A Unified Approach to the Elastic Lateral Buckling of Beams

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THE LATERAL BUCKLING of beams is a complicated phenomenon, the maximum elastic bending stress necessary to cause buckling being dependent upon the geometrical and material properties of the beam, the type and position of the loading, and also the support conditions. This paper presents a design procedure which enables all of these factors to be allowed for, in an accurate yet simple and direct manner, for 41 different cases of loading and support. The proposed method is both more accurate and more versatile than existing methods used in design codes.

NOMENCLATURE

A = Coefficient
A' = Coefficient
aL = Distance from support at which load is applied
B = Coefficient
B' = Coefficient
b = Flange breadth
C_1 = Coefficient
C_2 = Coefficient
D' = Coefficient
d = Depth of beam
E = Modulus of elasticity
F = Coefficient
F' = Coefficient
f = Height of point of application of load above tension flange
G = Modulus of rigidity
h = Distance between flange centroids
I_o = Second moment of area of compression flange
I_x = Second moment of area of beam about its major axis
I_y = Second moment of area of beam about its minor axis
J = Torsion constant
K = Coefficient
K_T = GJ/L = torsional stiffness of beam
k = Plate buckling coefficient
k_2 = Coefficient
L = Span
M_{cr} = Critical moment for lateral buckling
R^2 = L^2GJ/EC_w = torsional parameter for beam
r = Least radius of gyration
r_y = Radius of gyration about the minor axis
t_f = Flange thickness
t_w = Web thickness
α = Lateral buckling coefficient
β = End moment ratio
C_w = Warping constant
C'_w = "Equivalent" warping constant for monosymmetrical I-sections
ϵ = I_c/I_y = area ratio for monosymmetrical I-sections
μ = Poisson's ratio
σ_{cr} = Critical stress

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INTRODUCTION

Since stability considerations greatly influence the design of structural components, it is not surprising that a great deal of research has been conducted on the various buckling problems which are encountered in civil engineering. As a result of these studies, in the case of both column and plate buckling, there has evolved a standard form of presentation of the relationship which exists between the buckling stress and the physical proportions of the structural component. Unfortunately this has not occurred to the same extent in respect to the lateral buckling of beams. In this paper, the authors present a summary of the available theoretical solutions in a form which it is hoped will lead to a better understanding of the stability problem and facilitate the use of the extensive data which are available.

For column buckling the critical elastic buckling stress is normally expressed in the form of Eq. (1):

$$\sigma_{cr} = k(\pi^2 E)(r/L)^2 \qquad (1)$$

where

σ_{cr} = critical compressive stress
k = non-dimensional buckling coefficient
L/r = ratio of length of member to the least radius of gyration

Similarly for plate buckling, the critical elastic stress is given by Eq. (2):

$$\sigma_{cr} = k\left(\frac{\pi^2 E}{12(1-\mu^2)}\right)\left(\frac{t}{b}\right)^2 \qquad (2)$$

where

k = non-dimensional buckling coefficient
b/t = ratio of width of plate to thickness of plate (see Fig. 1)

It will be noted that Eqs. (1) and (2) are similar in form, each involving three terms, a non-dimensional buckling coefficient k, a term containing the material properties of the component, and a third term involving a "slenderness ratio."

Now with plate structures, buckling does not automatically result in collapse, since the plates are capable of developing a membrane type action. However, in the case of the lateral buckling of beams this is not the case, the elastic buckling load providing a close upper limit to the beam's load carrying capacity. The availability of a design procedure which will provide an accurate assessment of this buckling load is therefore desirable.

Because of its importance, the phenomenon of lateral buckling has received a great deal of attention from researchers, and solutions are available for most of the important loading and support conditions. Unfortunately much of this information has been presented in either graphical or tabular form and cannot, therefore, be used directly in any *computerized* design procedure.

Much of the work published prior to 1959 has been briefly summarized by Lee[1], while a more recent and extensive survey is presented in Ref. 2. In addition, in 1960 Clark and Hill[3] made a critical appraisal of the current design procedures used in connection with the lateral buckling of beams.

In this paper, the authors present a method which permits a rapid estimation of a beam's resistance to lateral buckling. It is based upon the concept of using a lateral buckling coefficient α similar to the more familiar plate buckling coefficient k.

Now the value of α varies with the shape of the beam, the type of loading and the level of application of this loading and also with the conditions of lateral support. It was decided to use as a reference datum the simplest

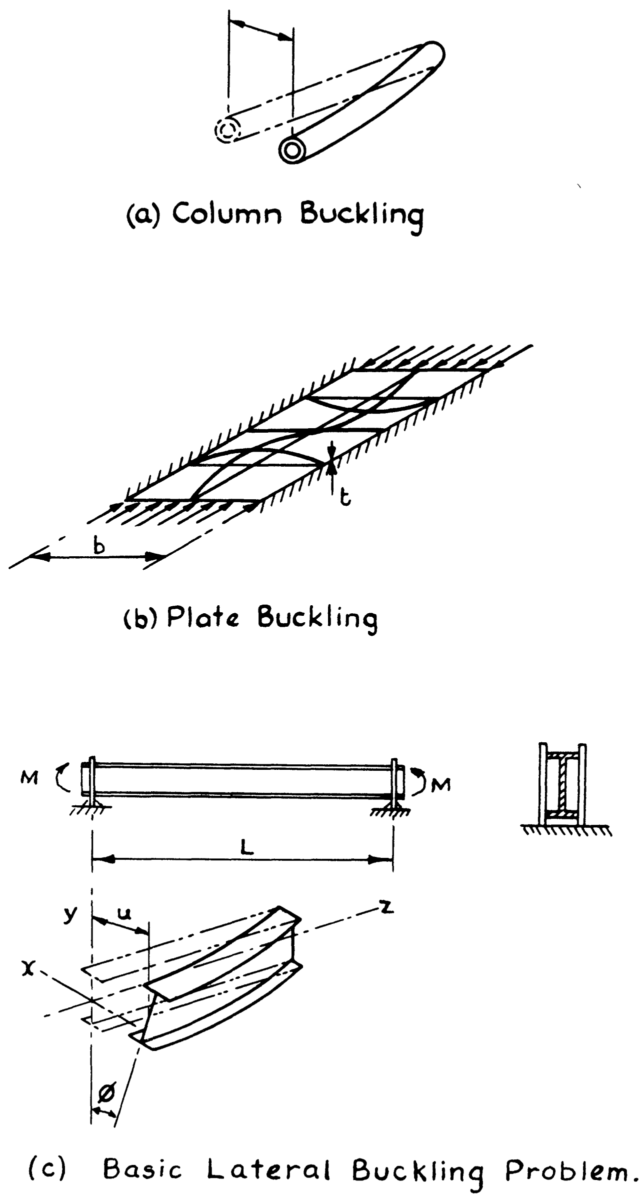


Fig. 1. Types of buckling

form of the lateral buckling phenomenon which is that of a simply supported bisymmetrical I-beam loaded by equal end moments applied in the plane of the major axis. This loading and support case is shown diagrammatically in Fig. 1(c).

Provided the beam is elastic and free from initial imperfections, then the critical moment is given by Eq. (3):

$$M_{cr} = \frac{\pi}{L}(EI_y GJ)^{1/2}\left[1 + \frac{\pi^2 EC_w}{L^2 GJ}\right]^{1/2} \qquad (3)$$

where

EI_y = minor axis flexural rigidity
GJ = torsional rigidity
EC_w = warping rigidity

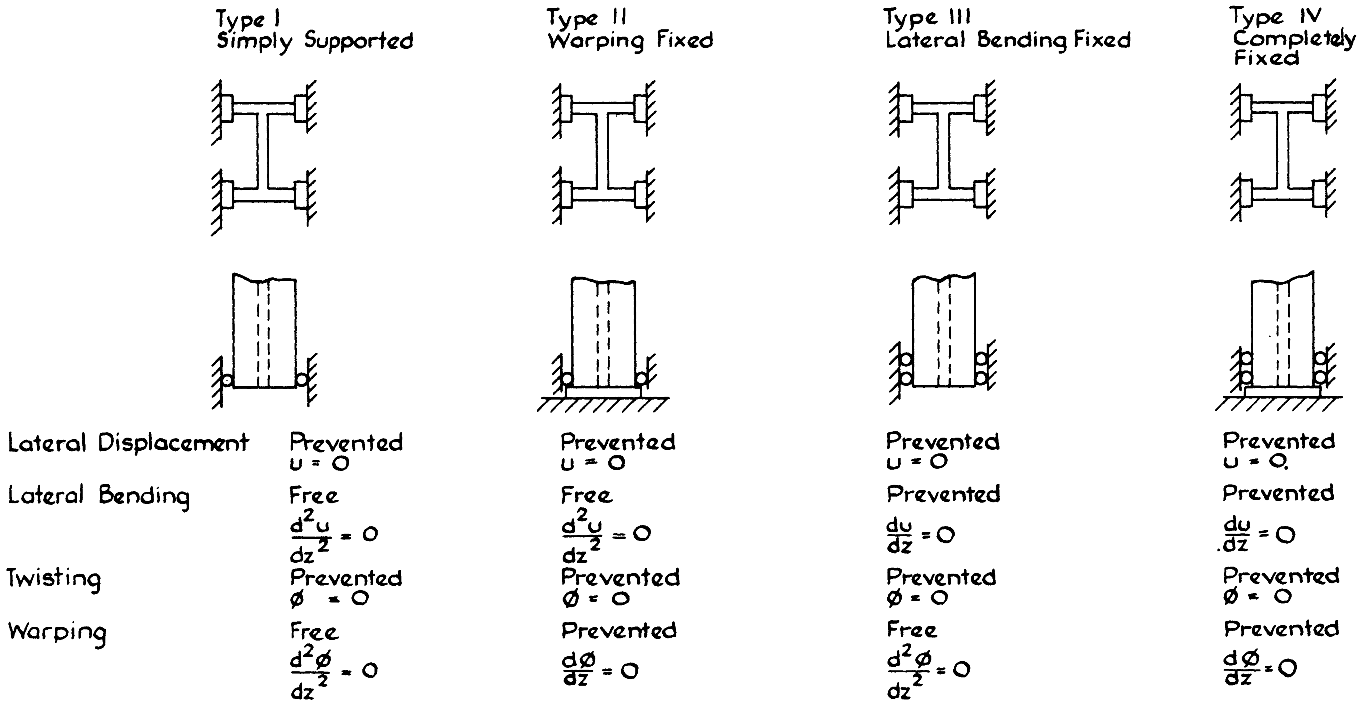


Fig. 2. End support conditions

The appearance of two terms in the square brackets is a consequence of the manner in which the beam resists twisting, part of this resistance being derived from the shear stresses set up by St. Venant torsion and part from the differential bending of the flanges. While the second term is negligible compared with the first for the majority of hot rolled sections, the reverse is true for light gage sections, which derive most of their resistance to torsional deformation from the warping action. Since the beam's length also appears in the second term, it too has a considerable influence upon the relative magnitudes of the two terms. Therefore, a formula which is to be of general application must involve all of the terms in Eq. (3). Equation (3) may, however, be rewritten in the simpler form shown in Eqs. (4) and (4a):

$$M_{cr} = \alpha(EI_yGJ)^{1/2}\left[\frac{\pi}{L}\left(1 + \frac{\pi^2}{R^2}\right)^{1/2}\right] \qquad (4)$$

where $R^2 = \dfrac{L^2GJ}{EC_w}$

or alternatively

$$M_{cr} = \alpha(EI_yGJ)^{1/2}\gamma \qquad (4a)$$

where $\gamma = \dfrac{\pi}{L}\left[1 + \dfrac{\pi^2}{R^2}\right]^{1/2}$

It should be noted that $\alpha = 1$ in the case of this particular loading and end support condition [see Fig. 1(c)]. The relationship between γ and R^2 has been plotted in Fig. 3.

Note that Eq. (4a) contains three terms similar to th of Eqs. (1) and (2). We shall see in the following sectic that the first term, α, varies with loading and supp conditions, while the second term varies with mater properties and the shape of the beam. The third, since it contains terms which vary with the length the beam, may be considered as a slenderness paramet

A study of the probable values of R^2 for a series commercial beams embracing both hot rolled and c formed sections reveals that elastic lateral buckling v occur when R^2 is greater than 4. Beams for which R

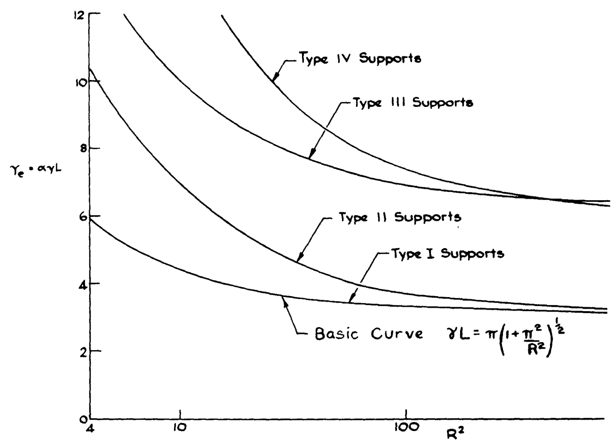


Fig. 3. Effect of variations in end support conditions on the relationship for beams loaded with equal end moments

Table 1

Symmetrical I-Beams Loaded with Equal End Moments

Type of Support Conditions	Formula for α	Maximum Error (%)
Simply supported Type I	$\alpha = 1$	0
Warping fixed Type II	$\alpha = 1 - \frac{0.304}{R^2} + \frac{1.778}{R}$	
Lateral bending fixed Type III	$\alpha = 2 - \frac{0.787}{R^2} + \frac{1.134}{R}$	1
Fixed Type IV	Use $\frac{L}{2}$ in place of L in Eq. (4)	0
Rigid central support Type V	Use $\frac{L}{2}$ in place of L in Eq. (4)	0

less than 4 will probably start to yield before they become unstable.

Now providing M_{cr} *is taken as the maximum moment occurring in the beam* and the value of α is chosen so that it allows for the loading and support conditions, then Eq. (4a) may be regarded as the basic equation for lateral buckling.

In Fig. 3, the relationship between $\gamma_e = \alpha(\gamma L)$ and R^2 is given for the case of beams loaded by equal end moments and supported at each end in four different ways. It will be noted that as the restraining action of the supports is progressively increased, the value of γ_e increases significantly. Table 1 gives the simple relationship between α and R^2 for the cases shown in Fig. 3. These relationships will be discussed in detail in later sections.

Figure 4 shows how γ_e varies with R^2 for different loading conditions, the support conditions being kept

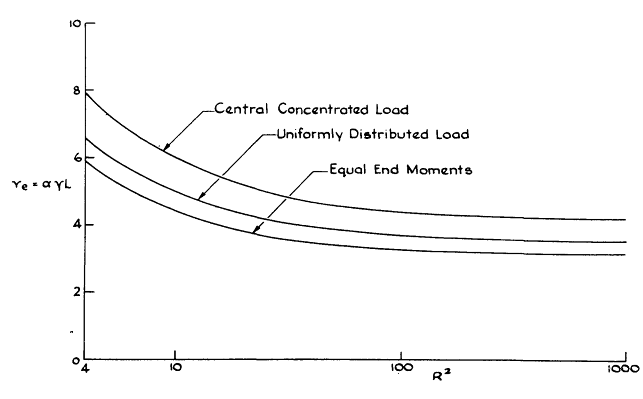


Fig. 4. Effect of variations in the type of loading on the γ_e-R^2 relationship for simply supported beams

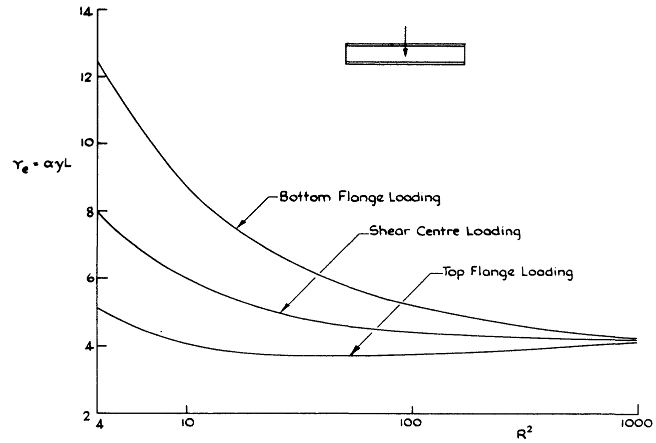


Fig. 5. Effect of variations in the level of application of the loading on the γ_e-R^2 relationship for simply supported, centrally loaded beams

constant, while Fig. 5 shows the effect of altering the level of application of the load with respect to the shear center (see also Table 2).

It will be noted that all of the curves in Figs. 3, 4, and 5 are similar in form and it is this fact which enables one to use the second coefficient α in conjunction with the basic expression for M_{cr} given in Eqs. (3) and (4a).

In the following sections, expressions for the values of α will be developed for a variety of loading and support conditions.

INFLUENCE OF SUPPORT CONDITIONS ON BEAMS WITH EQUAL END MOMENTS

The support conditions assumed in the derivation of Eq. (3) provide the lowest measure of lateral restraint and consequently yield the lowest value of α (= 1). It is possible, however, that the beam may be supported in such a manner that other, more beneficial, support conditions may be assumed. By allowing for the increased stability which results from the use of such support conditions, considerable economies in structural weight can be achieved. In this section the variation of the buckling coefficient α with the support conditions is examined for beams loaded in pure bending.

Since lateral buckling implies three kinds of deformation (twisting, lateral bending and warping), it is possible to imagine several types of end conditions. If the support's resistance to each of these actions is such as to only provide a partial restraint, then the problem becomes exceedingly complex. Several solutions to this problem exist (Refs. 4–8), but for the purposes of this paper only supports which may be assumed either completely to prevent, or alternatively offer no resistance to, each type of deformation will be considered.

It is usual in all problems of lateral buckling to assume that the end supports completely prevent end twisting (see Fig. 2). It may be shown (Refs. 7 and 8)

Table 2

Symmetrical I-Beams Simply Supported at Each End

Type of Loading	Formula for α		Maximum Error (%)
(a) Beams loaded by end moments			
M —— βM, $-1 \geqslant \beta \geqslant 1$	$\alpha = 1.16 + [0.6-\beta] - [\beta-0.6]^2$ for $1 \geqslant \beta \geqslant -0.8$ and $\alpha = 2.56$ for $\beta \leqslant -0.8$. N.B. Terms in square brackets are used only when positive.		2
(b) Beams loaded with transverse loads			
Load at top flange	$\alpha = A/B$		
Load at shear center	$\alpha = A$		
Load at bottom flange	$\alpha = A \times B$		
	A	B	
$\downarrow$	1.35	$1 - \dfrac{1.779}{R^2} + \dfrac{2.039}{R}$	2
	1.123	$1 - \dfrac{1.522}{R^2} + \dfrac{1.681}{R}$	2
$aL\downarrow \quad \downarrow aL$	$1 + a^2$	$1 - \dfrac{4.59a}{R^2} + \dfrac{5.14a}{R}$	5

that supports which possess a torsional stiffness in excess of 20 times the torsional stiffness of the beam, defined as

$$K_T = GJ/L$$

may be assumed to provide complete support with regard to end twisting. The multiple of 20 is not large and it is felt that practical supports will be capable of exerting at least this measure of torsional restraint. If the end supports exert no resistance to end twisting, then clearly the beam will be highly unstable and the work of Flint[7] and Schmidt[8] shows it to be incapable of supporting any load. Therefore, in all subsequent work reported in this paper it will be assumed that end twisting is prevented completely.

The four types of end supports that are considered in this paper are illustrated diagrammatically in Fig. 2 and consist of:

1. Those which prevent completely both lateral deflection and twist but offer no restraint either to warping or lateral bending—termed *simply supported* (Type I).
2. Those which prevent completely lateral deflection, twisting and warping but offer no restraint to lateral bending—termed *warping fixed* (Type II).
3. Those which prevent completely lateral deflection, twisting and lateral bending but offer no restraint to warping—termed *lateral bending fixed* (Type III).
4. Those which prevent completely lateral deflection, twisting, warping and lateral bending, termed *completely fixed* (Type IV).

If the ends are fixed completely and the loading corresponds to a uniform moment, then Timoshenko[9] has shown that the beam buckles laterally in a mode which requires the formation of inflection points at its quarter points. This situation is identical to the Euler buckling of a strut which is built-in at each end and may be analyzed in the same manner, i.e., by using the basic formula for simple supports employing a length equal to one-half of the span.

The analysis of beams for which end warping is prevented but no restraint is offered to lateral bending is rather more complicated. Flint[10] has obtained solutions to this problem both by solving the governing differential equation and also by the application of the energy method of analysis. These solutions agree quite well with a different energy solution obtained by Trahair.[4] Flint's solution may be expressed in closed form as:

$$M_{cr} = \frac{\pi}{L}\,(EIGJ)^{1/2}\left(1 + \frac{4\pi^2 E}{L^2 GJ}\right)^{1/2} \tag{5}$$

If instead of Eq. (5), Eq. (4a) is used together with the corresponding expression for α given in Table 1, then in no case is the error in M_{cr} greater than 5 percent.

If lateral bending of the beam's end sections is prevented and end warping is unrestrained (see Fig. 2), then the relationship between γ_e and R^2 will be of the form shown in Fig. 3. Again, this may be described by using Eq. (4a) together with the appropriate expression for α from Table 1, the maximum error in this case being less than 1 percent.

A convenient method of increasing any beam's lateral stability consists of providing it with a number of intermediate lateral supports. Provided these supports are sufficiently rigid, then buckling will occur in a mode requiring the formation of nodes at the supports.

Timoshenko[9] has considered the lateral buckling of a beam simply supported at each end, loaded with a uniform moment, and provided with a central support which is assumed to be capable of completely preventing lateral deflection and twisting. In this case each span buckles with no interaction at the central support, i.e., as if simply supported at that section. This behavior is analogous with that observed for a simple pin-ended Euler strut provided with a rigid central support and may be investigated by using a span of $L/2$ in place of L.

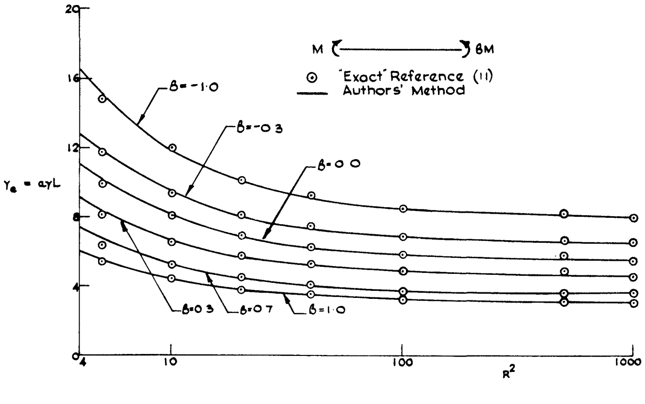


Fig. 6. The γ_e-R^2 relationship for simply supported beams loaded with unequal end moments

INFLUENCE OF TYPE OF LOADING ON SIMPLY SUPPORTED BEAMS

Unequal End Moments—The lateral stability of a simply supported beam loaded with end moments of magnitude M and βM, where β may take any value between $+1$ and -1, has been studied by both Horne[11] and Trahair,[12] the latter having given the expression for the critical moment as:

$$M_{cr} = \frac{\pi}{L} (EI_yGJ)^{1/2} \left(F + \frac{\pi^2 EC_w}{L^2GJ} F'\right)^{1/2} \tag{6}$$

where F and F' are constants, values of which are given in Ref. 11 for a series of values of β.

The relationship between γ_e, β and R^2 is given in Fig. 6. For every value of β, the ratio of γ_e to the value of γ_e for $\beta = 1$ remains nearly constant over the complete range of R^2 under consideration. In fact the ratio varies by no more than 5 percent at a value of β equal to -0.7 and becomes constant for values of β in excess of 0.3.

It is proposed, therefore, that in this case α be taken as a function of β only. A suitable expression for α is given in Table 2. The resulting values of γ_e are given in the graphs in Fig. 6, where they are compared with the original values obtained by Horne.[11] It will be noted that the proposed procedure gives excellent results.

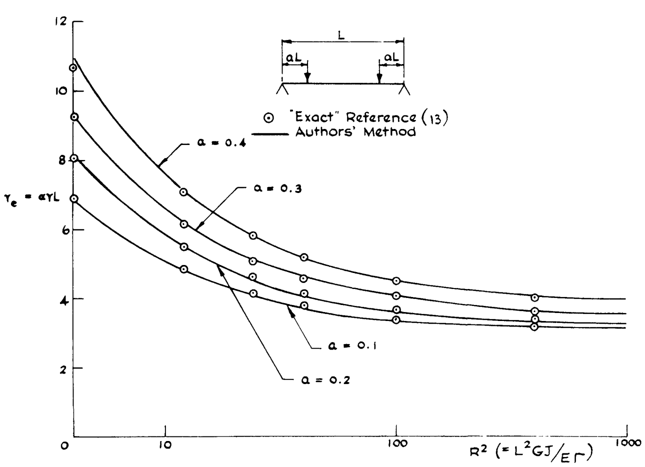


Fig. 7. The γ_e-R^2 relationship for a simply supported beam loaded with two concentrated loads applied at the level of the bottom flange

Transverse Loads—For the case of a beam loaded by a concentrated vertical load, Timoshenko[9] has tabulated values of the buckling load. Provided the load is applied at the level of the shear center, then the values of γ_e corresponding to the centrally loaded beam are in a constant ratio with those for the uniformly bent beam, whatever the value of R^2 (see Fig. 4). Therefore, α has a constant value A as shown in Eq. (7), the value of A being 1.35 in this particular case, as shown in Table 2.

$$\alpha = A \tag{7}$$

Raising or lowering the point of application of the load results in decreased or increased stability, respectively. This effect is illustrated in Fig. 5 for the two extreme loading positions, i.e., load applied at the level of the top flange or load applied at the level of the bottom flange. Since the ratios of γ_e (top flange loading) to γ_e (shear center loading) and γ_e (shear center loading) to γ_e (bottom flange loading) are almost identical for any particular value of R^2, though the actual ratio varies with R^2, this may be allowed for in the expression for α as shown below:

$$\left.\begin{aligned} \alpha &= A/B \text{ for top flange loading} \\ \alpha &= A \times B \text{ for bottom flange loading} \end{aligned}\right. \tag{7a}$$

Because the magnitude of this effect is itself dependent on the value of R^2, the coefficient B will be a function of R^2. Expressions for B for the three loading conditions are given in Table 2.

A similar case exists when the loading is uniformly distributed over the span. The γ_e-R^2 relationship for this type of loading may again quite accurately be defined by employing α as given by Eqs. (7) and (7a), the expressions for A and B being given in Table 2. The maximum deviation between the value of M_{cr} so obtained and the theoretical value as given in Ref. 9 is no more than 2 percent.

Schrader[13] has presented a solution in which the loading consists of two point loads, each of magnitude P, applied at a distance aL from the end supports. Schrader's[13] expression for the critical load is given in Eq. (8).

$$P_{cr} = A' \frac{E}{L^3} \left[\frac{I_y}{2}(h - 2f) + \left((h + 2f)^2 \frac{I_y^2}{4} + B'h^2 \frac{I_y^2}{4} + D' \frac{GJ}{E} I_y L^2 \right)^{1/2} \right] \tag{8}$$

where f is the height of the point of application of the loads above the bottom flange and A', B', and D' are coefficients whose value depends on a.

The relationship between γ_e and R^2 has been evaluated and plotted for four different values of a in Fig. 7 for the case where the loads are applied at the level of the lower flange. Once again Eqs. (7) and (7a) may be used to calculate the appropriate value of α, the expressions for A and B being obtained from Table 2. In Fig. 7, values of γ_e, as determined from Eq. (4a) used in conjunction with the appropriate equations for α, are compared with those calculated from Eq. (8) for four different values of a. In all cases excellent agreement may be observed, the maximum deviation being less than 2 percent. The error does, however, increase to 5 percent in certain cases of top flange loading.

BEAMS WITH ENDS COMPLETELY FIXED

Earlier in this paper, for the specific case of a beam loaded by equal moments at its ends, the increased lateral stability that results from fixing completely the beam's ends in the lateral plane was discussed. It was shown in that particular case that the appropriate value of M_{cr} could be obtained simply by using $L/2$ in Eq. (4a).

Salvadori[14] has considered the stability of beams whose ends are fixed in the lateral plane and which are loaded by unequal end couples of magnitude M and βM. This solution may be incorporated in the present procedure provided α is determined using the expression given in Table 3. This method is identical to that proposed earlier for a simply supported beam subjected to unequal end moments, except that $L/2$ must be used in place of L in Eq. (4a).

For a fixed-ended beam loaded with either a central concentrated load or a uniformly distributed load, the solutions given in Refs. 9 and 12 may be used to calculate the critical load. These solutions include the effect of altering the level of the load with respect to the beam's shear center. When the present method is applied to these cases, the form of α for each type of loading and each of the load positions considered corresponds to that given in Eqs. (7) and (7a), the expressions for A and B being given in Table 3. In these particular cases it will be noted that the coefficient A, like B, now varies with the ratio R.

Although complications due to both load and support conditions are now involved, using the proposed method it is still possible to obtain results which are in error by no more than 3 percent.

BEAMS WITH ENDS RESTRAINED AGAINST WARPING

This type of end condition is illustrated in Fig. 2 and in Fig. 3 its stabilizing influence is shown to lie between that of a simple support and a fully fixed support.

When the beam is loaded with either a central concentrated load or a uniformly distributed load, the energy solution of Trahair[4] may be used to obtain a

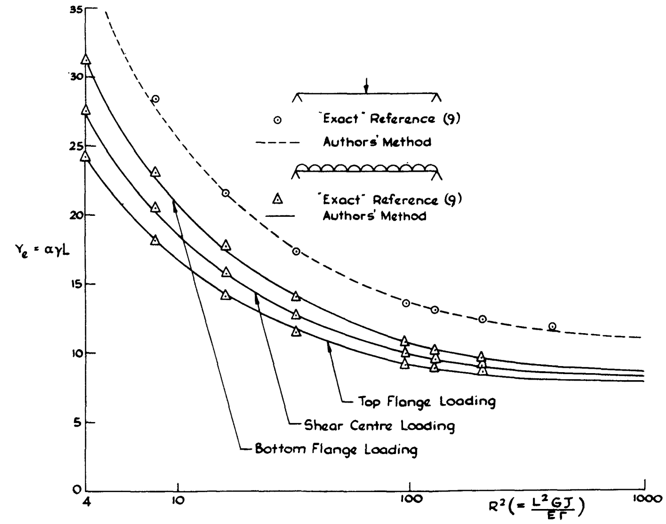


Fig. 8. γ_e-R^2 relationship for simply supported, transversely loaded beams with a central lateral support

reasonable estimate of its lateral stability. Provided the appropriate expressions for A and B given in Table 3 are used, then the lateral buckling coefficient α corresponding to either type of loading may be obtained from Eqs. (7) and (7a), Eq. (7) being used for loading applied at the level of the shear center and Eq. (7a) for loading applied to either flange. The results of using this procedure are again extremely accurate, being in error by no more than 4 percent.

BEAMS WITH ENDS RESTRAINED AGAINST LATERAL BENDING

When the beam's ends are restrained against lateral bending (see Fig. 2), the condition of lateral support is again somewhere between fully fixed and simply supported; this is illustrated in Fig. 3 for the case of a beam loaded in pure bending.

The results for the case of the buckling of a beam supported in the above manner and loaded by either a concentrated load or a uniformly distributed load are given in Ref. 4. Once again Eqs. (7) and (7a), in conjunction with Table 3, may be used to calculate the lateral buckling coefficient α.

LATERALLY SUPPORTED BEAMS

When loaded with a uniform moment, a simply supported beam which is rigidly laterally supported at its center buckles into two half-waves with no interaction at the support; this case was discussed earlier.

When the loading consists of transverse loads, although the beam still buckles in two half-waves, a certain amount of interaction takes place. Timoshenko[9] has listed values of the buckling load for such beams when loaded with a central concentrated load or a uniformly distributed load, and has shown the γ_e-R^2 relationship to be of the form given in Fig. 8. It is of interest to note that for the centrally loaded beam the loaded cross-

Table 3

Symmetrical I-Beams Loaded and Supported in Various Ways

Type of Loading	Type of Supports	Formula for α		Maximum Error (%)
(a) Beams loaded by end moments				
M ——— βM $-1 \geqslant \beta \geqslant 1$	IV	$\alpha = 1.16 + [0.6 - \beta] - [\beta - 0.6]^2$ for $1 \geqslant \beta \geqslant -0.8$ and $\alpha = 2.56$ for $\beta \leqslant -0.8$ Use $\frac{L}{2}$ in place of L in Equation (4) N.B. Terms in square brackets are used only when positive.		3
(b) Beams loaded with transverse loads				
Load at top flange Load at shear center Load at bottom flange		$\alpha = A/B$ $\alpha = A$ $\alpha = A \times B$		
		A	B	
↓	IV	$1.916 - \frac{4.186}{R^2} + \frac{5.814}{R}$	$1 - \frac{4.602}{R^2} + \frac{2.899}{R}$	4
	IV	$1.643 - \frac{4}{R^2} + \frac{5.563}{R}$	$1 - \frac{3.342}{R^2} + \frac{1.964}{R}$	3
↓	II	$1.43 + \frac{4.788}{R^2} + \frac{1.455}{R}$	$1 - \frac{3.13}{R^2} + \frac{1.945}{R}$	4
	II	$1.2 + \frac{4.106}{R^2} + \frac{1.263}{R}$	$1 - \frac{2.217}{R^2} + \frac{1.794}{R}$	4
↓	III	$2.0 - \frac{0.726}{R^2} + \frac{0.955}{R}$	$1 - \frac{2.045}{R^2} + \frac{3.289}{R}$	3
	III	$1.9 - \frac{1.184}{R^2} + \frac{0.02}{R}$	$1 - \frac{0.991}{R^2} + \frac{2.531}{R}$	4
↓	V	$2.95 - \frac{11.284}{R^2} + \frac{12.787}{R}$	1	3
	V	$2.093 - \frac{9.344}{R^2} + \frac{9.792}{R}$	$1.073 + \frac{0.137}{R}$	4

section cannot deflect laterally. Consequently, altering the level of application of the load with respect to the shear center has no effect and α will always be given by Eq. (7), where A is given in Table 3.

However, if the loading is uniformly distributed over the span, then its position does affect the beam's lateral stability. In this case α can be determined using either Eq. (7) or (7a), the appropriate A and B being given in Table 3. A series of values of γ_e so calculated are plotted in Fig. 8, together with the theoretical values given by Timoshenko; the degree of agreement is excellent.

CANTILEVERS

In all of the cases studied so far, the beam has been assumed to be supported against lateral deflection and twisting at both ends. However, lateral buckling may also be of importance in connection with the design of cantilevers. Solutions[9,15] exist for two cases of loading: point load at the free end and a uniformly distributed load. The γ_e-R^2 relationship for each of these cases may be described by using Eq. (4a) in conjunction with α as determined from Eq. (7), the expression for A corresponding to each type of loading being given in Table 4.

Table 4

Cantilevers		
Type of Loading	Formula for α	Maximum Error (%)
	$\alpha = 2.054 - \dfrac{6}{R^2} + \dfrac{5.88}{R}$	5
	$\alpha = 1.28 - \dfrac{1.8}{R^2} + \dfrac{1.75}{R}$	4

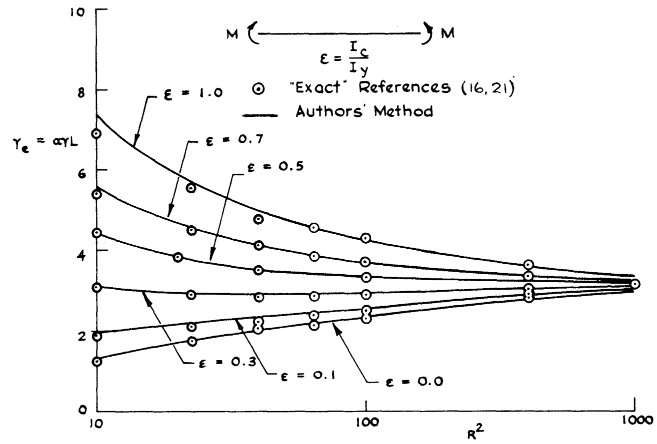


Fig. 9. The γ_e-R^2 relationship for simply supported monosymmetrical I-beams loaded with equal end moments

UNSYMMETRICAL I-SECTIONS IN PURE BENDING

Although the majority of both hot-rolled and cold-formed I-sections in common use are symmetrical about both the major and minor axes, it is sometimes necessary to use a section which is symmetrical about the minor axis only. An example of this is the heavy built-up sections used as crane girders. Indeed, provided the larger flange is in compression, then the monosymmetrical section will always be more stable than a bisymmetrical one having the same total flange area (see Fig. 9).

The analysis of the lateral buckling of such sections is considerably more complex than that for a bisymmetrical section, even for the simplest conditions of loading and support. In view of this it is hardly surprising that much less work has appeared on this subject.

Because the shear center and the centroid no longer coincide, the warping constant C_w may not be taken as $I_y h^2/4$, and in the limiting case of a tee section disappears altogether. It is therefore not possible to automatically relate γ to R^2 for these sections. However, if an "equivalent warping constant", C'_w, equal to $I_y h^2/4$ is used in place of C_w in calculating R^2, then the relationship between γ_e and R^2 will be of the form shown in Fig. 9.

Several solutions exist to the problem of the lateral buckling of a simply supported monosymmetrical I-section acted upon by equal end moments (Refs. 16–20), the most exact of these solutions being those of Goodier and Timoshenko. However, their solutions were not given in closed form, although the γ_e-R^2 relationship has been plotted in Ref. 21 for values of R^2 less than 40. Therefore, in the present work the Goodier-Timoshenko values have been used for the range $4 < R^2 < 40$ and values of γ for higher values of R^2 evaluated from Winter's formula for M_{cr}, which is given below:

$$M_{cr} = \frac{\pi}{L}\sqrt{EI_yGJ}\left[\left(1 + \pi^2\frac{EJ_yh^2}{4GJL^2}\right)^{1/2} + (2\epsilon - 1)\frac{\pi h}{2L}\sqrt{GJ}\right]$$

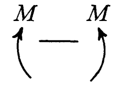
The resulting γ_e-R^2 relationships are plotted in Fig. 9 for six different values of ϵ. These relationships may be expressed in the form of Eq. (4b), where the values of A and B depend only upon the shape of the section as characterized by the parameter ϵ and may be calculated with the aid of the expressions given in Table 5.

$$\gamma = \pi + \frac{A}{R^2} + \frac{B}{R} \tag{4b}$$

The values of γ obtained from Eq. (4b) are plotted in Fig. 9 for comparison with the more exact values. Over most of the range of both R^2 and ϵ the proposed method yields values of γ which are in error by no more than ± 5 percent. However, for extreme values of ϵ and low values of R^2, the errors do increase to between 13 and 15 percent in a few instances, the error always being on the unsafe side; these are, however, much less than the errors involved in the current British Standard. This will be discussed in detail later.

When $\epsilon = 0.5$ and the section is bisymmetrical, the difference between the values of γ_e given by Eqs. (4b) and (4a) is nowhere greater than 1 percent. In fact, should anyone prefer to use Eq. (4b) instead of Eq. (4a) in all of the preceding work, then the resulting error in so doing will be very small.

Table 5

Unsymmetrical I-Sections			
Type of Loading	Type of Supports	Formula for γ	Maximum Error (%)
M M 	I	$\gamma = \pi + \dfrac{A}{R^2} + \dfrac{B}{R}$ where $A = 8.85 - 0.942(0.5 - \epsilon)$ $B = 1.126 + 18.78(0.5 - \epsilon)$ N.B. $\alpha = 1$	5

IMPLICATIONS FOR DESIGN

In the preceding sections it has been shown that the theoretical moment which causes a beam to buckle laterally may be calculated quite simply for a wide variety of loading and support conditions. In every case the critical moment is given by Eq. (4a), while the corresponding maximum bending stress may be obtained from Eq. (9).

$$M_{cr} = \alpha(EI_yGJ)^{1/2}\gamma \tag{4a}$$

and

$$\sigma_{cr} = \frac{M_{cr}d}{2I_x} \tag{9}$$

where I_x is the major second moment of area and d is the depth of the beam.

Now with the exception of the warping constant C_w and the torsion constant J, each of the properties occurring in Eqs. (4a) and (9) is normally given in tables of sections. It is felt that in future each of these quantities might also be tabulated, but until such tables are available both J and C_w may be calculated using the methods indicated below.

For any section composed of a number of flat plate elements, the value of J may be obtained from Eq. (10):

$$J = 1/3 \Sigma bt^3 \tag{10}$$

where t is the thickness of each plate element and b is the width of each plate element.

Therefore for a symmetrical I- or channel-section, J will be given by

$$J = 1/3(2bt_f^3 + ht_w^3)$$

If a more accurate value of the torsion constant is required, then the methods given in Ref. 22 may be used.

For a symmetrical I-section, the warping constant may be taken simply as

$$C_w = \frac{I_yh^2}{4} \tag{11}$$

It has previously been shown that Eq. (11) may also be used to calculate an "equivalent warping constant" in the case of a monosymmetrical I-section. Equation (11) is not strictly correct for channel sections, but Hill[23] has shown that its use will not result in an error in M_{cr} of more than 6 percent. If greater accuracy is required, the exact formula for C_w given in Eq. (12) should be used.

$$C_w = \frac{t_fb^3h^2}{12}\,\frac{3bt_f + 2ht_w}{6bt_f + ht_w} \tag{12}$$

Since all of the equations used to calculate the critical stress have been given in closed form, the proposed method is ideally suited to programming for a digital computer. Such a program has been written in FORTRAN IV, and the flow chart employed is given in Fig. 10. Its availability reduces the labor required of a designer to specify simply a control number corresponding to the conditions of loading and lateral support together with the necessary material and geometrical properties of the beam. Alternatively, the beam's dimensions may be input and the sectional properties evaluated directly by the computer, this feature being particularly useful if tables of sectional properties are not available.

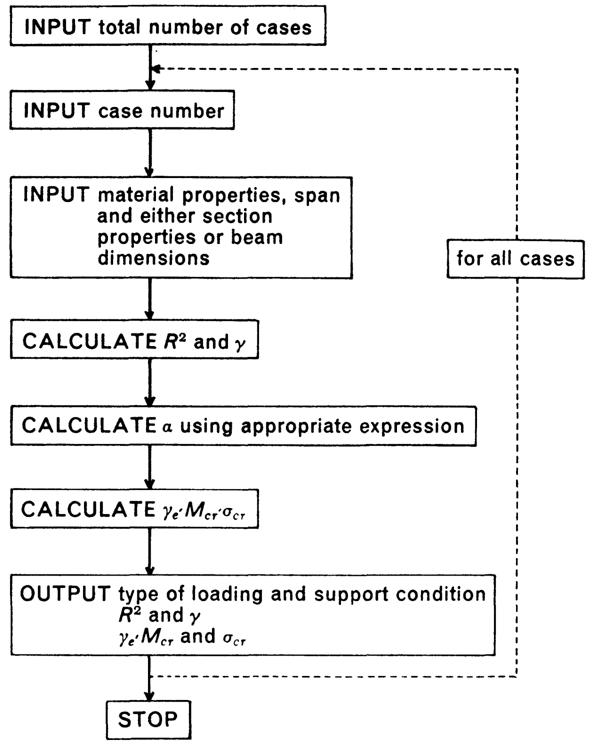


Fig. 10. Details of computer program

While it is appreciated that the equation for lateral buckling [see Eq. (13)], which forms the basis of the design method[21] incorporated in the current *British Standard 153* is rather simpler than Eq. (4a), it was obtained by rewriting Eq. (4) in terms of the critical stress and then substituting a series of approximate expressions for the sectional properties, a device which decreases its accuracy.

$$\sigma_{cr} = \frac{170{,}000}{(L/r_y)^2}\left\{\left[1 + \frac{1}{20}\left(\frac{Lt_f}{r_yd}\right)^2\right]^{1/2} + K_2\right\} \tag{13}$$

where K_2 is a coefficient whose value depends upon the value of ϵ ($K_2 = 0$ for $\epsilon = 0.5$).

Indeed, Table 1 of Ref. 21 shows that it predicts values of σ_{cr} which may be in error by as much as 35 percent for the basic loading and support conditions, and when it is applied to unsymmetrical I-sections Tables 2 and 3 of the same paper[21] show errors approaching 100 percent in certain cases.

Variations in loading were provided for by simply specifying an effective length of $1.2L$ for all cases of top flange loading, no provision being made for bottom flange loading, unequal end moments, or any of the other cases considered. The effect of different types of end supports was allowed for by specifying an effective length of $0.7L$ for fixed ends and $0.85L$ for partially fixed ends. Even when used in conjunction with the extended range of effective length factors proposed by Trahair[12], this method is neither as accurate nor as versatile as that outlined in this paper, due mainly to its neglect of the influence of R^2 on the effective length factors.

In the United States, a general equation [see Eq. (14)] for elastic lateral buckling has been presented by Clark and Hill[3]:

$$\sigma_{cr} = C_1 \frac{\pi^2 E I_y d}{2 I_x (KL)^2} \left[C_2 \left(f - \frac{d}{2} \right) + \left\{ C_2 \left(f - \frac{d}{2} \right)^2 + \frac{C_w}{I_y} \left(1 + \frac{GJ(KL)^2}{\pi^2 E C_w} \right) \right\}^{1/2} \right] \quad (14)$$

In Eq. (14) values of the coefficients C_1, C_2, and K all depend upon the conditions of loading and support, but in certain instances are also dependent upon the beam's geometry. Clark and Hill provided values of these coefficients for several types of loading and simply supported and fixed end conditions, but in instances where their value depends upon R^2 only, the range of values was given; no relationship between the value of each particular coefficient and R^2 was quoted.

It is felt that Eq. (4a) is considerably simpler to use than Eq. (14), since the value of γ_e corresponding to any of the conditions of loading and support considered may be calculated quite simply from the expressions given in Tables 1–4. Moreover the variety of loading and support conditions considered in this paper, numbering 41 different cases in all, is much more comprehensive than that dealt with in Ref. 3 or any other previous work.

CONCLUSIONS

A method has been advanced which enables a designer to calculate rapidly the maximum bending stress which will cause a beam to buckle laterally for an extensive range of loading and support conditions. The method only requires a knowledge of certain standard geometrical properties and, in those cases where certain of these properties are unavailable in current sectional tables, methods for evaluating them have been given. Because the critical stress may always be expressed directly in terms of the beam's material and geometrical properties, the method may easily be programmed for a digital computer.

The proposed design procedure permits a more accurate assessment of the influence of the type of loading and the support conditions than is at present possible using the methods of the Codes of Practice.

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