Lateral Instability of Castellated Beams

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CASTELLATED BEAMS are sometimes used to satisfy aesthetic, economic, and functional aspects of design in structures. Several analytical and experimental studies pertaining to stress distribution and flexural strength of castellated beams have been undertaken in recent years. In the conventional method of analysis, the flexural behavior of these structural members is considered to be similar to that of Vierendeel frames. Consequently, an "exact" analysis for castellated beams under such idealization is difficult and, as a result, a very simplified approach identical to the "cantilever method" in the analysis of multistory frames is adopted for practical design problems. Hence, in castellated beams the horizontal members, consisting of tee portions and vertical members known as web posts, are assumed to contain points of contraflexure at their centers. This behavior of the castellated beam in the manner of a Vierendeel frame has been established in several investigations,[1–4] and is being used in practice[5] for flexural analysis.

A problem of lateral instability may be associated with castellated beams, as with other flexural members comprised of open sections; its solution is necessary to determine the critical span lengths and the spacing of lateral bracing on a rational basis. In general, the design criterion against lateral buckling is accentuated in the presence of weak torsional resistance of the cross section and when the flexural rigidity of the beam in its plane of loading is considerably larger than that in the lateral direction. These conditions are inherent with castellated beams and the possibility of lateral buckling is greater for these open-web beams in comparison with conventional rolled steel beams.

In this paper, a general method of lateral instability analysis for castellated beams will be presented along with applications to certain particular cases.

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NOMENCLATURE

A_1,A_2	Coefficients of deflection functions
A_T	Area of the tee section
b	Width of flange
b_1	Width of web post along the mid-depth longitudinal axis
b_2	Maximum width of web post
c	Constant defined by the ratio, b_1/b_2
C_0	Geometrical parameter of the web post
d	Depth of castellated beam
E	Modulus of elasticity
G	Shear modulus of elasticity
h_1	Effective length of web post, the distance between centers of gravity of flange tee-sections
I_1	Moment of inertia of tee section
I_2	Moment of inertia of the web post at its maximum width
J_1	Torsional constant of the tee section
J_2	Torsional constant of the web post at its maximum width
k_{11},k_{12}	Dimensionless parameters of the stiffness coefficients
K_{11},K_{12}	Stiffness coefficients of the web post
L	Effective span
M_0	End moments of the web post
$M(x)$	Bending moment at distance x
n	Number of buckling modes
N	Total number of web posts in the span length
r_t	Radius of gyration of the tee section
t_f	Thickness of flange
t_w	Thickness of web
u	Lateral displacement of the mid-depth longitudinal axis
U	Total strain energy
U_1	Bending strain energy of the tee portions
U_2	Twisting strain energy of the tee portions
U_3	Twisting strain energy of the web posts
U_4	Bending strain energy of the web posts
U_w	Potential energy of the external forces
U_{w_1}	Potential energy of the uniformly distributed load

U_{w_2}	Potential energy of the concentrated load at mid-span
w	Uniformly distributed load
w_{cr}	Critical uniformly distributed load for lateral buckling
$\overline{w}_{cr}$	Dimensionless critical load parameter
$w(x)$	Rate of loading at distance x
W	Concentrated load at mid-span
W_{cr}	Critical load at mid-span for lateral buckling
$\overline{W}_{cr}$	Dimensionless critical load parameter
x,y	Rectangular coordinates
α_1,α_2	Coefficients associated with the critical load
β,ϕ	Angular deformations due to lateral buckling
ϕ_0	Dimensionless parameter of angular deformation
δ	Dirac's delta function
μ_1, μ_2	Geometrical parameters of the web

THEORETICAL ANALYSIS

Because of several complications involved in the geometry of the beam with discontinuities in its web, energy principles will be used to solve the lateral stability problem. In some earlier studies on castellated beams,[4,6,7] the experimental observations on those specimens which reached their ultimate capacity due to lateral torsional buckling provide a qualitative insight into the nature of their buckling mode. At the laterally buckled configuration, castellated beams exhibited a smooth, continuous geometrical profile over their entire span length.

The conventional elastic lateral stability theory of rolled steel beams[8] assumes that the section does not change its shape due to buckling. This is particularly suited for W-shape members which are rather stocky. However, if the section components are slender, the "Goodier-Barton effect"[9] becomes significant and the deformation of web takes place as a consequence of the warping torsional moment. Because rolled shapes are expanded to create them, castellated beams are usually relatively deep and slender. Further, the reduction of web area due to the presence of openings enhances the flexibility. Thus, the extent of web deformation may be large, depending upon the properties of cross section, and must be taken into account in the buckling mode to be used in this analytical solution.

In view of the foregoing considerations, the following displacements satisfying geometric and static boundary conditions are used for the case of a castellated beam with simply supported ends.

$$\phi = \phi_0 \sin (n\pi x/L) \tag{1a}$$

$$u = A_1 \phi_0 \sin (n\pi x/L) \tag{1b}$$

$$\beta = A_2 \phi_0 \sin (n\pi x/L) \tag{1c}$$

In Eqs. (1a)–(1c), L is the effective span, n is the number of buckling modes over the span length, ϕ_0 is an undetermined dimensionless parameter, and the displacement components ϕ, u, and β of the laterally buckled configuration, as shown in Fig. 2, are the angle of rotation of the centroidal axis, lateral displacement and the angle of twist of the tee portions respectively. The unknown coefficients, A_1 (having the dimension of length) and A_2 (which is dimensionless) are to be evaluated.

Each of the displacements in Eqs. (1a)–(1c) contains a single term of the Fourier series expansion of the respective displacement function. Other terms are not considered in recognition of the fact that, both in elastic[10] and in elastic-plastic buckling[11] of rolled steel beams, consideration of only a single term in the displacement function introduces negligible error in the final result. Also in the case of light weight trusses,[12] the assumption of single term displacement functions has been made.

The effects of residual stresses, which are intractable at present for castellated beams, and of local yielding due to stress concentration at the corners are neglected in this analysis. Further, it is assumed that any other type of buckling, including the out-of-plane instability of the web posts, does not take place before the critical load due to general lateral buckling is reached. In the laterally buckled configuration, the external loads are considered as remaining parallel to their original direction. Following the conclusion of Kolosowski from his studies on flexural behavior of castellated beams,[3] the common parts of tee portions and web posts are included in both components; and with two distinct parts hatched in different styles as shown in Fig. 1, the entire region of the beam has been included in the analysis.

Strain Energy in the Beam—The change of strain energy in the state of lateral instability is reckoned from the vertical equilibrium position of the beam under the critical load. The two distinct regions consisting of the tee portions and of the web posts are laterally bent and twisted, which results in the total change of strain energy.

The strain energy stored due to lateral bending of both top and bottom tee portions is given by

$$U_1 = \frac{EI_1}{2}\int_0^L \left[\frac{d^2\left(u + \frac{h_1}{2}\phi\right)}{dx^2}\right]^2 dx + \frac{EI_1}{2}\int_0^L \left[\frac{d^2\left(u - \frac{h_1}{2}\phi\right)}{dx^2}\right]^2 dx = \frac{EI_1}{2}\frac{\phi_0^2 n^4\pi^4}{L^3}\left(A_1^2 + \frac{h_1^2}{4}\right) \tag{2a}$$

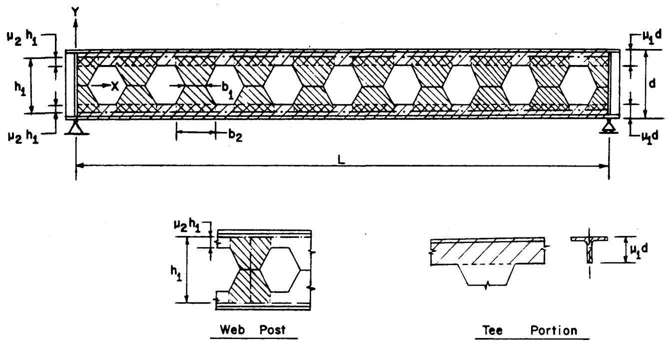

Fig. 1. Details of the castellated beam

where E is the modulus of elasticity, I_1 is the moment of inertia of each tee section about the vertical axis of symmetry, and h_1 denotes the effective length of web post as shown in Fig. 1.

In the twisting of these tee portions, the strain energy stored is

$$U_2 = 2\,\frac{GJ_1}{2}\int_0^L \left(\frac{d\beta}{dx}\right)^2 dx = \frac{GJ_1}{2}\,\frac{A_2^2\phi_0^2 n^2\pi^2}{L} \qquad \text{(2b)}$$

in which J_1 represents St. Venant's torsion constant of the tee section and G stands for the shear modulus.

Now, considering a web post at distance x from the origin, the strain energy due to twisting of this web post is

$$\frac{1}{2}\int_{-h_1/2}^{+h_1/2} GJ(y)\left[\frac{1}{h_1}\left\{\frac{d\left(u+\frac{h_1}{2}\phi\right)}{dx} - \frac{d\left(u-\frac{h_1}{2}\phi\right)}{dx}\right\}\right]^2 dy = \frac{1}{2}\,G\left(\frac{d\phi}{dx}\right)^2\int_{-h_1/2}^{+h_1/2} J(y)dy$$

in which $J(y)$ is the St. Venant's torsion constant of the non-prismatic web post at distance y from the longitudinal axis. The quantity

$$\int_{-h_1/2}^{+h_1/2} J(y)dy = \int_{-h_1/2}^{-(h_1/2)+\mu_2 h_1} \frac{b_2 t_w^3}{3}\,dy + \int_{-(h_1/2)+\mu_2 h_1}^{0}\left[b_2 - \frac{(b_2-b_1)\left(y+\frac{h_1}{2}-\mu_2 h_1\right)}{\frac{h_1}{2}-\mu_2 h_1}\right]\frac{t_w^3}{3}\,dy + \int_0^{(h_1/2)-\mu_2 h_1}\left[b_2 + \frac{(b_2-b_1)\left(y-\frac{h_1}{2}+\mu_2 h_1\right)}{\frac{h_1}{2}-\mu_2 h_1}\right]\frac{t_w^3}{3}\,dy + \int_{(h_1/2)-\mu_2 h_1}^{(h_1/2)} \frac{b_2 t_w^3}{3}\,dy = J_2 h_1 C_0$$

where,

$$J_2 = \frac{b_2 t_w^3}{3}$$

$$C_0 = \frac{(1+2\mu_2)+c(1-2\mu_2)}{2}$$

$$c = b_1/b_2$$

The notations b_1, b_2 are minimum and maximum widths of web post, t_w is the thickness of web and μ_2 is a coefficient as shown in Fig. 1. Thus, the strain energy due to twisting of the web post at distance x from the origin is:

$$\frac{1}{2}\,GJ_2 h_1 C_0\left(\frac{d\phi}{dx}\right)^2$$

If there are N numbers of web posts in the span length of L of the castellated beam, then the total strain energy stored because of twisting of all the posts is:

$$U_3 = \frac{1}{2}\,GJ_2 h_1 C_0\int_0^L \frac{N}{L}\left(\frac{d\phi}{dx}\right)^2 dx = \frac{1}{4}\,\frac{\phi_0^2 n^2\pi^2 N C_0}{L^2}\,GJ_2 h_1 \qquad \text{(2c)}$$

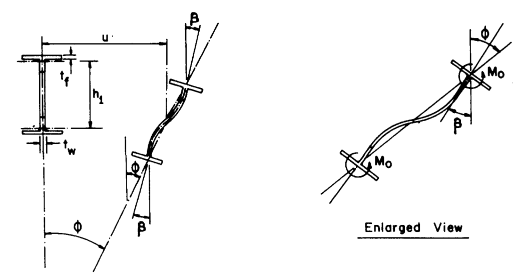

Fig. 2. Lateral displacements and geometry of cross section

Again, considering the web post at distance x, the two equal end moments M_0, as illustrated in Fig. 2, are determined from the slope-deflection equation for non-prismatic members. Thus:

$$M_0 = -\left[K_{11}\beta + K_{12}\beta - \frac{\left(u+\frac{h_1}{2}\phi\right)-\left(u-\frac{h_1}{2}\phi\right)}{h_1}(K_{11}+K_{12})\right]$$

If the dimensionless quantities are denoted by

$$k_{11} = \frac{K_{11}}{\left(\frac{EI_2}{h_1}\right)} \quad \text{and} \quad k_{12} = \frac{K_{12}}{\left(\frac{EI_2}{h_1}\right)}$$

with $$I_2 = \frac{b_2 t_w^3}{12}$$

then the end moment becomes

$$M_0 = (k_{11} + k_{12})(\phi - \beta)EI_2/h_1$$

Thus, the strain energy due to bending of this web post is:

$$(k_{11} + k_{12})(\phi - \beta)^2 \frac{EI_2}{h_1}$$

Considering all the N web-posts in the unsupported span length L, the strain energy stored as a result of bending of the web posts is given by

$$U_4 = \int_0^L \frac{N}{L}\frac{EI_2}{h_1}(k_{11} + k_{12})(\phi - \beta)^2 dx = \frac{1}{2}\frac{EI_2}{h_1} N\phi_0^2(k_{11} + k_{12})(1 - A_2)^2 \quad (2d)$$

The sum of all four expressions, Eqs. (2a) through (2d), represented by

$$U = U_1 + U_2 + U_3 + U_4 \quad (3)$$

constitutes the total energy stored in the laterally buckled configuration of the castellated beam.

Potential Energy of the External Forces—In a practical application, the external load may be applied on the top flange, along the centroidal axis, or may be suspended from the bottom flange of the beam. Accordingly, the potential energy due to external forces is given[8]:

$$U_w = \int_0^L M(x)\phi\left(\frac{d^2u}{dx^2}\right)dx \pm \frac{d}{4}\int_0^L w(x)\phi^2 dx \quad (4)$$

where, $M(x)$ is the bending moment at a distance x from the left support due to externally applied load $w(x)$. The second term of the above energy expression vanishes when the load acts along the centroidal axis of the beam and has a negative or positive sign when the load is applied on the top flange or suspended from the bottom flange, respectively. Without any intermediate lateral support, two types of externally applied loads are considered.

(a) Uniformly Distributed Load—In this case $w(x) = w$, constant over the span. Substituting in Eq. (4), the potential energy of the applied load becomes

$$U_{w_1} = -w\phi_0^2 L\left(\frac{A_1 n^2\pi^2}{24} + \frac{A_1}{8} \pm \frac{d}{8}\right) \quad (5a)$$

(b) Concentrated Load W at the Center—In this case, $w(x) = W\delta[x - (L/2)]$, where δ denotes Dirac's delta function of the expression in brackets which follows the δ. Therefore, using Eq. (4),

$$U_{w_2} = -W\phi_0^2\left[\frac{A_1 n^2\pi^2}{16} - \frac{A_1 \cos n\pi}{8} + \frac{A_1}{8} \pm \frac{d}{4}\right] \quad (5b)$$

In both of the cases considered, the critical or smallest load at instability corresponds to the first buckling mode, so that $n = 1$ should be used in Eqs. (5a) and (5b). Here it may be added that, when lateral supports are provided at equal intervals and the applied concentrated forces act at those same locations, there will be no tipping or stabilizing effect. The potential energy of the external forces will be contributed by only the first term of Eq. (4), in which the integration must be performed at appropriate intervals and the number of buckling modes n will be interpreted accordingly.

Determination of the Critical Load—In the laterally buckled configuration, the beam is in a state of neutral equilibrium, so that the total potential energy is zero. Thus:

$$U + U_w = 0 \quad (6)$$

The undetermined dimensionless parameter ϕ_0 is eliminated in the above equation. Considering uniformly distributed load, the minimum value of w in Eq. (6) is the critical load w_{cr}. In order that w be minimum, the necessary conditions are $\partial w/\partial A_1 = 0$ and $\partial w/\partial A_2 = 0$. With these relations, A_1 and A_2 are obtained from Eq. (6) as follows:

$$A_1 = \frac{wL^4}{EI_1\pi^4}\left(\frac{\pi^2}{24} + \frac{1}{8}\right) \quad (7)$$

$$A_2 = \frac{EN(k_{11} + k_{12})I_2/h_1}{EN(k_{11} + k_{12})\frac{I_2}{h_1} + GJ_1\pi^2/L} \quad (8)$$

A_1 and A_2 as obtained above, satisfy the sufficient condition for the minimum value of w, which is w_{cr}. Defining the dimensionless quantities:

$$\bar{w}_{cr} = \frac{w_{cr}L^3}{EI_1} \quad (9)$$

$$\alpha_1 = \frac{d}{8L} \quad (10)$$

$$\alpha_2 = \left[\left(\frac{h_1}{L}\right)^2 \frac{\pi^4}{8} + \frac{GJ_1}{EI_1}\frac{A_2^2\pi^2}{2} + \frac{h_1}{L}\frac{GJ_2}{EI_1}\frac{\pi^2 NC_0}{4} + \frac{I_2 L}{2I_1 h_1} N(1 - A_2)^2(k_{11} + k_{12})\right] \quad (11)$$

and substituting the values of A_1 and A_2 from Eqs. (7) and (8), the solution of Eq. (6) leads to:

$$\bar{w}_{cr} = \begin{cases} \left[-\alpha_1 + \left\{\alpha_1^2 + 2\left(\dfrac{1}{24} + \dfrac{1}{8\pi^2}\right)^2 \alpha_2\right\}^{1/2}\right] \Bigg/ \left(\dfrac{1}{24} + \dfrac{1}{8\pi^2}\right)^2 & \text{(12a)} \\ (2\alpha_2)^{1/2} \Bigg/ \left(\dfrac{1}{24} + \dfrac{1}{8\pi^2}\right) & \text{(12b)} \\ \left[\alpha_1 + \left\{\alpha_1^2 + 2\left(\dfrac{1}{24} + \dfrac{1}{8\pi^2}\right) \alpha_2\right\}^{1/2}\right] \Bigg/ \left(\dfrac{1}{24} + \dfrac{1}{8\pi^2}\right)^2 & \text{(12c)} \end{cases}$$

depending on whether the uniformly distributed load w is applied on the top flange, along the centroidal axis, or under the bottom flange respectively.

Similarly, considering the concentrated load at mid-span the critical load W_{cr} is expressed through the dimensionless quantity

$$\overline{W}_{cr} = \frac{W_{cr}L^2}{EI_1} \tag{13}$$

and the solution of Eq. (6) is obtained as

$$\overline{W}_{cr} = \begin{cases} \left[-2\alpha_1 + \left\{4\alpha_1^2 + 2\left(\dfrac{1}{16} + \dfrac{1}{4\pi^2}\right)^2 \alpha_2\right\}^{1/2}\right] \Bigg/ \left(\dfrac{1}{16} + \dfrac{1}{4\pi^2}\right)^2 & \text{(14a)} \\ (2\alpha_2)^{1/2} \Bigg/ \left(\dfrac{1}{16} + \dfrac{1}{4\pi^2}\right) & \text{(14b)} \\ \left[2\alpha_1 + \left\{4\alpha_1^2 + 2\left(\dfrac{1}{16} + \dfrac{1}{4\pi^2}\right)^2 \alpha_2\right\}^{1/2}\right] \Bigg/ \left(\dfrac{1}{16} + \dfrac{1}{4\pi^2}\right)^2 & \text{(14c)} \end{cases}$$

Equations (14a), (14b), and (14c) pertain to the mid-span concentrated load applied on the top flange, at the centroidal axis, and under the bottom flange, respectively.

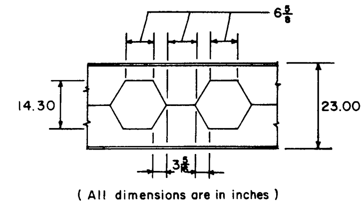

Fig. 3. Specimen 1: Original beam W16×36, *expanded to 23.0× 36-3*

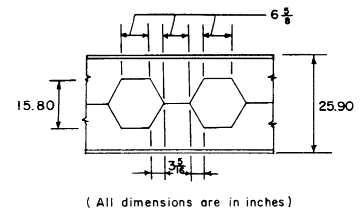

Fig. 4. Specimen 2: Original beam W18×50, *expanded to 25.9× 50-3*

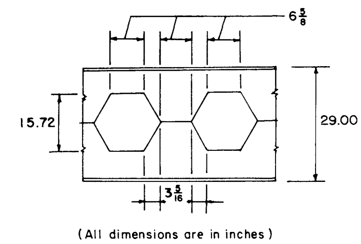

Fig. 5. Specimen 3: Original beam W21×96, *expanded to 29.0× 96-3*

NUMERICAL EXAMPLES

Three different types of commercially available castellated beams are considered for numerical illustration. The details of openings are shown in Figs. 3, 4, and 5 and the necessary geometrical properties are tabulated in Table 1.

The quantity $r_t = \sqrt{I_1/A_T}$, where A_T is the area of each tee section. In all the specimens, the spacing of web posts is 19.875 in. center to center, and $c = 0.5$. For the uniformly distributed load, the values of $\bar{w}_{cr}$ are

Table 1

Specimen number	Original Depth	Expanded Depth d	h_1	μ_2	I_1	J_1	r_t	I_2	J_2
1	15.85	23.00	21.35	0.165	11.04	0.263	1.628	0.0295	0.118
2	18.00	25.90	23.95	0.170	18.58	0.609	1.778	0.0507	0.202
3	21.14	29.00	26.21	0.200	54.58	3.181	2.157	0.2099	0.839

Note: All the dimensional quantities are in inch units.

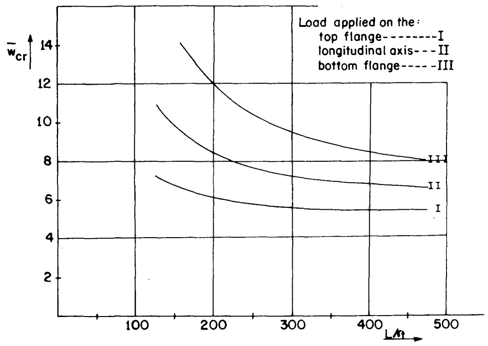

Fig. 6 Stability curves of the castellated beam 23×36-3

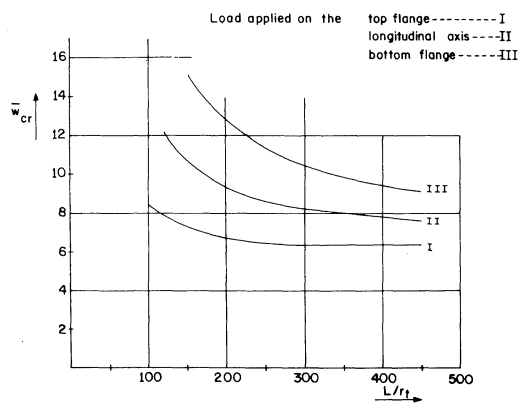

Fig. 7. Stability curves of the castellated beam 25.9×50-3

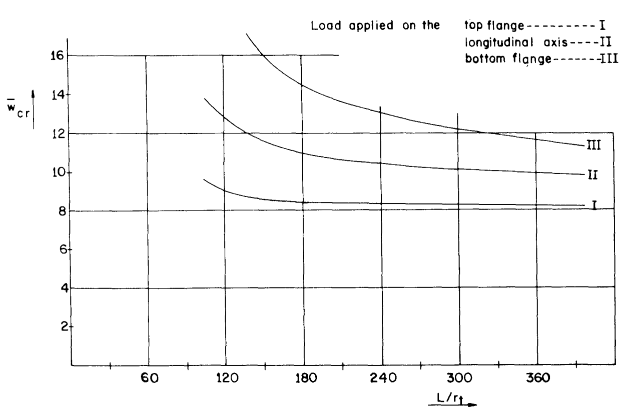

Fig. 8 Stability curves of the castellated beam 29.0×96-3

calculated from Eqs. (12a), (12b), and (12c) for different values of L/r_t, and the stability characteristic curves are plotted in Figs. 6, 7, and 8.

SUMMARY AND CONCLUSION

The study reported herein deals with the lateral instability analysis of castellated beams using the principle of minimum potential energy. Several experiments available from the literature provide information for selecting the deformed configuration which satisfies the conditions of both statics and geometry at the boundary; certain idealizations are then made to characterize the geometrical features necessary for the purpose of analysis. The sets of Eqs. (12) and (14) provide the solution for lateral buckling of castellated beams under uniform load and concentrated load at mid-span, respectively. These equations include the solution for the external load when it is applied on the top flange, at the mid-depth longitudinal axis, or suspended from the bottom flange. It is observed that the critical load depends upon several parameters representing the geometrical features of castellated beams. In the case of commercially available products, the relations between critical loads and the governing parameters can be plotted with the help of the derived formulas for convenient use in practical design. The values of k_{11} and k_{12} can be suitably obtained by the method of Column Analogy frequently used for determining the stiffness influence coefficients of non-prismatic members. For $c = \frac{1}{2}$, as in many applications, the values of k_{11} and k_{12} are plotted in Figs. 9 and 10 by using the method of Column Analogy. Even though several assumptions are made in the analysis, the method of solu-

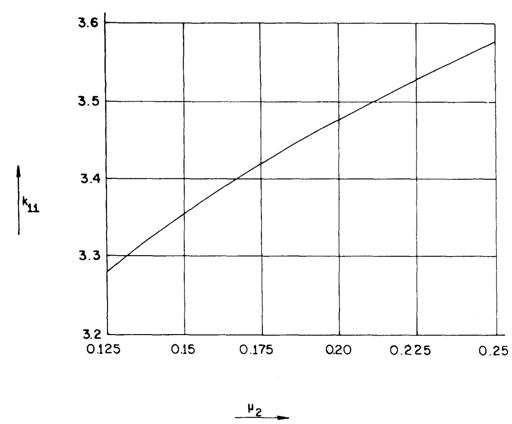

Fig. 9. Data for k_{11}, dimensionless parameter of the stiffness coefficient K_{11}

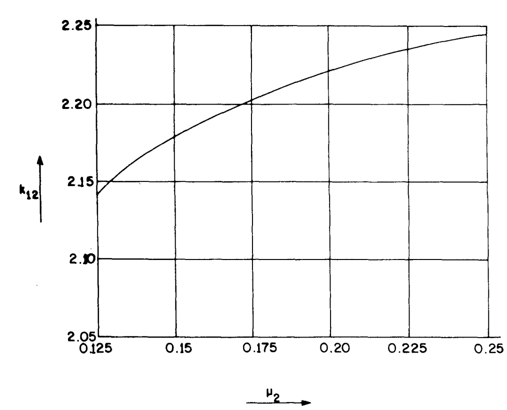

Fig. 10. Data for k_{12}, dimensionless parameter of the stiffness coefficient K_{12}

tion presented herein is by far the best for castellated beams because of its simplicity and its closed-form presentation of the answers in precise expressions. Although all types of boundary conditions have not been treated in this analysis, the method is general. Corresponding modifications which would be necessary for different loading or support conditions have been outlined in the analysis.

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