Calculation of Effective Lengths and Effective Slenderness Ratios of Stepped Columns

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THE ANALYSIS of stepped columns arises in the design of heavy mill buildings. Such columns are generally loaded at the top and at the section where the cross section changes. The application of the AISC Specification requires that the engineer determine the effective length of each section of the column. This is a problem in elastic stability theory, and the results are dependent not only on the end fixities, but also on the ratio of the end axial load to the intermediate axial load, the ratio of the length of the upper segment to the length of the lower segment, and the ratio of the upper moment of inertia to the lower moment of inertia.

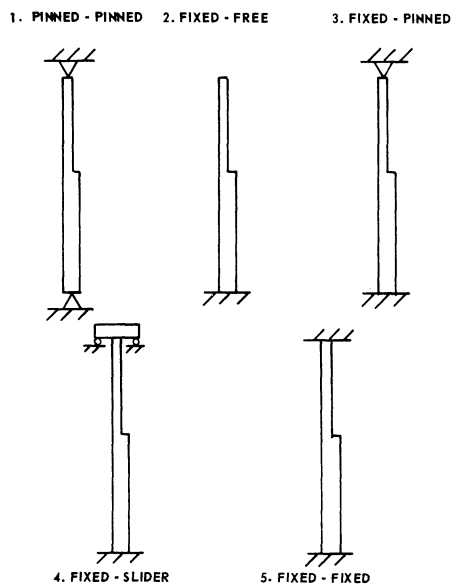

Fig. 1. End fixity conditions considered

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In this paper the authors have extended the analysis to three cases of practical interest, not previously considered. Consistent nondimensional characteristic equations are given for all five cases shown in Fig. 1.

The characteristic equations are complicated transcendental equations that must be solved for the effective lengths. Because of the complexity of the equations, the authors believe that this problem is especially suited for solution on a digital computer.

A comprehensive flow chart for a computer program which calculates effective lengths for any of the five cases is included. The program features simplified input and low running time, and is particularly well suited for a time-sharing computer. In the authors' experience it has proved to be a convenient and accurate engineering tool.

EFFECTIVE LENGTHS OF STEPPED COLUMNS

To illustrate the method of analysis, the *fixed-slider case* will be presented in detail. Let E be constant for upper and lower segments; and within each segment, assume the axial load and the cross section geometry do not vary. The terms I_1 and I_2 represent the moments of

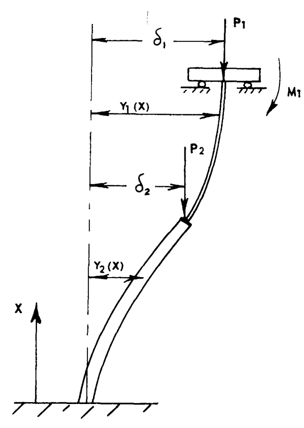


Fig. 2. Parameters for solution of fixed-slider case

inertia of the upper and lower portions, respectively. The load applied at the top is P_1, and at the step (or crane rail) is P_2. The upper length is l_1, and l_2 is the length of the lower segment. The total length is denoted by l.

During buckling the slider will translate; however, the column is not allowed to rotate at that boundary. Let M_1 be the moment developed at the top, δ_1 the slider translation, and δ_2 the deflection at the step. These parameters along with the coordinate system are illustrated in Fig. 2.

The moment distribution in the top portion is

$$M(X) = M_1 + P_1[\delta_1 - y_1(X)]; \qquad l_2 \leq X \leq l$$

and in the lower portion is

$$M(X) = M_1 + P_1[\delta_1 - y_2(X)] + P_2[\delta_2 - y_2(X)]; \qquad 0 \leq X \leq l_2$$

Since $EIy'' = M(X)$, the governing differential equations are

$$\begin{aligned} y_1'' + \gamma_1^2 y_1 &= \frac{M_1 + P_{1\delta 1}}{EI_1}; \qquad l_2 \leq X \leq l \\ y_2'' + \gamma_2^2 y_2 &= \frac{M_1 + P_{1\delta 1} + P_{2\delta 2}}{EI_2}; \qquad 0 \leq X \leq l_2 \end{aligned} \tag{1}$$

where

$$\gamma_1^2 = \frac{P_1}{EI_1}; \qquad \gamma_2^2 = \frac{P_1 + P_2}{EI_2} \tag{2}$$

The boundary conditions are

$$\begin{aligned} y_1(l) &= \delta_1 \\ y_1'(l) &= 0 \\ y_1(l_2) &= \delta_2 \\ y_2(l_2) &= \delta_2 \\ y_1'(l_2) &= y_2'(l_2) \\ y_2(0) &= 0 \\ y_2(0) &= 0 \end{aligned} \tag{3}$$

Solution of the differential equations and evaluation of the constants of integration so that the boundary conditions are satisfied eventually leads to the following characteristic equation:

$$\frac{l_1}{l_2}\cdot\gamma_2 l_2\cdot\cos(\gamma_1 l_1)\sin(\gamma_2 l_2) + \left(1 + \frac{P_2}{P_1}\right)\cdot\gamma_1 l_1\cdot\sin(\gamma_1 l_1)\cos(\gamma_2 l_2) = 0 \tag{4}$$

Now let

$$\gamma_1 l_1 = Z \tag{5}$$

and from Eq. (2),

$$\begin{aligned} \gamma_2 l_2 &= \gamma_1 l_1\cdot\frac{l_2}{l_1}\cdot\sqrt{\frac{I_1}{I_2}\left(1 + \frac{P_2}{P_1}\right)} \\ &= Z\beta \end{aligned} \tag{6}$$

where

$$\beta = \frac{l_2}{l_1}\cdot\sqrt{\frac{I_1}{I_2}\left(1 + \frac{P_2}{P_1}\right)} \tag{7}$$

Eq. (4) can now be written as

$$\frac{l_1}{l_2}\cdot\beta\cdot\cos Z\sin(Z\beta) + \left(1 + \frac{P_2}{P_1}\right)\sin Z\cos(Z\beta) = 0 \tag{8}$$

For given length, load, and moment of inertia ratios, the lowest root (say Z_{rt}) of this equation must be found. The computer program uses an iteration scheme to determine Z_{rt}.

Z_{rt} is the lowest value of $\gamma_1 l_1$ for which buckling can occur. From Eq. (6) the *corresponding* value of $\gamma_2 l_2$ is $Z_{rt}\cdot\beta$. Using Eq. (2), we see that at *buckling*

$$\begin{aligned} P_1 &= \left(\frac{Z_{rt}}{l_1}\right)^2\cdot EI_1 \\ P_1 + P_2 &= \left(\frac{Z_{rt}\cdot\beta}{l_2}\right)^2\cdot EI_2 \end{aligned} \tag{9}$$

Suppose we now define the effective lengths of the upper and lower segments (KL_1 and KL_2) to be values such that at *buckling*,

$$\begin{aligned} P_1 &= \frac{\pi^2 EI_1}{(KL_1)^2} \\ P_1 + P_2 &= \frac{\pi^2 EI_2}{(KL_2)^2} \end{aligned} \tag{10}$$

In terms of the root Z_{rt} of the characteristic equation, the effective lengths are

$$\begin{aligned} KL_1 &= \pi l_1/Z_{rt} \\ KL_2 &= \pi l_2/(Z_{rt}\cdot\beta) \end{aligned} \tag{11}$$

These are the effective lengths that must be inserted into Eqs. (1.5-1) or (1.5-2) of the AISC Specification in order to obtain the allowable stresses in the upper and lower segments.

The concept of buckling load is sometimes difficult to grasp for a column subjected to more than a single end load. With a stepped column, for example, there are two loads applied. One interpretation is to assume the ratio of P_2 to P_1 to be fixed, and gradually increase P_1. Because the load ratio and geometry are fixed, the only parameter which changes in the characteristic Eq. (8) is Z, since

$$Z = \gamma_1 l_1 = l_1\sqrt{\frac{P_1}{EI_1}}$$

The column buckles at the lowest value of Z (hence, P_1) for which Eq. (8) is satisfied. The corresponding P_2 is then found from the specified load ratio. Then the ef-

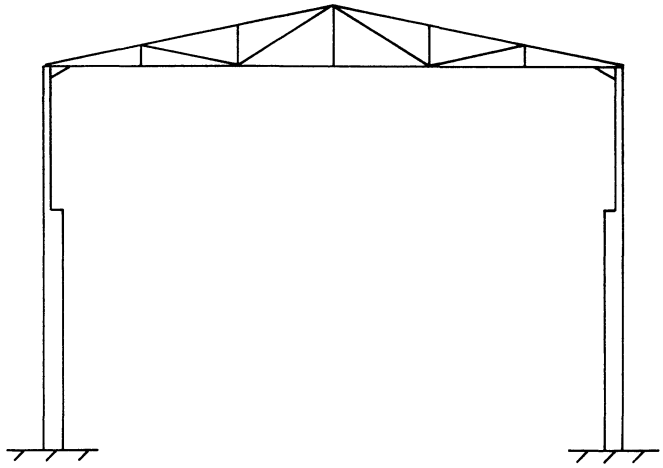


Fig. 3. Representative cross section of a heavy mill building

fective lengths of the upper and lower segments can be found using Eq. (10).

Appendix A contains the characteristic equations for each of the columns illustrated in Fig. 1.

SELECTION OF END FIXITIES

Close consideration must be given to the end fixities in an actual column, since they strongly influence the value of the effective length and, hence, effective slenderness ratio.

The program based on this analysis will accommodate five different sets of end conditions, all of which can be found in structural columns. For crane columns in a mill building, however, either the fixed-pinned case or the fixed-slider case would normally be selected.

A typical cross section of a heavy mill building is shown in Fig. 3. The first decision is whether or not the top of the column can undergo sidesway. The sidesway of importance is that resulting from vertical loads only, not wind. That is, as a result of a large, vertical crane load at a single column, will it tend to buckle so that the top translates horizontally, or will this translation be prevented, as would be the case in a long building with columns tied together as illustrated in Fig. 4.

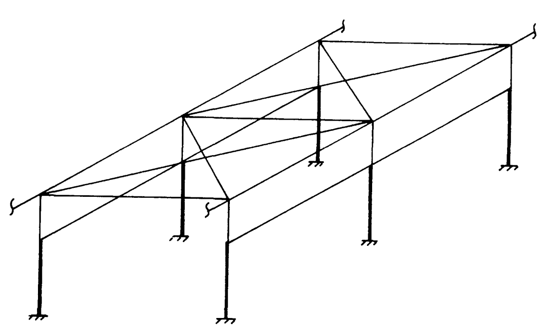


Fig. 4.

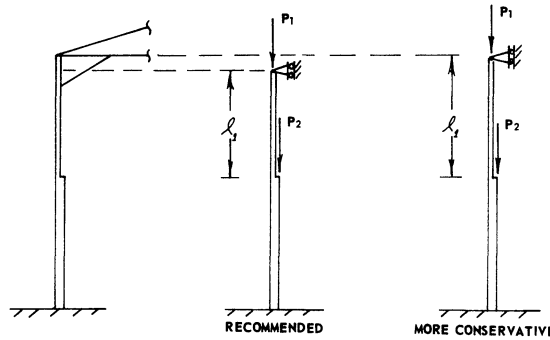


Fig. 5. Top prevented from translating during buckling

With lower chord roof bracing, a single column is prevented from translating by the other building columns. However, in a short building or if there were no roof bracing, buckling of a single column would be accompanied by sidesway.

When the top of the column is braced, another problem is to determine what length to take for the upper segment. This was discussed by Murray and Graham[1] in relation to finding the moment distribution in a stepped column subjected to a lateral loading. Even though the buckling problem is basically different, much of their discussion concerning end fixities is applicable.

For a column braced at the top and prevented from translating, it is recommended that the top be assumed pinned midway between the knee-brace and the bottom chord of the truss. A more conservative procedure is to ignore the knee-brace and assume the column is pinned at the bottom chord.

For a column braced at the top, but, for such reasons as those stated above, permitted to translate during buckling, it is recommended that the fixity at the top be modeled by a slider located at the bottom of the knee-brace. If placed at the bottom of the truss, a more conservative design would result.

These recommendations are illustrated in Figs. 5 and 6.

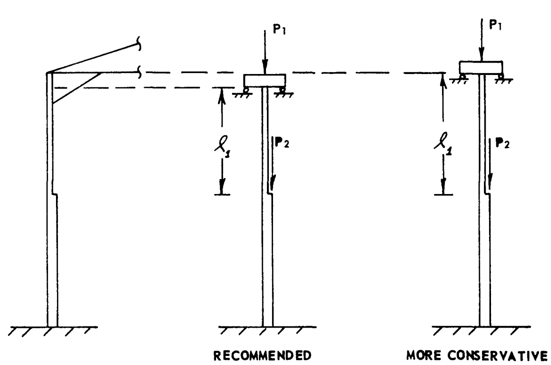


Fig. 6. Top permitted to translate during buckling

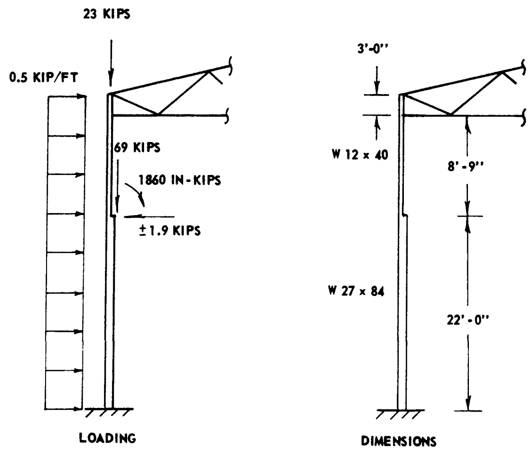


Fig. 7. Example

EXAMPLE

A crane column in a multi-bay building is subjected to the loading shown in Fig. 7. The building has roof bracing between columns.

Problem: For both upper and lower segments, determine the first term in the interaction formula for combined axial compression and bending, Eq. (1.6-1a) of the AISC Specification:

$$\frac{f_a}{F_a} + \frac{C_{mx} f_{bx}}{\left(1 - \frac{f_a}{F_{ex}'}\right) F_{bx}} + \frac{C_{my} f_{by}}{\left(1 - \frac{f_a}{F_{ey}'}\right) F_{by}} \leq 1.0$$

Solution: Since the structure has roof bracing, assume the top is pinned, with the pin midway between the lower and upper chords of the roof truss. The first step is to find the effective lengths of both upper and lower segments. *Only the vertical loads enter into this calculation.* Figure 8 summarizes the required data.

For the authors' time-sharing program, the first input line is simply the number of problems to be solved. This is followed by an input line for each prob-

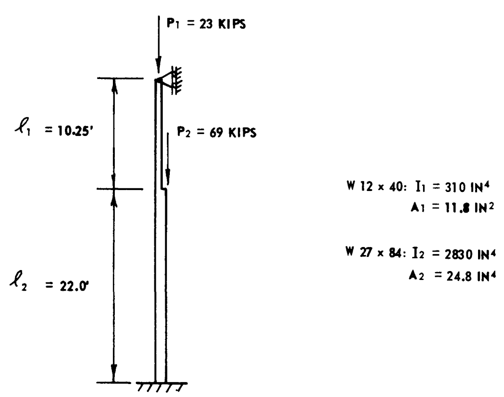


Fig. 8. Required input data for solution of example

lem to be solved, containing in order the parametei P_1, P_2, l_1, l_2, I_1, I_2, A_1, A_2, EFC. The last term, EFC, i the end fixity code for that particular problem and ha a value of 1 through 5.

The input data for the example under consideratio is:

```
100  1
110  23, 69, 10.25, 22, 310, 2830, 11.8, 24.8, 3
```

A complete listing of the output follows:

```
STEPCO              11:45 G              05/14/71
******PROBLEM NO.— 1
P1 =  23.000      P2 =  69.000       (KIPS)
L1 =  10.250      L2 =  22.000       (FEET)
I1 =     310.000  I2 =    2830.000   (INCHES**4
A1 = 11.80        A2 = 24.80         (INCHES**2
END FIXITY CODE = 3
   THE EFFECTIVE LENGTHS FOLLOW:
      KL1 = 19.243     KL2 = 29.070     (FEET
   THE EFFECTIVE SLENDERNESS RATIOS
    FOLLOW:
      UPPER KL/R  = 45.05
      LOWER KL/R = 32.66
PROGRAM STOP AT 550
USED     2.34 UNITS
```

Upper segment:

$$Kl/r = 45.05 < C_c = 126.1$$

The allowable stress can be determined using Ec (1.5-1) or Table 1-36 of the AISC Specification. Eithe gives

$$F_a = 18.78 \text{ ksi}$$
$$f_a = P_1/A_1 = 23/11.8 = 1.95 \text{ ksi}$$
$$\therefore f_a/F_a = 1.95/18.78 = \mathbf{0.104}$$

Lower segment:

$$Kl/r = 32.66 < C_c$$

From Table 1-36,

$$F_a = 19.75 \text{ ksi}$$
$$f_a = (P_1 + P_2)/A_2 = 92/24.8 = 3.71 \text{ ksi}$$
$$f_a/F_a = 3.71/19.75 = \mathbf{0.188}$$

The design check would be completed by determin ing the contribution due to bending in Eq. (1.6-1) c the AISC Specification, and making the unity check.

Note: If the W12×40 and W27×84 are found to b unacceptable and different sections are to be tried, th program must be run again to find the new effectiv lengths and effective slenderness ratios. These quantitie are dependent on the moment of inertia ratio, the lengtl ratio, and the load ratio, as well as the end fixities.

REFERENCES

1. *Murray, J. J. and Graham, Thomas C.* The Design of Mill Buildings *Iron and Steel Engineers, Feb. 1957*, pp. 159–172.
2. *Timoshenko, S. P. and Gere, J. M.* Theory of Elastic Stability, *McGraw-Hill Book Company, Inc., 1961, pp. 98–114.*
3. *Dalal, Suresh T.* Some Non-Conventional Cases of Column Design *Engineering Journal, AISC, January, 1969, pp. 28–29.*
4. *Huang, H. C.* Determination of Slenderness Ratios for Design of Heavy Mill Building Stepped Columns *Iron and Steel Engineering Yearbook, 1968, pp. 773–784.*

APPENDICES

A. Summary of Characteristic Equations—A summary of the characteristic equations for the stepped columns with the various end fixities shown in Fig. 1 is given in this appendix. An extensive search revealed only two of the cases presented. Those equations not specifically referenced were derived by the authors. All of the equations have been rewritten so that the notation is consistent throughout, and all equations are nondimensionalized.

In most cases the characteristic equations for the loadings of $P_1 = 0$ or $P_2 = 0$ are not readily obtained by reduction of the general equation. For this reason, the special cases of either the top or intermediate load being equal to zero are given separately.

The parameters in the equations have the following definitions:

$$\mathrm{IR} = I_1/I_2 \qquad \mathrm{Z} = \gamma_1 l_1$$
$$\mathrm{LR} = l_1/l_2 \qquad \mathrm{ZB} = \gamma_2 l_2$$
$$\mathrm{PR} = P_1/P_2$$

Case 1—Pinned-Pinned:

a. General (Refs. 2 and 3):

$$(1 + \mathrm{PR})(1 + \mathrm{PR} + \mathrm{PR/LR})\cdot \mathrm{Z}\cdot \mathrm{COS(Z)\ SIN(ZB)} + \mathrm{PR}[\mathrm{PR}(1 + \mathrm{LR}) + \mathrm{LR}]\cdot \mathrm{ZB}\cdot \mathrm{SIN(Z)}\cdot \mathrm{COS(ZB)} - \mathrm{SIN(Z)\ SIN(ZB)} = 0 \tag{A-1}$$

b. $P_1 = 0$:

$$[2 + 1/\mathrm{LR} - (\mathrm{LR}\cdot \mathrm{ZB})^2/(3\cdot \mathrm{IR})]\ \mathrm{SIN(ZB)} + \mathrm{LR}\cdot \mathrm{ZB}\cdot \mathrm{COS(ZB)} = 0 \tag{A-2}$$

c. $P_2 = 0$

$$\mathrm{ZB}\cdot \mathrm{SIN(Z)\ COS(ZB)} + (\mathrm{Z/LR})\cdot \mathrm{COS(Z)}\cdot \mathrm{SIN(ZB)} = 0 \tag{A-3}$$

Case 2—Fixed-Free:

a. General:

$$(1 + \mathrm{PR})\cdot \mathrm{Z}\cdot \mathrm{COS(Z)\ COS(ZB)} - \mathrm{LR}\cdot \mathrm{PR}\cdot \mathrm{ZB}\cdot \mathrm{SIN(Z)\ SIN(ZB)} = 0 \tag{A-4}$$

b. $P_1 = 0$:

$$\mathrm{COS(ZB)} = 0 \tag{A-5}$$

c. $P_2 = 0$ (Ref. 2):

$$\mathrm{ZB}\cdot \mathrm{SIN(Z)\ SIN(ZB)} - (\mathrm{Z/LR})\cdot \mathrm{COS(Z)}\cdot \mathrm{COS(ZB)} = 0 \tag{A-6}$$

Case 3—Fixed-Pinned:

a. General (Ref. 4):

$$2\cdot \mathrm{ZB}\cdot \mathrm{SIN(Z)} + (\mathrm{PR} + 1/\mathrm{PR})\cdot \mathrm{ZB}\cdot \mathrm{SIN(Z)}\cdot \mathrm{COS(ZB)} + [\mathrm{PR} + \mathrm{LR}(1 + \mathrm{PR})]\cdot (\mathrm{ZB})^2\cdot \mathrm{SIN(Z)}\cdot \mathrm{SIN(ZB)} - (1 + 1/\mathrm{PR})(1 + \mathrm{PR} + \mathrm{PR/LR})\cdot \mathrm{Z}\cdot \mathrm{ZB}\cdot \mathrm{COS(Z)\ COS(ZB)} + (1 + \mathrm{PR})\cdot (\mathrm{Z/LR})\cdot \mathrm{COS(Z)\ SIN(ZB)} = 0 \tag{A-7}$$

b. $P_1 = 0$:

$$2 + [\mathrm{LR}\cdot \mathrm{ZB} + 1/(\mathrm{LR}\cdot \mathrm{ZB})]\ \mathrm{SIN(ZB)} - [2 + 1/\mathrm{LR} - (\mathrm{LR}\cdot \mathrm{ZB})^2/(3\cdot \mathrm{IR})]\ \mathrm{COS(ZB)} = 0 \tag{A-8}$$

c. $P_2 = 0$:

$$\mathrm{ZB}\cdot \mathrm{SIN(Z)\ COS(ZB)} + (1 + \mathrm{LR})\cdot (\mathrm{ZB})^2\cdot \mathrm{SIN(Z)}\cdot \mathrm{SIN(ZB)} + (\mathrm{Z/LR})\cdot \mathrm{COS(Z)\ SIN(ZB)} - (1 + 1/\mathrm{LR})\cdot \mathrm{Z}\cdot \mathrm{ZB}\cdot \mathrm{COS(Z)\ COS(ZB)} = 0 \tag{A-9}$$

Case 4—Fixed-Slider:

a. General:

$$\mathrm{LR}\cdot \mathrm{ZB}\cdot \mathrm{COS(Z)\ SIN(ZB)} + (1 + 1/\mathrm{PR})\cdot \mathrm{Z}\cdot \mathrm{SIN(Z)\ COS(ZB)} = 0 \tag{A-10}$$

b. $P_1 = 0$:

$$\mathrm{SIN(ZB)} + (\mathrm{LR}\cdot \mathrm{ZB/IR})\ \mathrm{COS(ZB)} = 0 \tag{A-11}$$

c. $P_2 = 0$:

$$\mathrm{LR}\cdot \mathrm{ZB}\cdot \mathrm{COS(Z)\ SIN(ZB)} + \mathrm{Z}\cdot \mathrm{SIN(Z)\ COS(ZB)} = 0 \tag{A-12}$$

Case 5. Fixed-Fixed:

a. General:

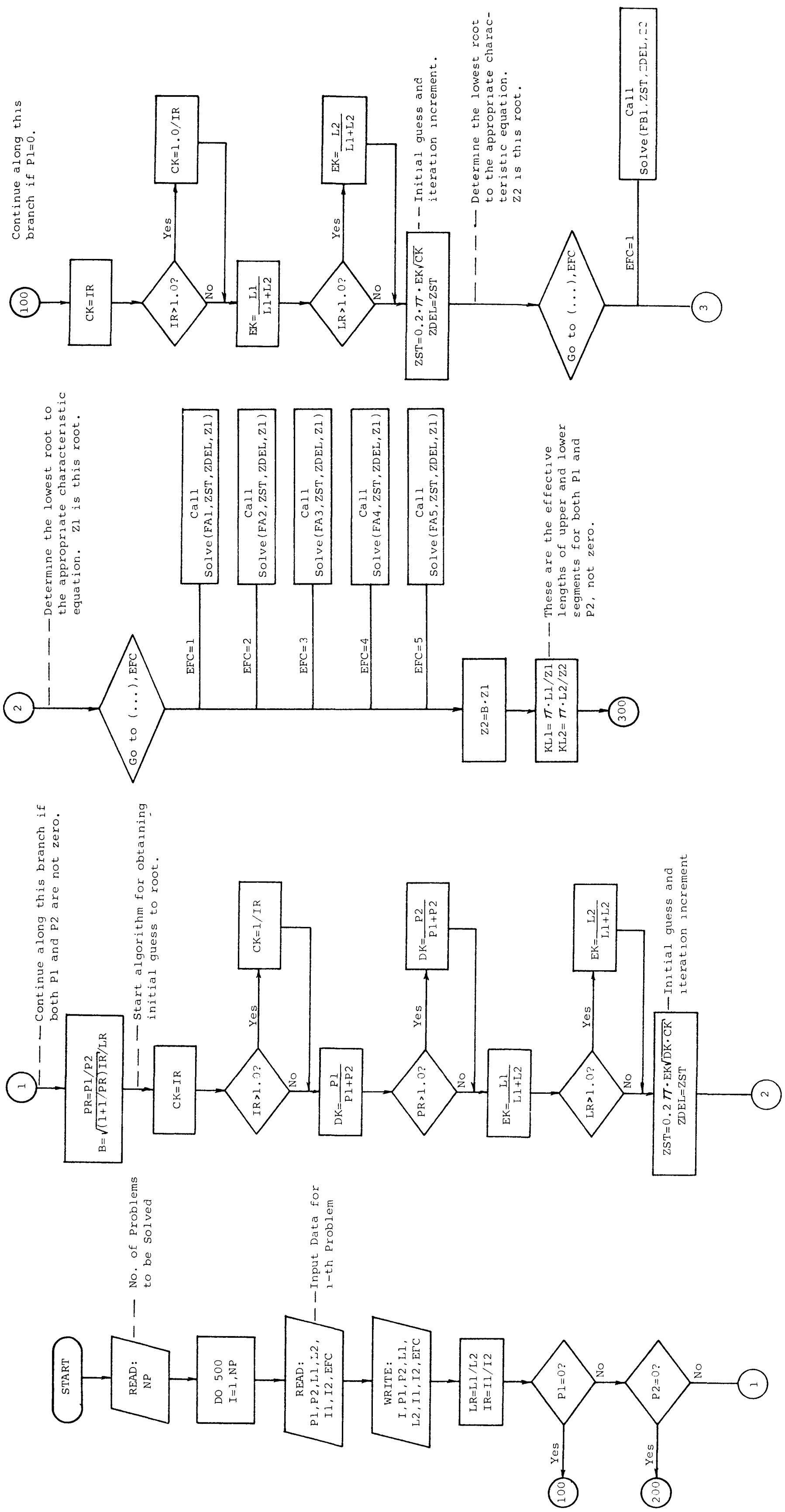
$$-2(1 + \mathrm{PR})\cdot \mathrm{Z}\cdot \mathrm{ZB} + 2\cdot \mathrm{Z}\cdot \mathrm{ZB}\cdot \mathrm{COS(Z)} - 2(1 + 1/\mathrm{PR})\cdot \mathrm{Z}\cdot \mathrm{ZB}\cdot \mathrm{COS(ZB)} + 2(1 + \mathrm{PR} + 1/\mathrm{PR})\cdot \mathrm{Z}\cdot \mathrm{ZB}\cdot \mathrm{COS(Z)\ COS(ZB)} - (1 + \mathrm{PR})[(\mathrm{Z})^2/\mathrm{LR} + (\mathrm{ZB})^2\cdot \mathrm{LR}]\ \mathrm{SIN(Z)}\cdot \mathrm{SIN(ZB)} + [\mathrm{PR} + (1 + \mathrm{PR})\cdot \mathrm{LR}]\cdot \mathrm{Z}\cdot (\mathrm{ZB})^2\cdot \mathrm{COS(Z)\ SIN(ZB)} + (1 + 1/\mathrm{PR})\cdot (1 + \mathrm{PR} + \mathrm{PR/LR})\cdot (\mathrm{Z})^2\cdot \mathrm{ZB}\cdot \mathrm{SIN(Z)\ COS(ZB)} = 0 \tag{A-13}$$

b. $P_1 = 0$:

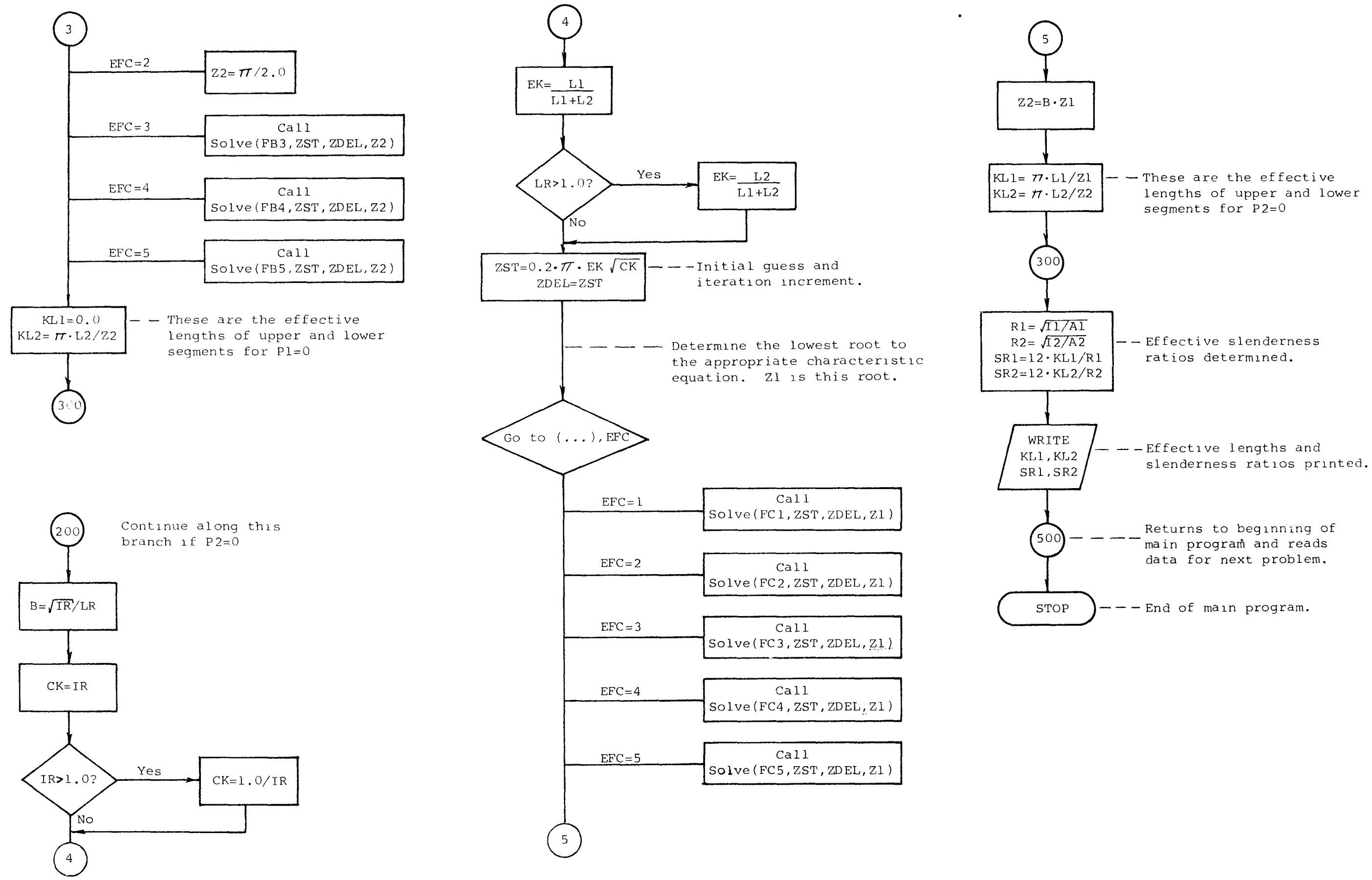
$$2 + (\mathrm{LR}\cdot \mathrm{ZB})^2/\mathrm{IR} - [2 - (\mathrm{LR}\cdot \mathrm{ZB})^4/(12\cdot \mathrm{IR}^2) + \mathrm{LR}\cdot (1 + \mathrm{LR})\cdot (\mathrm{ZB})^2/\mathrm{IR}]\ \mathrm{COS(ZB)} + [-\mathrm{ZB} + (\mathrm{LR}\cdot \mathrm{ZB})^3/(3\cdot \mathrm{IR}) + \mathrm{LR}\cdot \mathrm{ZB/IR}]\cdot \mathrm{SIN(ZB)} = 0 \tag{A-14}$$

c. $P_2 = 0$:

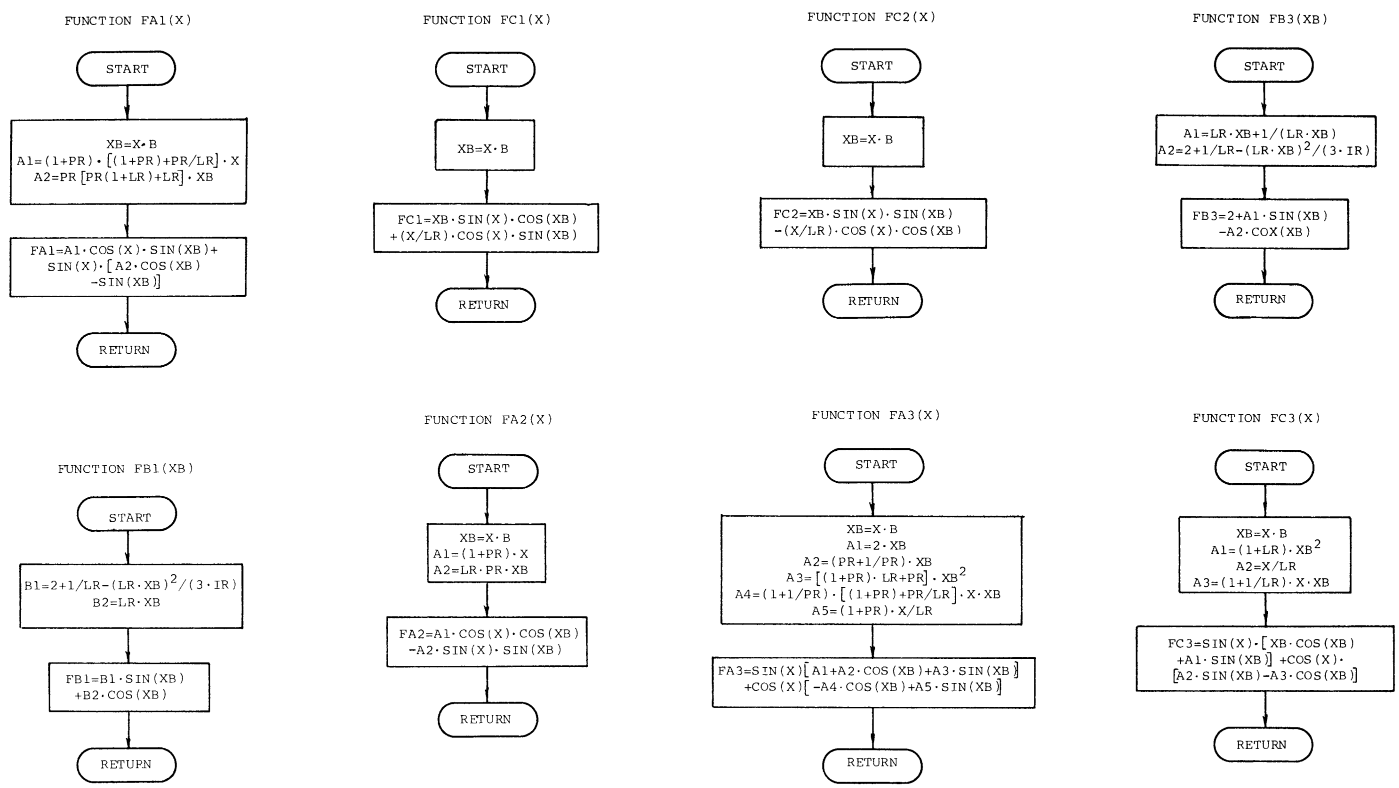
$$2\cdot \mathrm{Z} - 2\cdot \mathrm{Z}\cdot \mathrm{COS(Z)}\cdot \mathrm{COS(ZB)} + \mathrm{ZB}\cdot (1 + \mathrm{IR})/\mathrm{LR}\cdot \mathrm{SIN(Z)\ SIN(ZB)} - \mathrm{Z}\cdot \mathrm{ZB}\cdot (1 + \mathrm{LR})\cdot \mathrm{COS(Z)\ SIN(ZB)} - (\mathrm{Z})^2\cdot (1 + 1/\mathrm{LR})\cdot \mathrm{SIN(Z)}\cdot \mathrm{COS(ZB)} = 0 \tag{A-15}$$



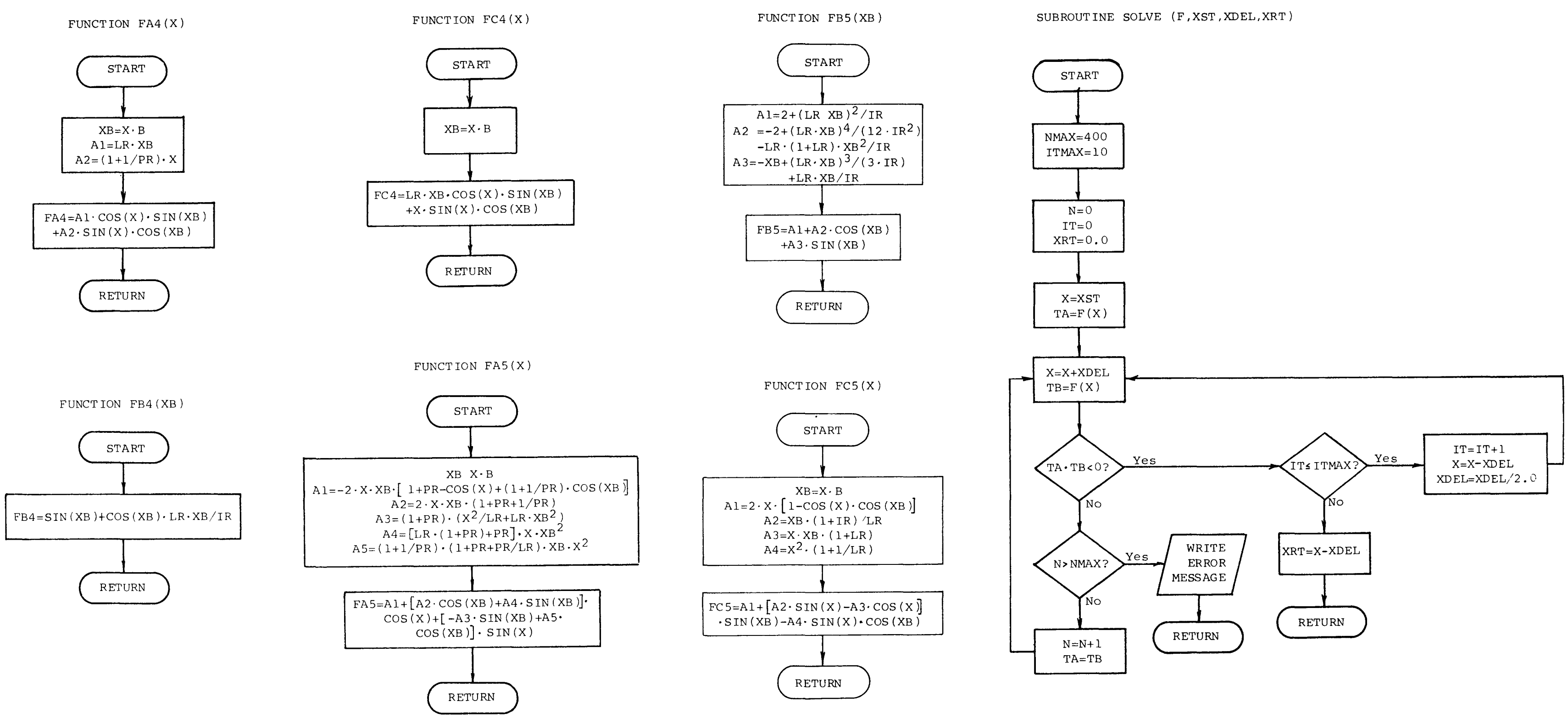
Flow chart of main program



Flow chart of main program (cont'd)



Flow chart of functions and subroutines



Flow chart of functions and subroutines (cont'd)

B. Program Flow Chart—The symbols appearing in the flow chart are defined as follows:

Main Program

A1, A2	Cross-sectional areas of upper and lower segments, respectively
B	β, Eq. (7)
CK, DK, EK	Temporary storage
EFC	End fixity code. (same as case no.)
I1, I2	Moments of inertia of upper and lower segments, respectively
IR	I1/I2
KL1, KL2	Effective lengths of upper and lower segments, respectively
L1, L2	Lengths of upper and lower segments, respectively
LR	L1/L2
P1	Axial load applied at top of column
P2	Axial load applied at step of column
PR	P1/P2
NP	Total number of problems
R1, R2	Radii of gyration of upper and lower segments, respectively
SR1, SR2	Effective slenderness ratios of upper and lower segments, respectively
Z1	$\gamma_1 l_1$ that solves characteristic equation
Z2	γ_2, l_2, Eq. (6)
ZST	Initial guess to root of characteristic equation
ZDEL	Initial iteration increment

Functions

FA1(X)	Eq. (A-1)
FB1(X)	Eq. (A-2)
FCl(X)	Eq. (A-3)
FA2(X)	Eq. (A-4)
FC2(X)	Eq. (A-6)
FA3(X)	Eq. (A-7)
FB3(X)	Eq. (A-8)
FC3(X)	Eq. (A-9)
FA4(X)	Eq. (A-10)
FB4(X)	Eq. (A-11)
FC4(X)	Eq. (A-12)
FA5(X)	Eq. (A-13)
FB5(X)	Eq. (A-14)
FC5(X)	Eq. (A-15)
X	Dummy argument
XB	X·B
A1, A2, A3, A4, A5	Temporary storage

SUBROUTINE SOLVE(F, XST, XDEL, XRT)

F	Function whose root is to be determined
XST	Initial guess to root
XDEK	Iteration increment
XRT	Root
IT	Number of times iteration increment halved
ITMAX	Maximum value of IT
N	Number of iteration steps
NMAX	Maximum number of iteration steps allowed
TA, TB	Temporary storage