Referring to Fig. 3, modulus of rigidity G_e is given by:

$$G_e = \frac{P/A}{\Delta/h} = \frac{Ph}{\Delta A} \tag{9}$$

Unit weight γ_e is given by:

$$\gamma_e = \frac{mg}{Ah} \tag{10}$$

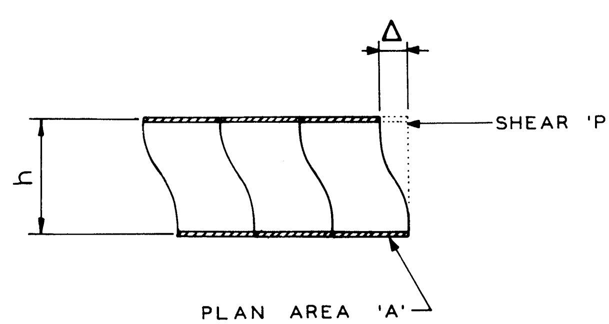

Fig. 3. Story shear and relative story deflection

The velocity of shear wave along the building is given by

$$a_e = \sqrt{\frac{G_e g}{\gamma_e}} \tag{11}$$

Substituting Eqs. (9) and (10) in Eq. (11) gives:

$$a_e = h\sqrt{\frac{P}{m\Delta}} \tag{12}$$

But P/Δ = story stiffness k.

Substituting $P/\Delta = k$ in Eq. (12) gives:

$$a_e = h\sqrt{\frac{k}{m}} \tag{13}$$

The natural periods of vibrations of the building are given by using Eqs. (6), (7), and (8). Thus, for the first three modes:

$$T_1 = 4\frac{L}{h}\sqrt{\frac{m}{k}} \tag{14}$$

$$T_2 = \frac{4}{3}\frac{L}{h}\sqrt{\frac{m}{k}} \tag{15}$$

$$T_3 = \frac{4}{5}\frac{L}{h}\sqrt{\frac{m}{k}} \tag{16}$$

Consider now the case of a shear building with stories having different heights, masses, and stiffnesses. Let

h_i = height of i^{th} story
k_i = stiffness of i^{th} story
m_i = mass of i^{th} story
a_i = shear wave velocity at i^{th} story level

Thus

$$a_i = h_i\sqrt{\frac{k_i}{m_i}} \tag{17}$$

Time taken by shear wave to travel distance h_i is given by

$$\frac{h_i}{a_i} = \sqrt{\frac{m_i}{k_i}} \tag{18}$$

Travel time of shear wave from base of building to top is given by

$$\sum_{i=1}^{n}\frac{h_i}{a_i} = \sum_{i=1}^{n}\sqrt{\frac{m_i}{k_i}} \tag{19}$$

Using the analogy of shear beam and Eqs. (6), (7), and (8), the periods of first three modes of vibrations are approximately given by:

$$T_1 = 4\sum_{i=1}^{n}\sqrt{\frac{m_i}{k_i}} \tag{20}$$

$$T_2 = \frac{4}{3}\sum_{i=1}^{n}\sqrt{\frac{mi}{k_i}} \tag{21}$$

$$T_3 = \frac{4}{5}\sum_{i=1}^{n}\sqrt{\frac{m_i}{k_i}} \tag{22}$$

Since Eqs. (20), (21), and (22) are derived by first considering an equivalent uniform shear beam, these will give exact values if all stories have equal masses and stiffnesses and will give fair approximations when story masses and stiffnesses are different.

FLEXURAL DEFLECTION

In the vibrations of tall buildings, the horizontal deflections are partly shear type and partly flexural type. It can be shown that Eqs. (20), (21), and (22) derived for shear buildings can also be applicable when deflections are due to combined shear and flexure. In this case the story stiffness k is equal to the story shear divided by relative horizontal story deflection due to both shear and flexure.

Consider first transverse vibrations of a uniform cantilever flexural beam with negligible shear deformation (Fig. 4). Let

I = moment of inertia of uniform beam section
E = Young's modulus of material of the beam
μ = mass per unit length of the beam
L = length of beam

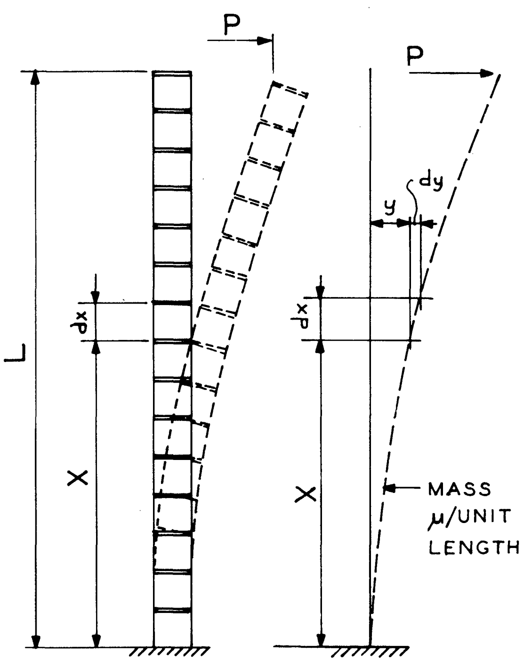

Fig. 4. Cantilever flexural building

From any textbook on structural dynamics, it is found that the natural period of first mode of transverse vibrations of the cantilever beam is given by:

$$T_1 = 1.78L^2 \sqrt{\frac{\mu}{EI}} \tag{23}$$

Consider this uniform beam to represent an ideal case of a multistory building have very rigid shear walls such that horizontal deflections of the building are of flexural type only and shear-type deflections are negligible. Also as an ideal case, assume all stories are of equal heights and have equal masses.

Consider the deflected shape of cantilever beam due to a horizontal force P at the top. Let

x = height above base of the i^{th} story
y = horizontal deflection at the i^{th} story from initial vertical position
dx = height of each story
m_i = mass of the i^{th} story
Δ_i = relative deflection of i^{th} story with respect to $(i-1)^{th}$ story
k_i = stiffness of i^{th} story as defined for flexural-type deformation
n = total number of stories

The equation of the deflected shape is given by

$$y = \frac{Px^3}{3EI} \tag{24}$$

Differentiating Eq. (24), we get the slope of the deflection curve:

$$\frac{dy}{dx} = \frac{Px^2}{EI} \tag{25}$$

$$\text{Relative story deflection } \Delta_i = dy = \frac{Px^2}{EI}dx \tag{26}$$

$$\text{Story stiffness } k_i = \frac{P}{\Delta_i} = \frac{EI}{x^2dx} \tag{27}$$

Applying the same reasoning as for a uniform shear beam representing a multistory shear building, let us assume that the natural period of first mode of the flexural building is given by:

$$T_1 = Q \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} \tag{28}$$

where Q is a constant to be evaluated.

For a uniform building of equal story heights and equal story masses;

$$m_i = \mu dx \tag{29}$$

From Eqs. (27) and (29):

$$\sqrt{\frac{m_i}{k_i}} = \sqrt{\frac{\mu}{EI}}\, xdx \tag{30}$$

By summation over the full height of the building:

$$\sum_{i=1}^{n}\sqrt{\frac{m_i}{k_i}} = \int_0^L \sqrt{\frac{\mu}{EI}}\, xdx = \frac{L^2}{2}\sqrt{\frac{\mu}{EI}} \tag{31}$$

Therefore,

$$T_1 = Q\,\frac{L^2}{2}\sqrt{\frac{\mu}{EI}} \tag{32}$$

The exact value of T_1 is given by Eq. (23).

Hence, from Eqs. (23) and (32) we can evaluate the constant Q which is found to be 3.56.

Therefore, for a multistory building with flexural-type deformation, the natural period of first mode of vibration is given by:

$$T_1 = 3.56 \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} \tag{33}$$

If the deflections in a tall building are partly of shear type and partly flexural type, which is the case in actual buildings, the natural period of first mode of vibration is given by:

$$T_1 = Q' \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} \tag{34}$$

where Q' is a constant, its value lying between 3.56 and 4.0.

Assuming deflections are 50 percent shear type and 50 percent flexural type, we have, by taking the average of Eqs. (20) and (33):

$$T_1 = \frac{(4.0 + 3.56)}{2} \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} = 3.78 \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} \tag{35}$$

For practical purposes,

$$T_1 = 4 \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} \tag{36}$$

DEFLECTION DUE TO JOINT ROTATION

In multistory buildings there is additional horizontal deflection due to joint rotations. This additional deflection is comprised of both shear-type deformation and flexural deformation. The period of fundamental mode is fairly closely given by:

$$T_1 = 4 \sum_{i=1}^{n} \sqrt{\frac{m_i}{k_i}} \tag{37}$$

where k_i is the story stiffness for deformation due to all causes, i.e., shear, flexure, and joint rotation.

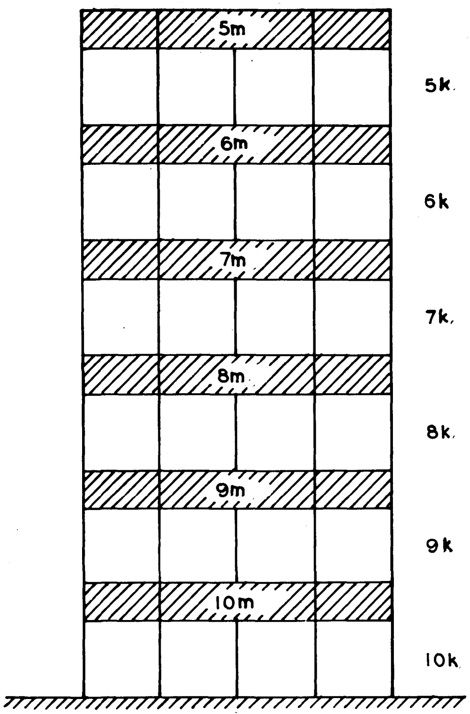

Fig. 5. Six-story shear building

The natural periods of 2nd and 3rd modes are approximately given by:

$$T_2 = \frac{4}{3}\sum_{i=1}^{n}\sqrt{\frac{m_i}{k_i}} \tag{38}$$

$$T_3 = \frac{4}{5}\sum_{i=1}^{n}\sqrt{\frac{m_i}{k_i}} \tag{39}$$

The story stiffness k_i is obtained by applying an arbitrary horizontal shear P at the top of the building and determining relative horizontal deflections Δ_i of adjacent stories. The mass and stiffness of all non-structural elements should be included to determine m_i and k_i.

FOUNDATION ROTATION AND TRANSLATION:

The simplified formulas for natural periods of multistory buildings are applicable when there is practically no rotation and translation of the foundation during vibrations. However, due to ground yielding there will be both rotation and translation of the foundation. The natural periods will be higher than those given by Eqs. (37), (38), and (39). For practical purposes, it is conservative to assume full fixity of the base of building. But if the field investigation of the foundation material indicate appreciable rotation and translation of the foundation, the fundamental periods due to foundation rotation and translation can be determined separately from the knowledge of soil properties, building mass and mass moment of inertia. These periods can be combined with the fundamental period obtained from Eq. (37) by the Southwell-Dunkerley equation, which states that the sum of squares of the natural periods of isolated systems is approximately equal to the square of the natural period of the combined system.

NUMERICAL EXAMPLE

The usefulness of the simplified methods developed in this paper is illustrated by their application to a numerical example. The example chosen is a six-story shear building. The solution of the example problem by more rigorous methods is given on pages 414–415 of Ref. 5.

Example: (Fig. 5)

A six-story shear building with fully-fixed foundation has story masses from top down equal to $5m$, $6m$, $7m$, $8m$, $9m$, and $10m$. The shear rigidities between stories are equal to $5k$, $6k$, $7k$, $8k$, $9k$, and $10k$.

Find the first and second mode natural frequencies of the building.

Using Eqs (20) and (21):

$$\text{First mode period } T_1 = 4\left(\sqrt{\frac{10m}{10k}} + \sqrt{\frac{9m}{9k}} + \sqrt{\frac{8m}{8k}} + \sqrt{\frac{7m}{7k}} + \sqrt{\frac{6m}{6k}} + \sqrt{\frac{5m}{5k}}\right)$$

$$= 24\sqrt{\frac{m}{k}}$$

$$\text{Circular frequency } \omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{24}\sqrt{\frac{k}{m}} = 0.262\sqrt{\frac{k}{m}}$$

radians/sec.

$$\text{Second mode period } T_2 = \frac{24}{3}\sqrt{\frac{m}{k}} = 8\sqrt{\frac{m}{k}} \text{ seconds}$$

$$\text{Circular frequency } \omega_2 = \frac{2\pi}{8}\sqrt{\frac{k}{m}} = 0.785\sqrt{\frac{k}{m}}$$

radians/sec.

The solution by rigorous methods in Ref. 5 gives the following circular frequencies:

$$\omega_1 = 0.276\sqrt{\frac{k}{m}} \text{ and } \omega_2 = 0.71\sqrt{\frac{k}{m}} \text{ radians/sec.}$$

CONCLUSIONS

The simplified methods for determining natural periods of vibrations of multistory buildings presented in this paper can be used in earthquake resistant design or for design to resist blast and shock loading. Also, these are useful for vibrational analysis of buildings and structures supporting unbalanced rotating machinery. Because of the simple calculations involved, these approximate methods can be easily programmed for a digital computer, thus keeping expensive computer time to a minimum. The numerical example checks the accuracy of the simplified formulas. These are found to be sufficiently accurate for practical purposes. They are more accurate for the first mode of vibration and only approximate for the second and third modes.

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