Stiffness Coefficients of Beam-Columns with Non-Prismatic Members

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THE OBJECTIVE of this paper is to determine the stiffness and carry-over factors of non-prismatic members, taking into consideration the effect of axial loads, and to construct charts that can be readily used.

The method of successive approximations was used for the numerical formulation of the problem, a program in Fortran IV was written, and a high speed digital computer was used for the determination of the numerical solutions.

Charts indicating the variation of stiffness and carry-over factors as a function of kL for various haunched members were plotted by the computer, where kL is a function of the axial load.

The values can be readily used either for the solution of frame problems with non-prismatic members including the effect of the axial loads, or for determining the buckling loads of frames with non-prismatic members.

GENERAL FORMULATION OF STIFFNESS AND CARRY-OVER FACTORS

If I_m is the moment of inertia of the minimum section of a non-prismatic member, then the deflection and slope due to lateral loading, using successive approximations, is[2] (see Fig. 1),

$$Y_o = q_d(M_i\Delta L^2/EI_m) \quad (1)$$

$$S_o = q_s(M_i\Delta L/EI_m) \quad (2)$$

where q_d and q_s are factors to be determined from successive approximations, M_i is the lateral load at point **i**, ΔL is the length of the equal segments into which the member has been divided and E is the modulus of elasticity.

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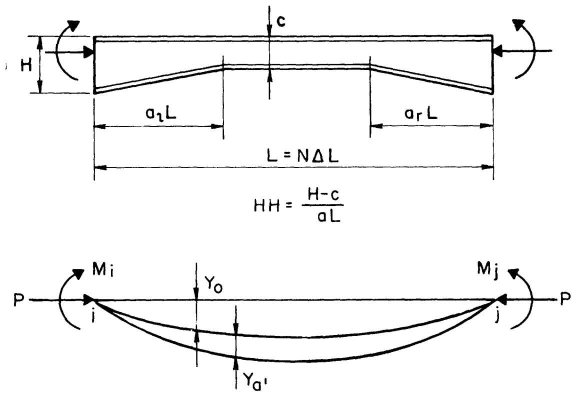

Fig. 1. Deflections and slopes due to lateral and axial loads

If an axial load P is applied, the additional deflection and slope are given by[2]:

$$Y_a' = q_d'[M_iP\Delta L^4/(EI_m)^2] \quad (3)$$

$$S_a' = q_s'[M_iP\Delta L^3/(EI_m)^2] \quad (4)$$

where q_d' and q_s' are factors to be determined from successive approximations.

If

$$k^2 = P/EI_m \quad (5)$$

$$L = N\Delta L \quad (6)$$

where N is the number of equal segments into which the member is divided, the solution at the point of convergence is:

$$k^2 = N^2\Sigma q_r/L^2\Sigma q_r' \quad (7)$$

or

$$Lk = N\sqrt{\Sigma q_r/\Sigma q_r'} \quad (8)$$

where r can take the values of d or s.

Let θ_{ij} = the slope at **i** due to a moment M_j at **j** and axial load P.

By using Eqs. (2) and (4) the slopes at **i** and **j** are:

$$\theta_{ii} = S_{oi} + S_i' = \left(\frac{M_iL}{EI_m}\right)\left(q_i\frac{1}{N} + q_i'k^2\frac{L^2}{N^2}\right) \quad (9)$$

$$\theta_{ji} = S_{oj} + S_j' = \left(\frac{M_iL}{EI_m}\right)\left(q_j\frac{1}{N} + q_j'k^2\frac{L^2}{N^2}\right) \quad (10)$$

From Eqs. (9) and (10), the stability functions ψ and ϕ are defined as follows:

$$\psi_i = (1/N)q_i + (L^2/N^2)q_i'k^2 \quad (11)$$

$$\phi_i = (1/N)q_j + (L^2/N^2)q_j'k^2 \quad (12)$$

Therefore, Eqs. (9) and (10) become:

$$\theta_{ii} = (M_iL/EI_m)\psi_i \quad (13)$$

$$\theta_{ji} = (M_iL/EI_m)\phi_i \quad (14)$$

For a moment at **j**:

$$\theta_{ij} = (M_jL/EI_m)\psi_j \quad (15)$$

$$\theta_{jj} = (M_jL/EI_m)\phi_j \quad (16)$$

Let K_{ij} be the stiffness at **i** due to a unit rotation at **i** while $\theta_j = 0$. Using Eqs. (13) and (14), the following relationships can be obtained:

$$K_{ij}L = EI_m[\psi_i/(\psi_i\psi_j - \phi_i\phi_j)] \quad (17)$$

$$\text{C.O.F.}_{ij} = \phi_i/\psi_j \quad (18)$$

where C.O.F._{ij} is the carry-over factor for member **ij**, from **i** to **j**.

In the same manner, it can be found that:

$$K_{ji}L = EI_m[\psi_j/(\psi_i\psi_j - \phi_i\phi_j)] \quad (19)$$

$$\text{C.O.F.}_{ji} = \phi_j/\psi_i \quad (20)$$

APPLICATION

As an example for determining the stiffness coefficients and carry-over factors of a structure with nonprismatic members, consider the structure given in Fig. 2.

First, consider member **AB**. From the geometry of this member (Fig. 2) and the notation used in Fig. 1, the following geometric parameters are known:

$$H = 12.48 \text{ in.}$$
$$c = 6.00 \text{ in.}$$
$$aL = 64.80 \text{ in.}$$

Therefore,

$$a = \frac{64.80}{18 \times 12} = 0.30$$

To find parameter HH, use the equation given in Fig. 1:

$$HH = \frac{H - c}{aL} = \frac{12.48 - 6.00}{64.80} = 0.1$$

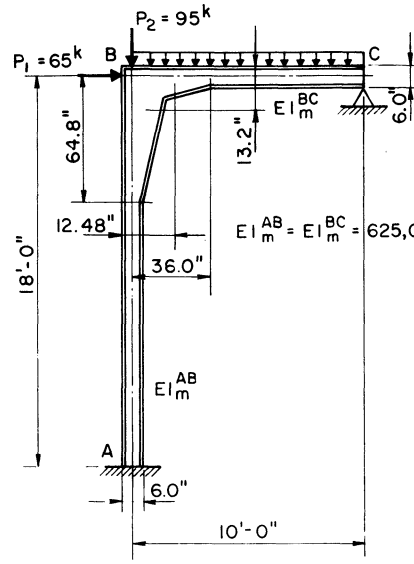

Figure 2

Besides the parameter HH, the other parameter required in the use of the tables is kL. To determine this parameter, work as follows: From Eq. (5):

$$k = \sqrt{P/EI_m}$$

The axial load P acting on member **AB**, as a first approximation, is:

$$P = P_2 + \tfrac{1}{2}(10)(1) = 95 + 5 = 100 \text{ kips}$$

From Fig. 2,

$$EI_m = 625{,}000 \text{ kip-in.}^2$$

Therefore,

$$k = \sqrt{100/625{,}000} = 0.0126$$

$$kL = 0.0126 \times 18 \times 12 = 2.72$$

With this value of $kL = 2.72$, enter the upper chart reproduced in Fig. 3, as shown by the dotted line. The corresponding value is:

$$K_{BA}L_{AB}/EI_m^{AB} = 3.2$$

or

$$K_{BA} = 3.2EI_m^{AB}/L_{AB} = \frac{3.2 \times 625{,}000}{18 \times 12}$$

$$= 9{,}250 \text{ kip-in.}$$

The carry-over factor is found similarly from the chart at the bottom of Fig. 3:

$$\text{C.O.F.}_{BA} = 0.66$$

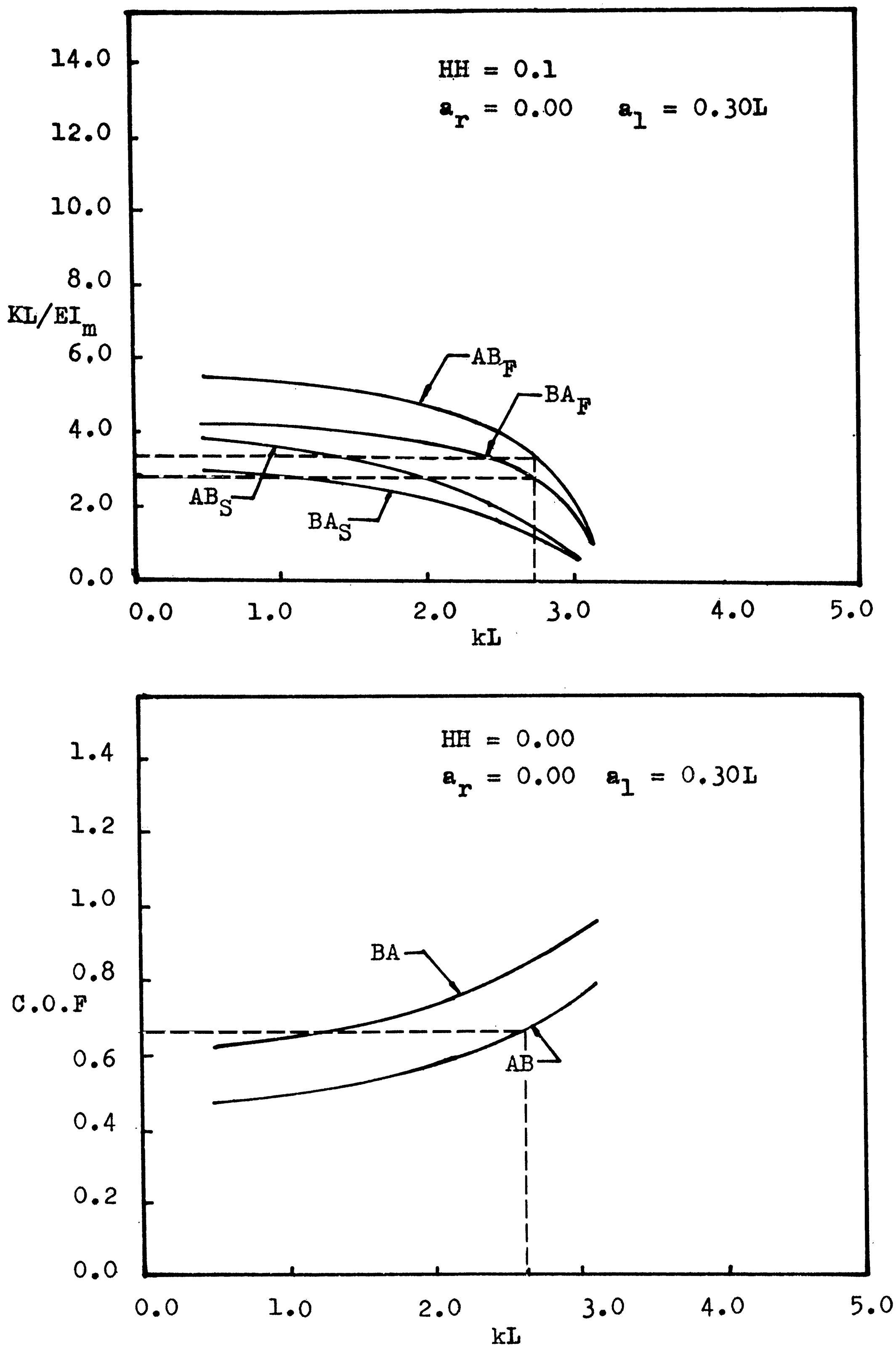

Fig. 3. Use of Chart 10 for Design Example

If it were necessary to find the stiffness K_{AB} of member **AB**, with the same value of $kL = 2.72$, using the curve designated $\mathbf{BA_F}$ of the upper table in Fig. 3 one obtains:

$$K_{AB}L_{AB}/EI_m{}^{AB} = 2.8$$

or

$$K_{AB} = \frac{2.8 \times 625{,}000}{18 \times 12} = 8{,}100 \text{ kip-in.}$$

It should be mentioned that in determining the curves of the tables for members with a haunch only on one side, **B** was designated as the end without the haunch. This explains the reversal of the subscripts between the curves and the member under consideration.

In determining the stiffness of member **BC**, the work is similar. In this case:

$$H = 13.2 \text{ in.}$$
$$c = 6.00 \text{ in.}$$
$$aL = 18.00 \text{ in.}$$

Therefore,

$$a = 18.00/(10 \times 12) = 0.15$$

$$HH = 13.20 - (6.00/18.00) = 0.4$$

From Eq. (5):

$$k = \sqrt{65/625{,}000} = 0.0102$$

$$kL = 0.0102 \times 10 \times 12 = 1.22$$

The parameters $HH = 0.4$, $a_r = 0$, $a_l = 0.15$ determine the chart to be used. This is the upper half of Chart 15. For $kL = 1.22$ the chart yields:

$$K_{BC}L_{BC}/EI_m{}^{BC} = 3.4$$

or

$$K_{BC} = (3.4 \times 625{,}000)/(10 \times 12) = 17{,}700 \text{ kip-in.}$$

Having determined the stiffnesses and the carry-over factors of the members of the structure, the stiffness coefficients necessary for the formulation of the stiffness matrix of the structure can be found. For instance, for the above example, if X_B is the redundant at **B** the stiffness coefficient f_{BB} is:

$$f_{BB} = K_{BA}EI_m{}^{BA}/L_{AB} + K_{BC}EI_m{}^{BC}/L_{BC}$$
$$= 9{,}250 + 17{,}700$$
$$= 26{,}950 \text{ kip-in.}$$

CONCLUSIONS

This paper provides a study of the stiffness and carryover factors of non-prismatic members subject to lateral axial loads.

A program in Fortran IV was written and an IBM 1130 with a plotter was used for the generation of the charts. The charts indicate the variation of the stiffness or carry-over factors as a function of kL, where kL is a function of the axial load. The stiffness and carry-over factors can be used either for the analysis of a frame with non-prismatic members including the effect of axial loads or for determining the buckling loads of frames with non-prismatic members. In the latter case, the application of the method of successive approximations limits the analysis to the fundamental mode of buckling which is sufficient for design purposes.

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LEGEND FOR CHARTS

a_lL = Length of left haunch (see Fig. 1)

a_rL = Length of right haunch (see Fig. 1)

HH = Tangent of the angle of the haunch (see Fig. 1)

AB_F = Indicates stiffness at end **A** while the other end **B** is fixed

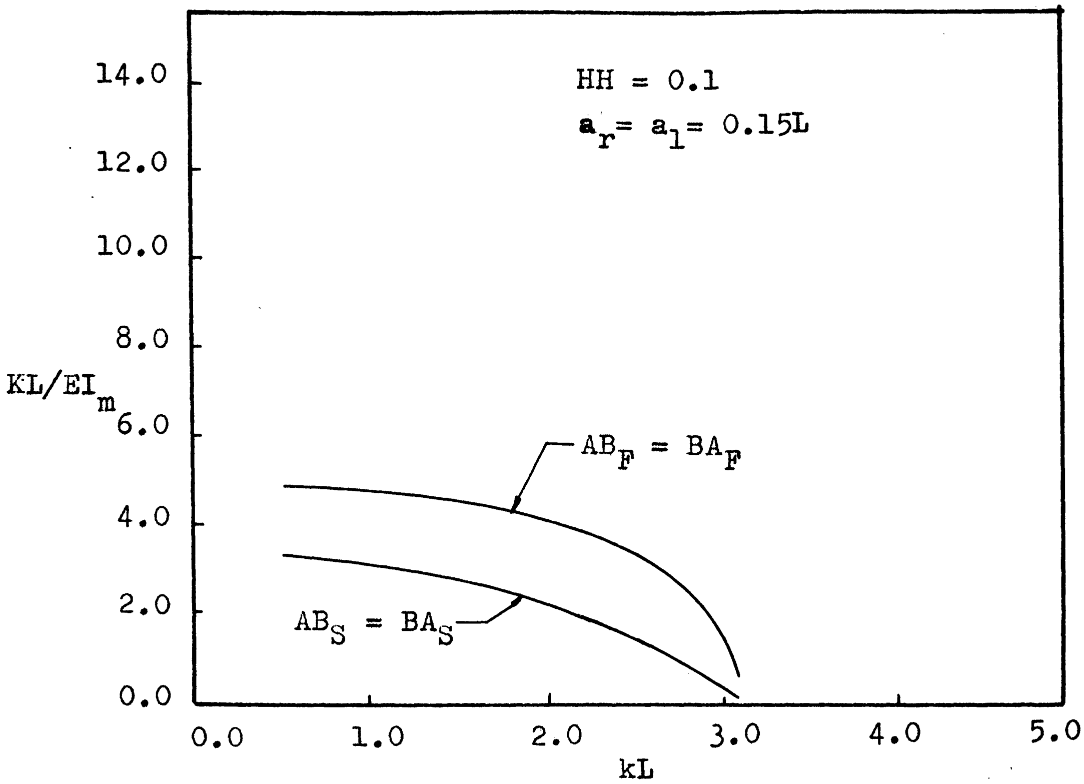
BA_F = Indicates stiffness at end **B** while the other end **A** is fixed

AB_S = Indicates stiffness at end **A** while the other end **B** is simply supported

BA_S = Indicates stiffness at end **B** while the other end **A** is simply supported

AB = Indicates the carry-over factor from **A** to **B**

BA = Indicates the carry-over factor from **B** to **A**



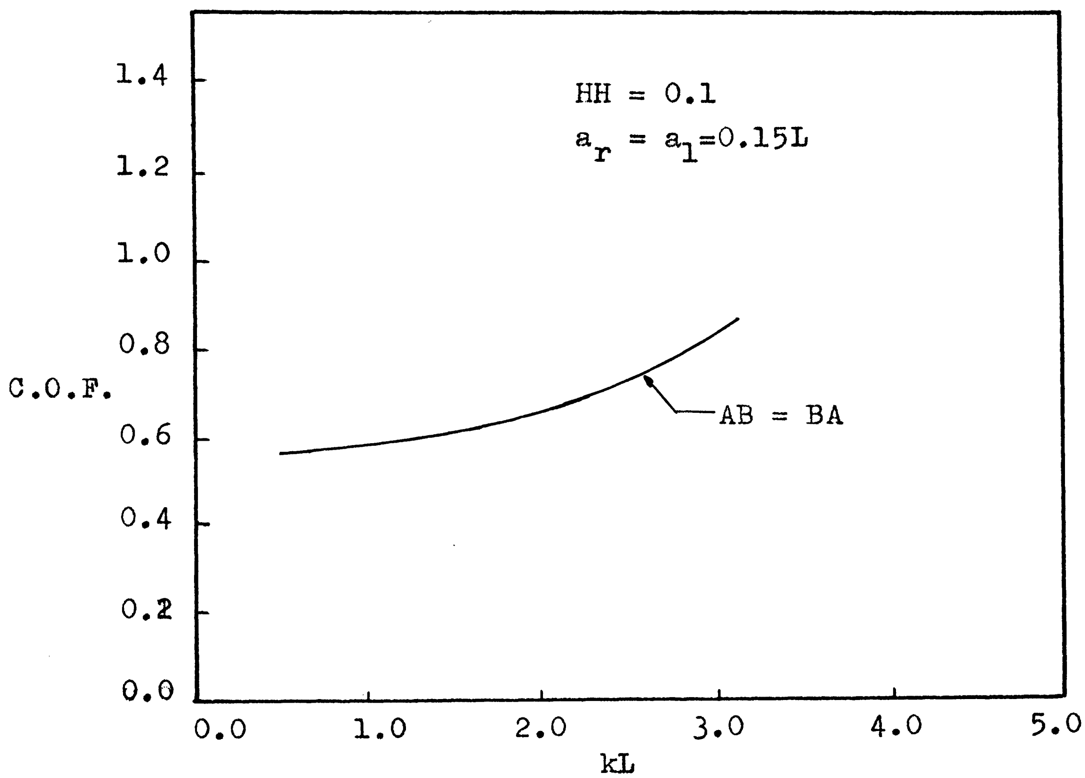
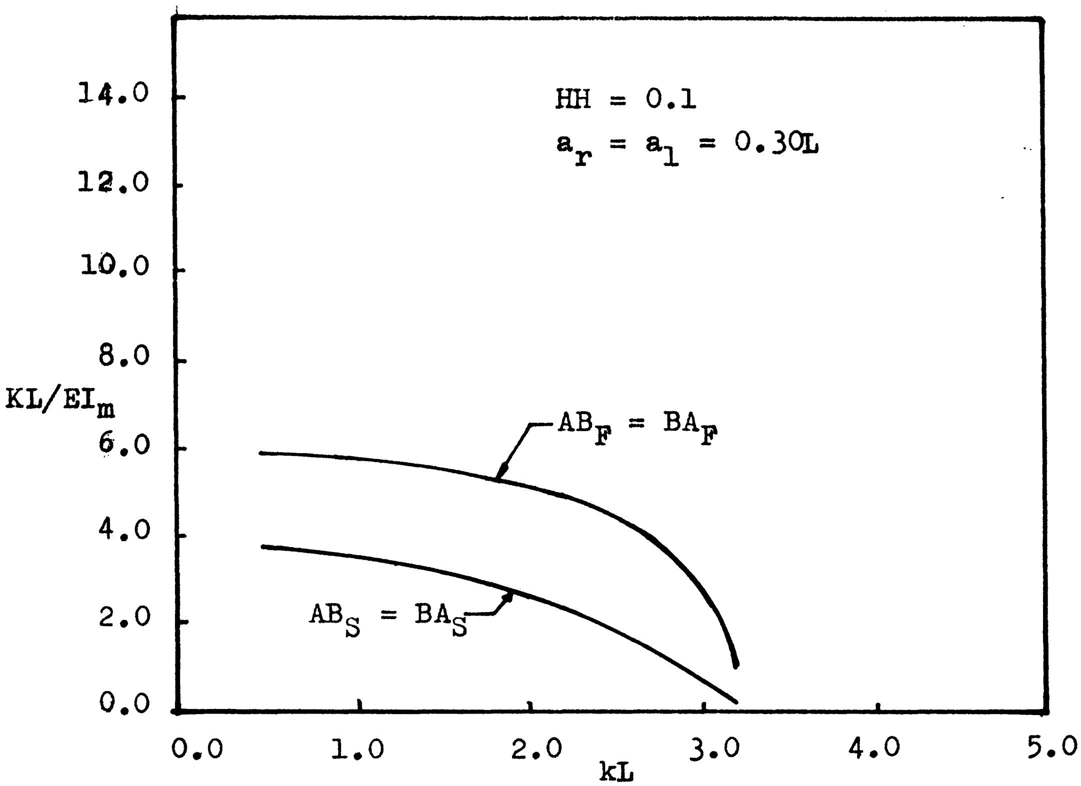


Chart 1. HH = 0.1, $a_r = a_l = 0.15L$



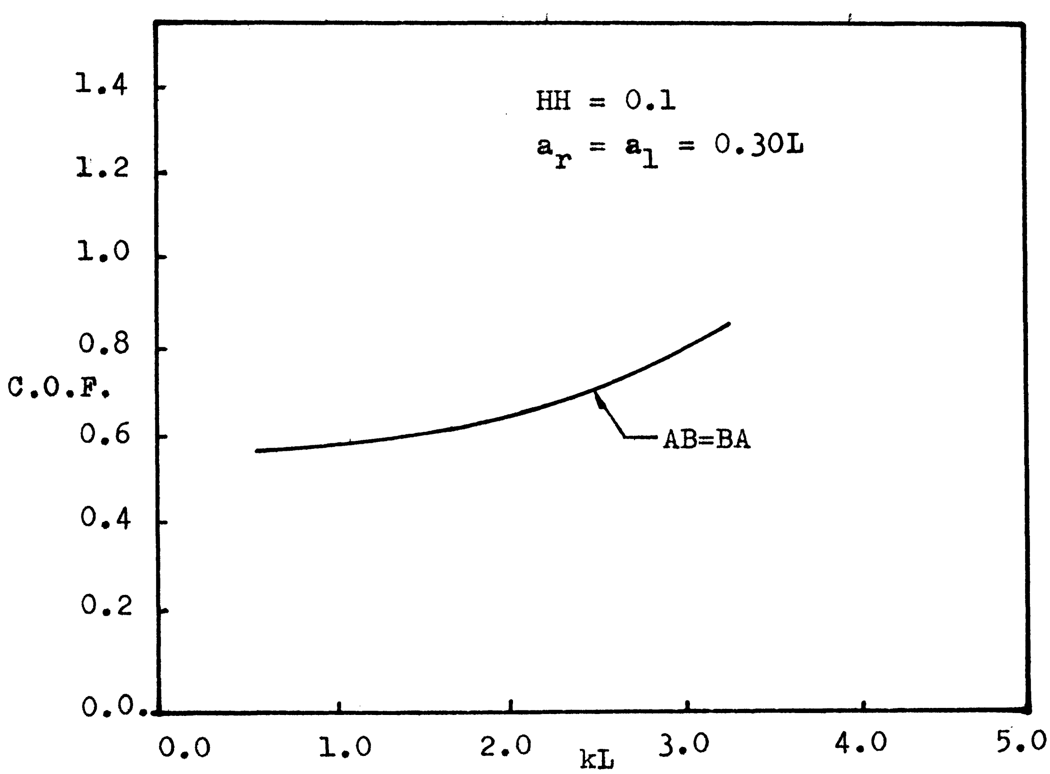
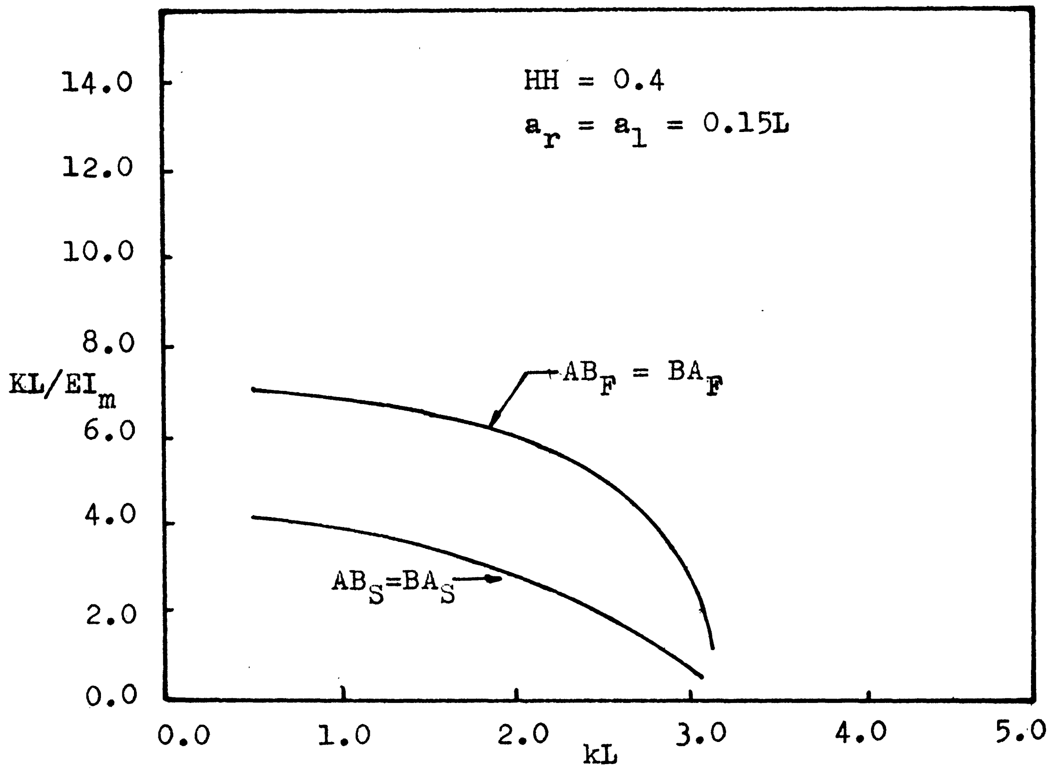


Chart 2. HH = 0.1, $a_r = a_l = 0.30L$



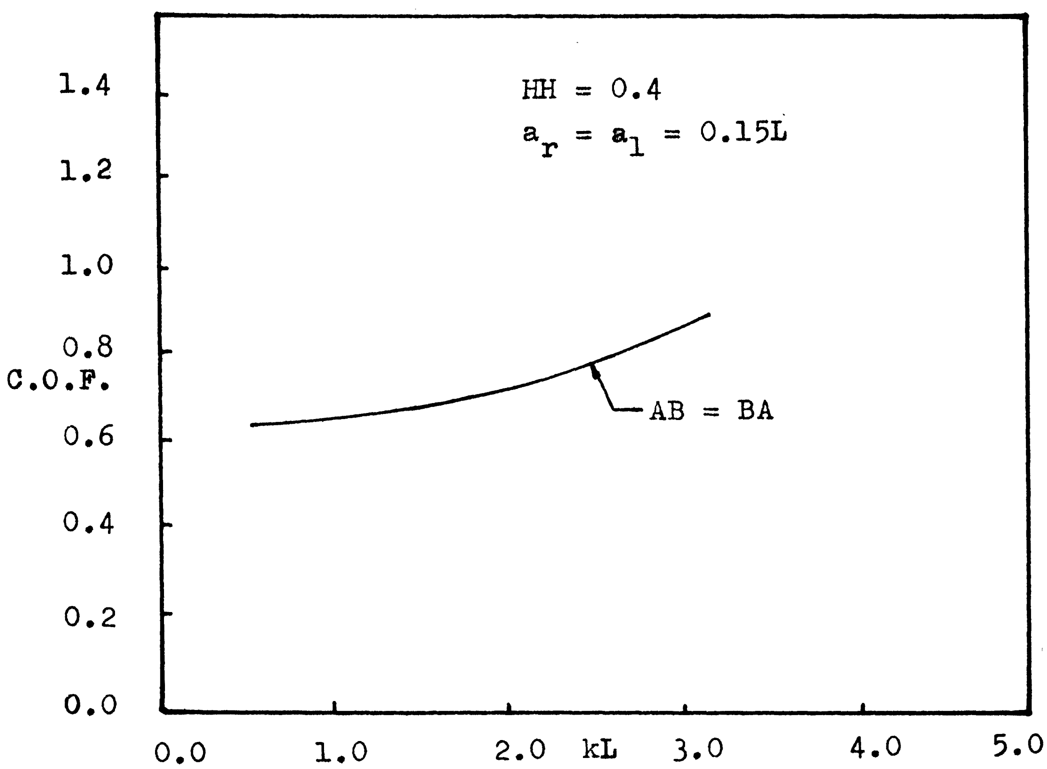


Chart 3. HH = 0.4, $a_r = a_l = 0.15L$

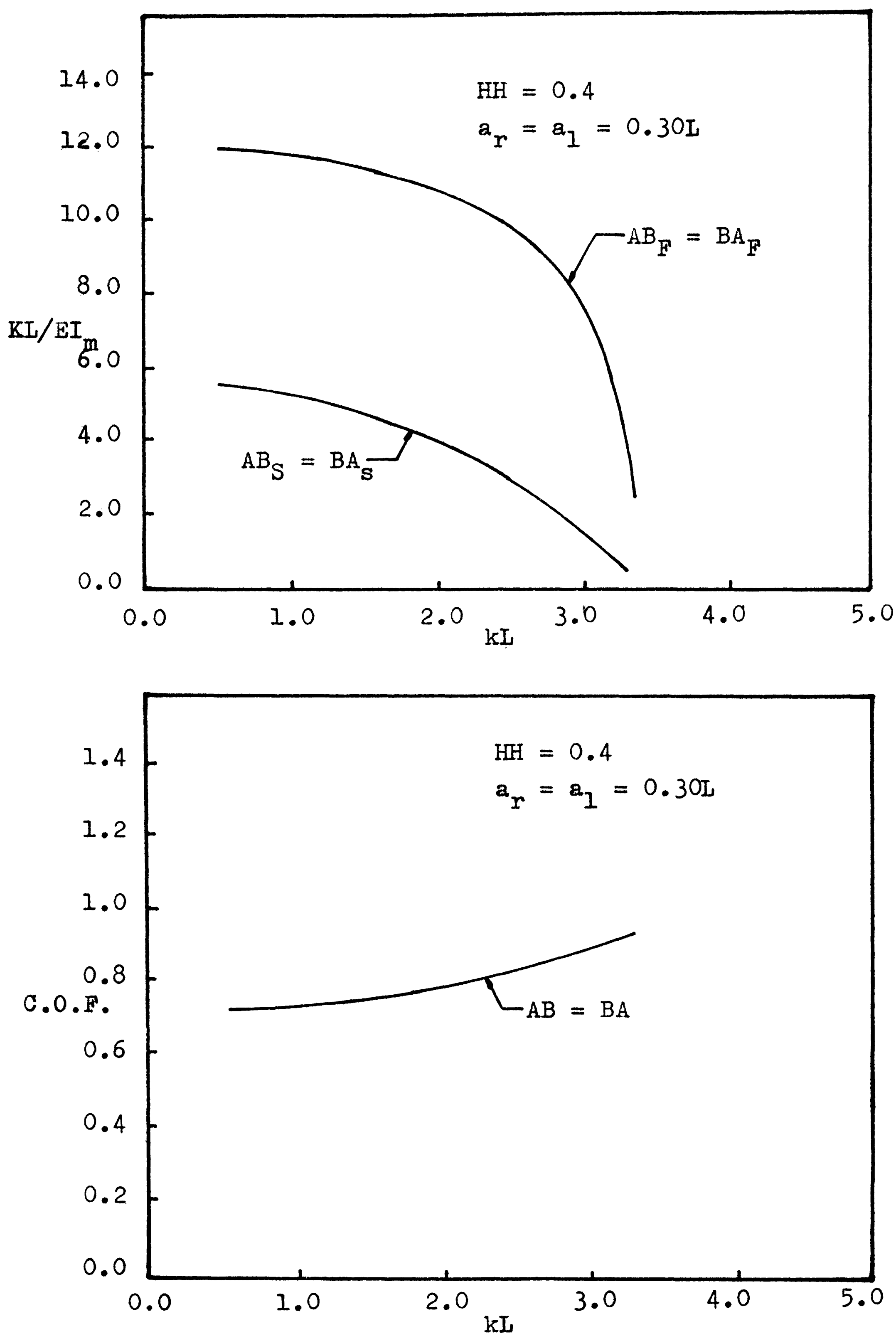
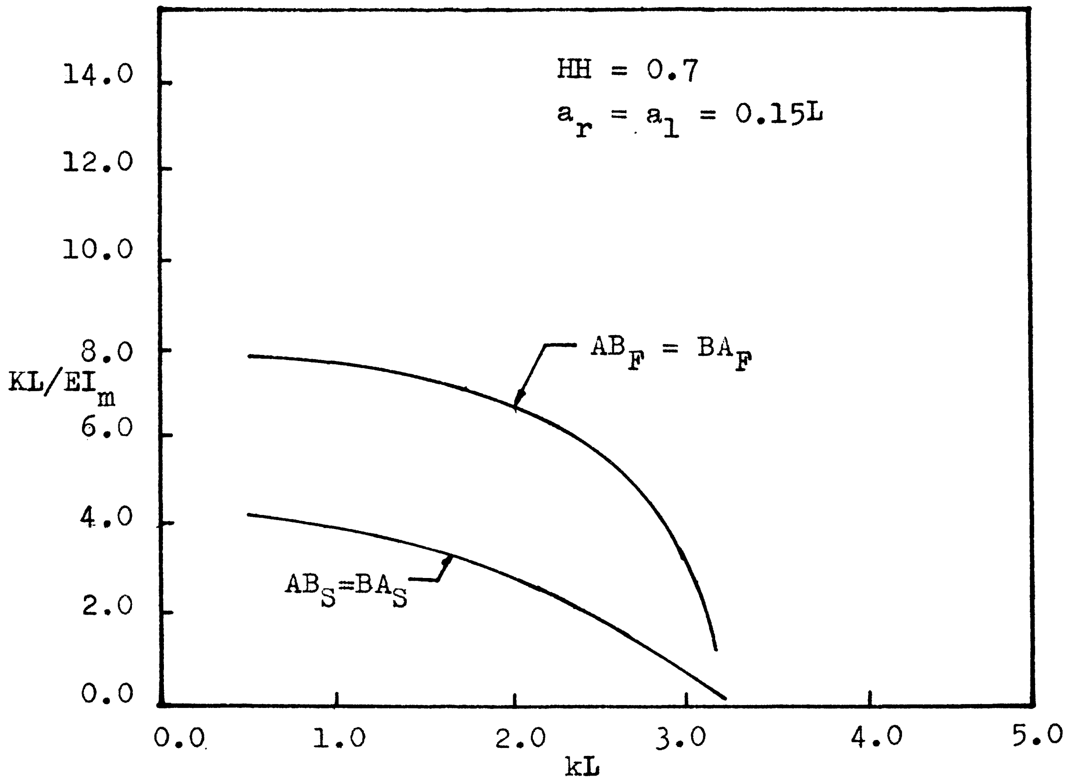


Chart 4. $HH = 0.4$, $a_r = a_l = 0.30L$



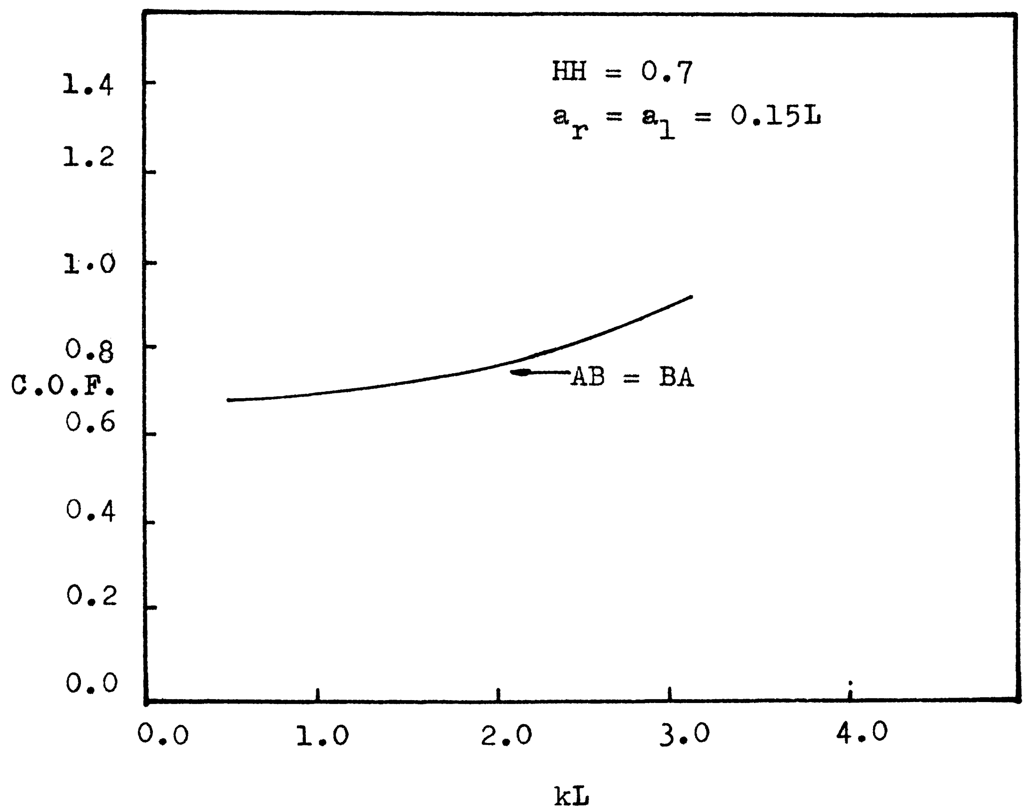
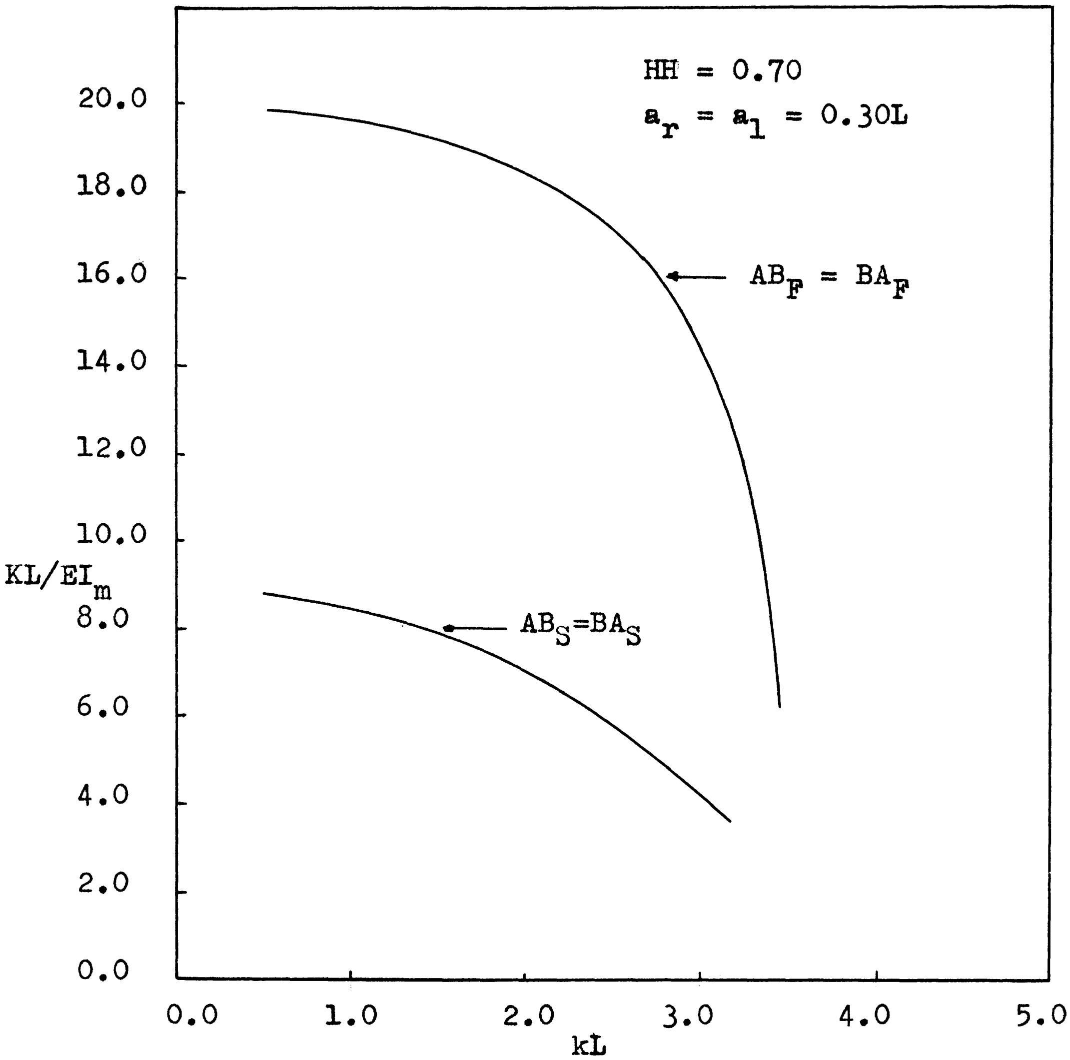


Chart 5. $HH = 0.7$, $a_r = a_l = 0.15L$



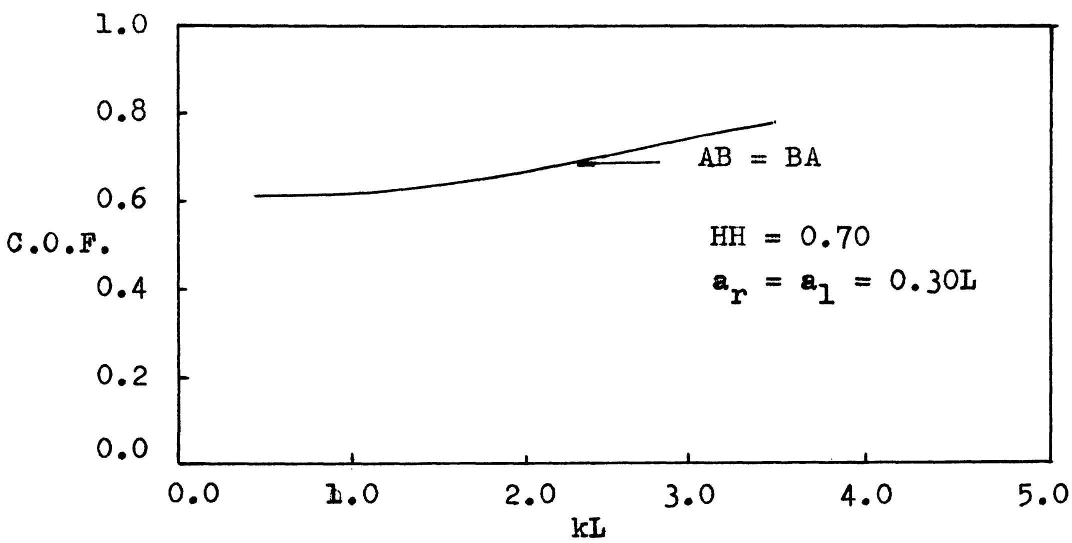
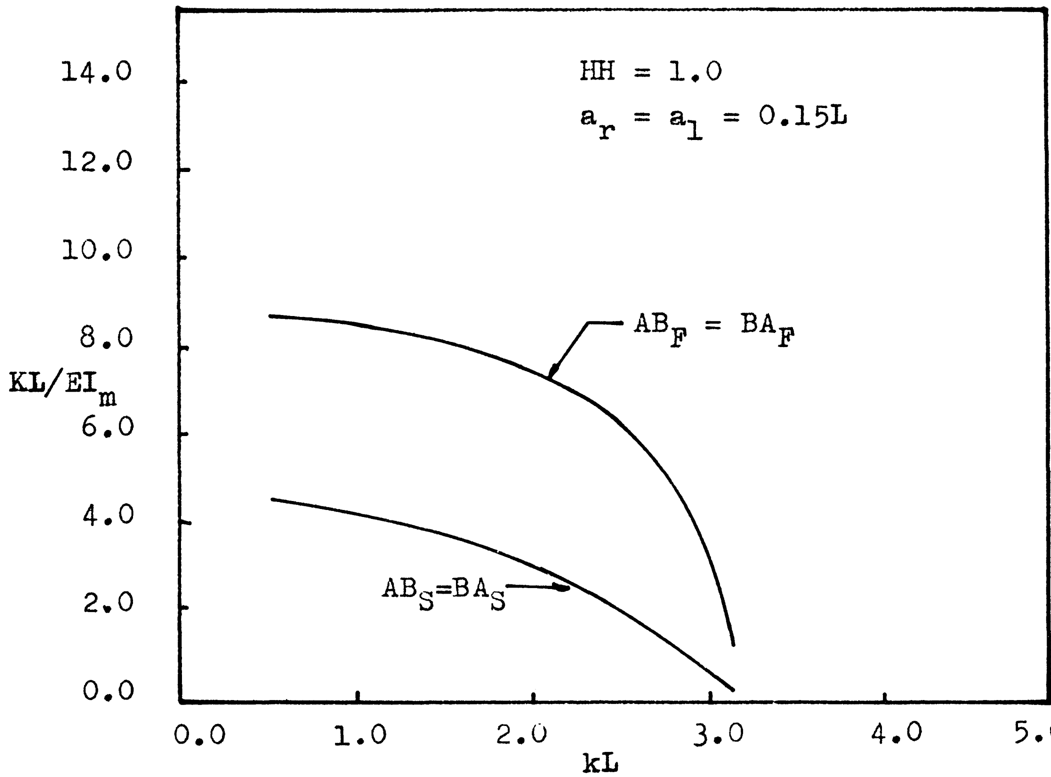


Chart 6. $HH = 0.7$, $a_r = a_l = 0.30L$



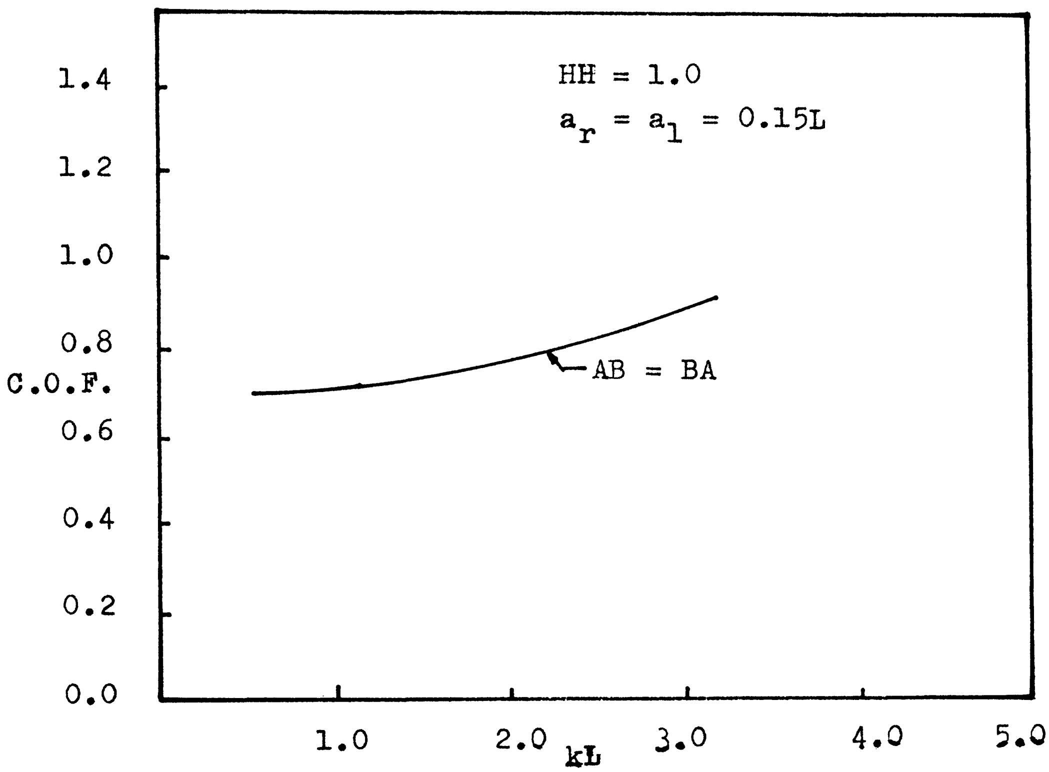
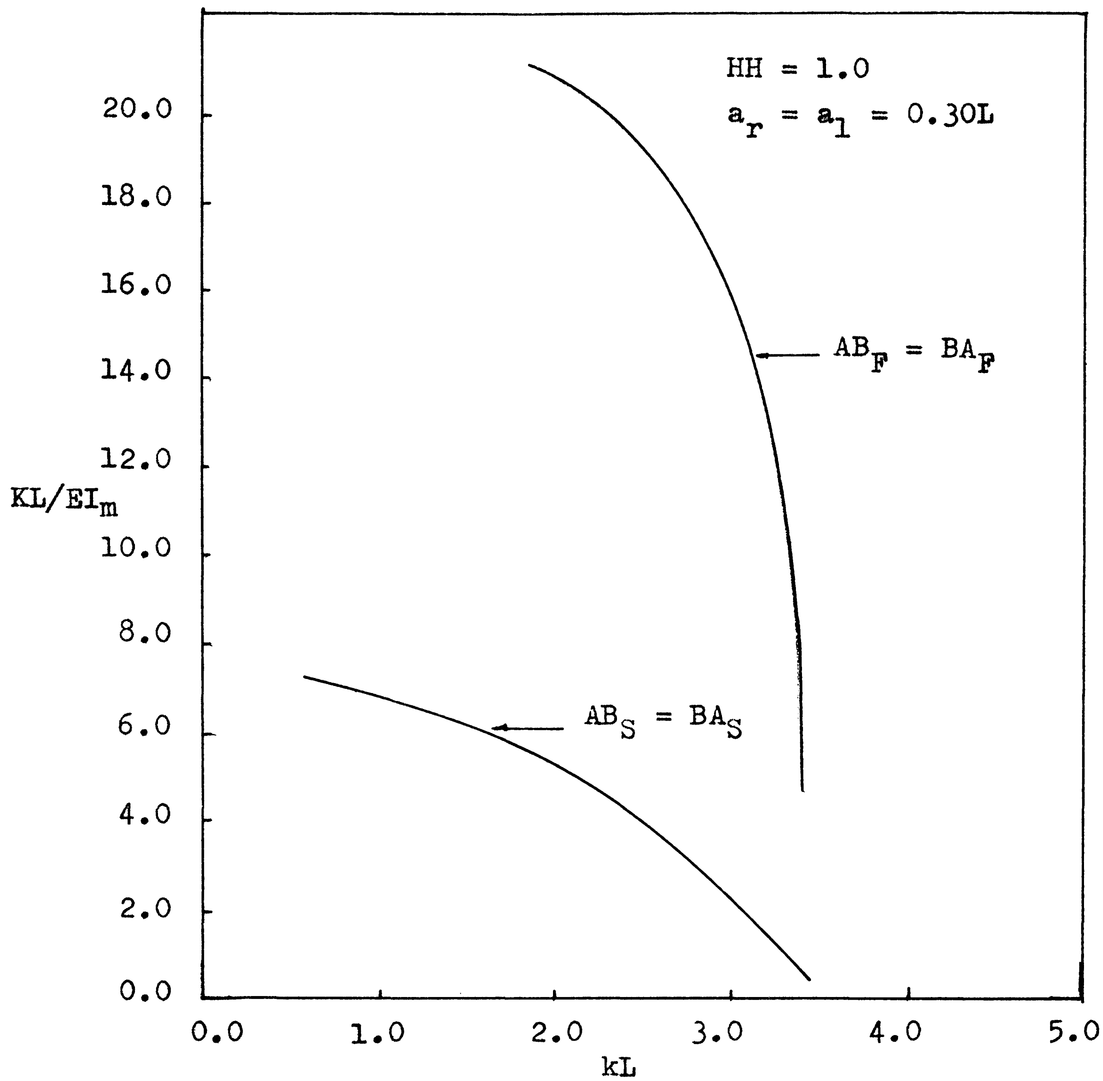


Chart 7. HH = 1.0, $a_r = a_l = 0.15L$



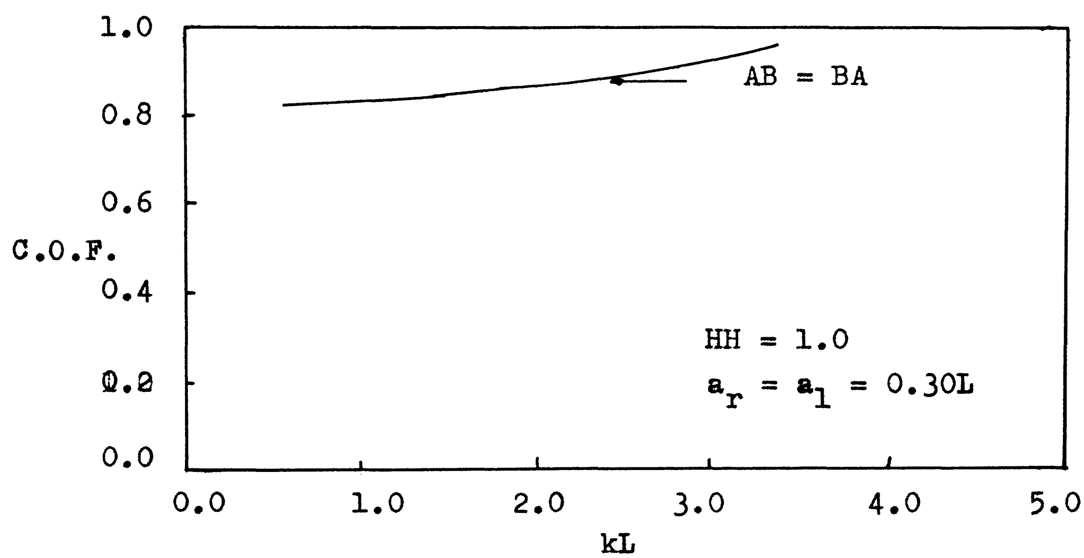
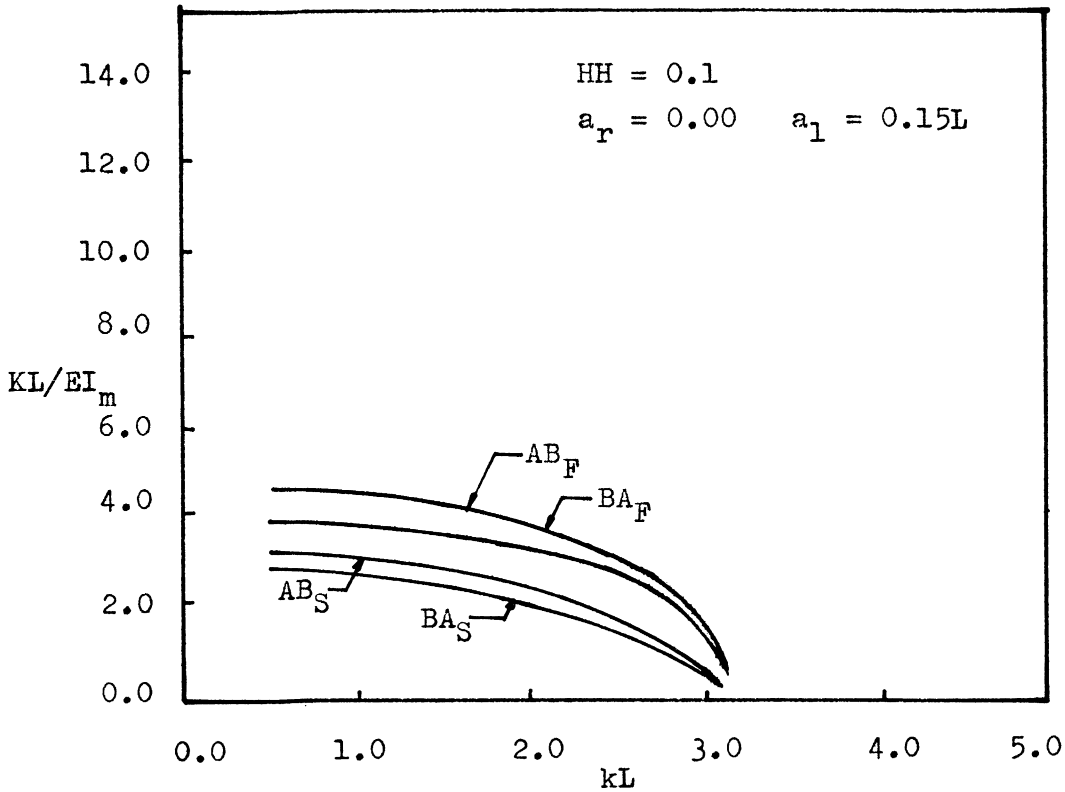


Chart 8. $HH = 1.0$, $a_r = a_l = 0.30L$



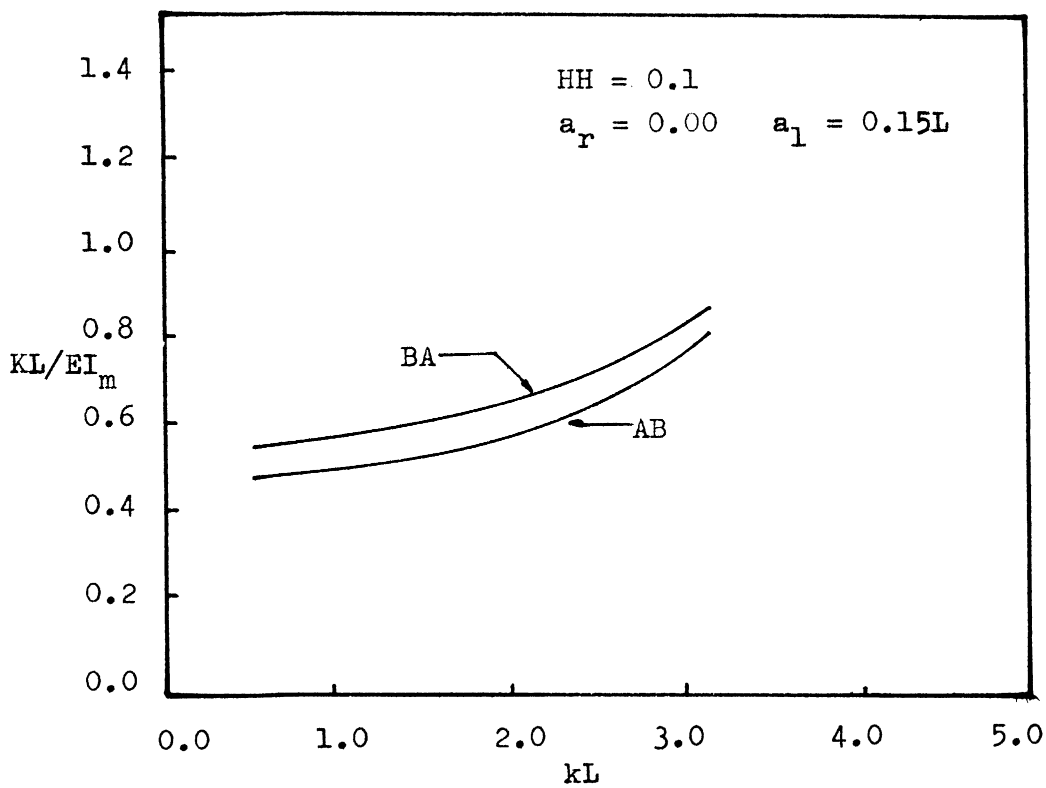
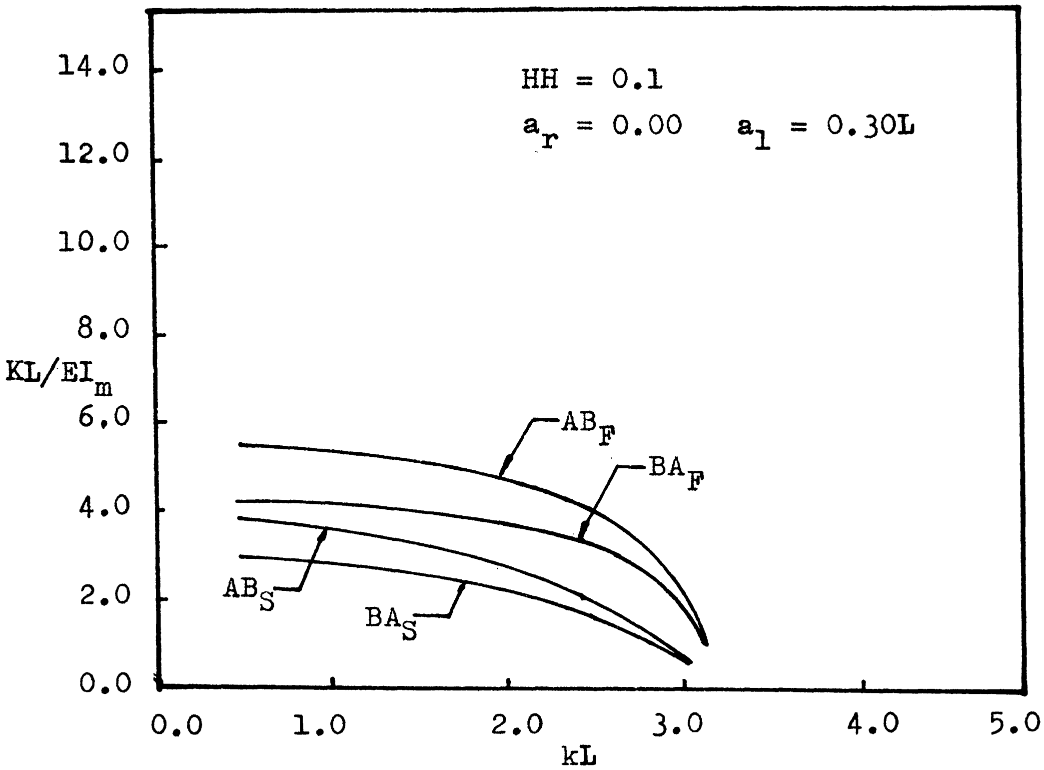


Chart 9. $HH = 0.1$, $a_r = 0.00$, $a_l = 0.15L$



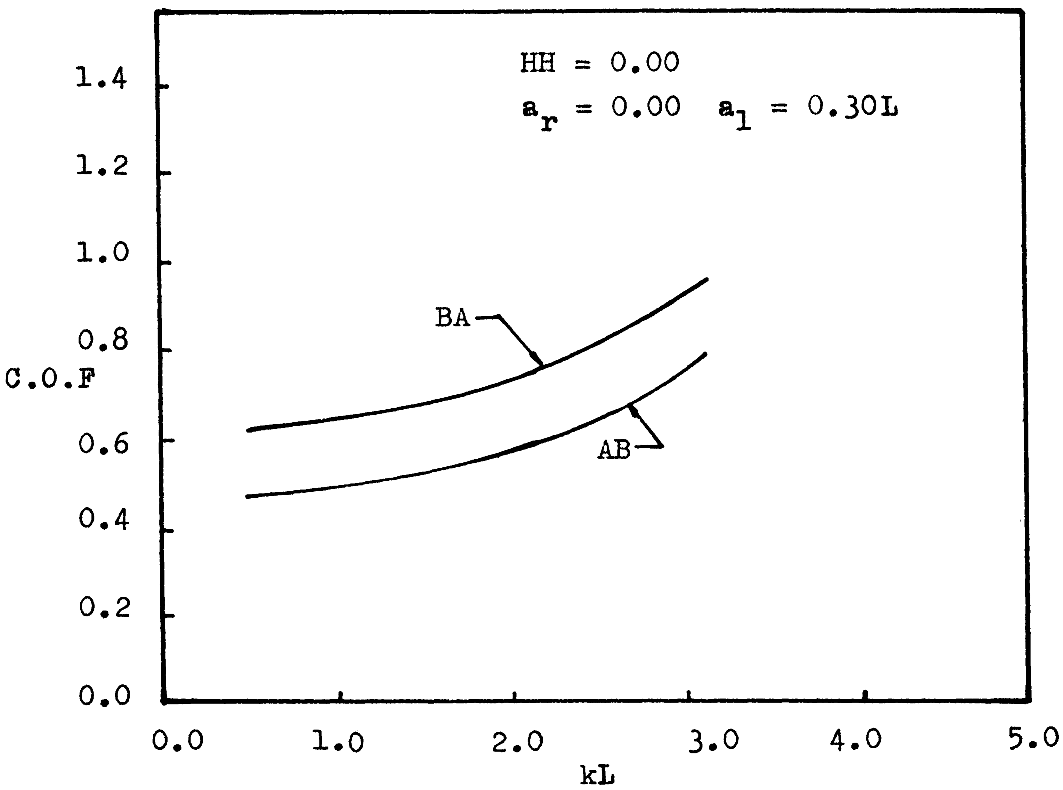
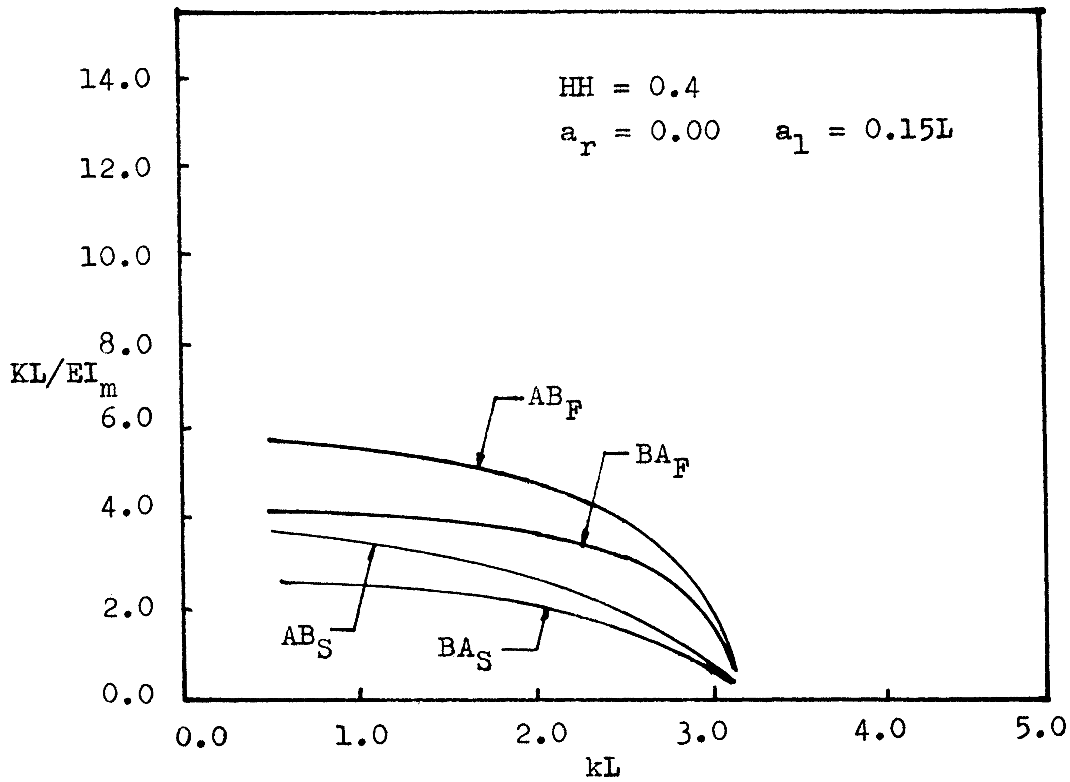


Chart 10. $HH = 0.1$, $a_r = 0.00$, $a_l = 0.30L$



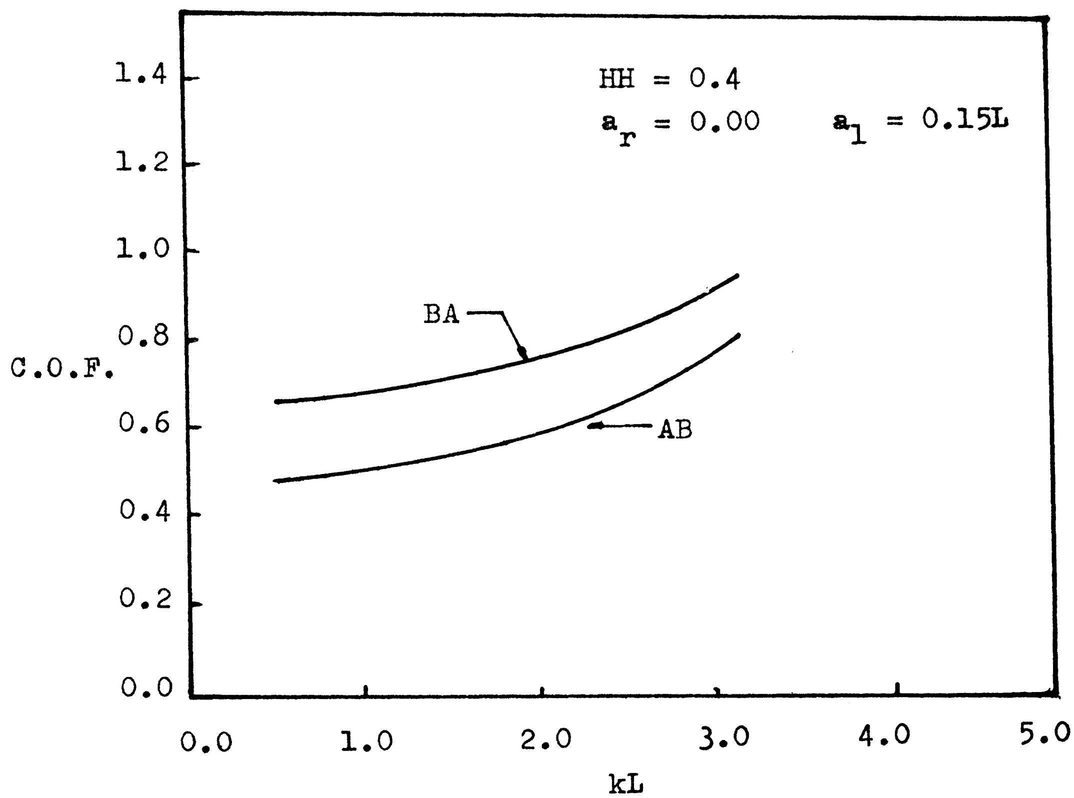
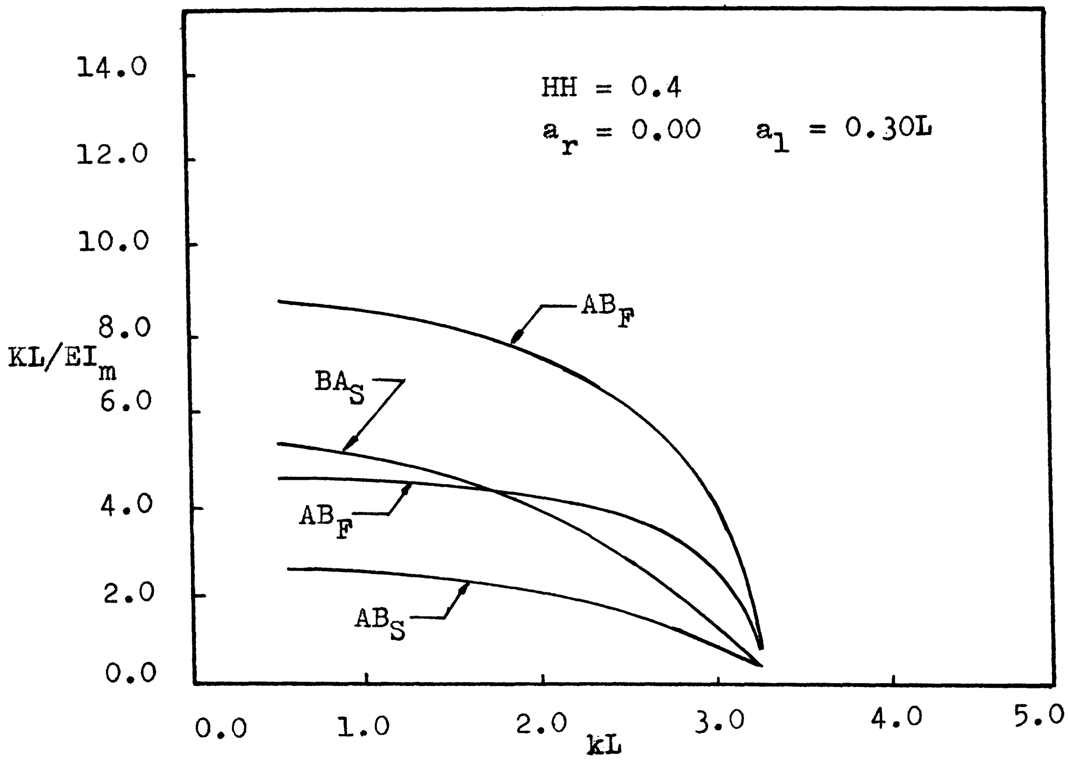


Chart 11. $HH = 0.4$, $a_r = 0.00$, $a_l = 0.15L$



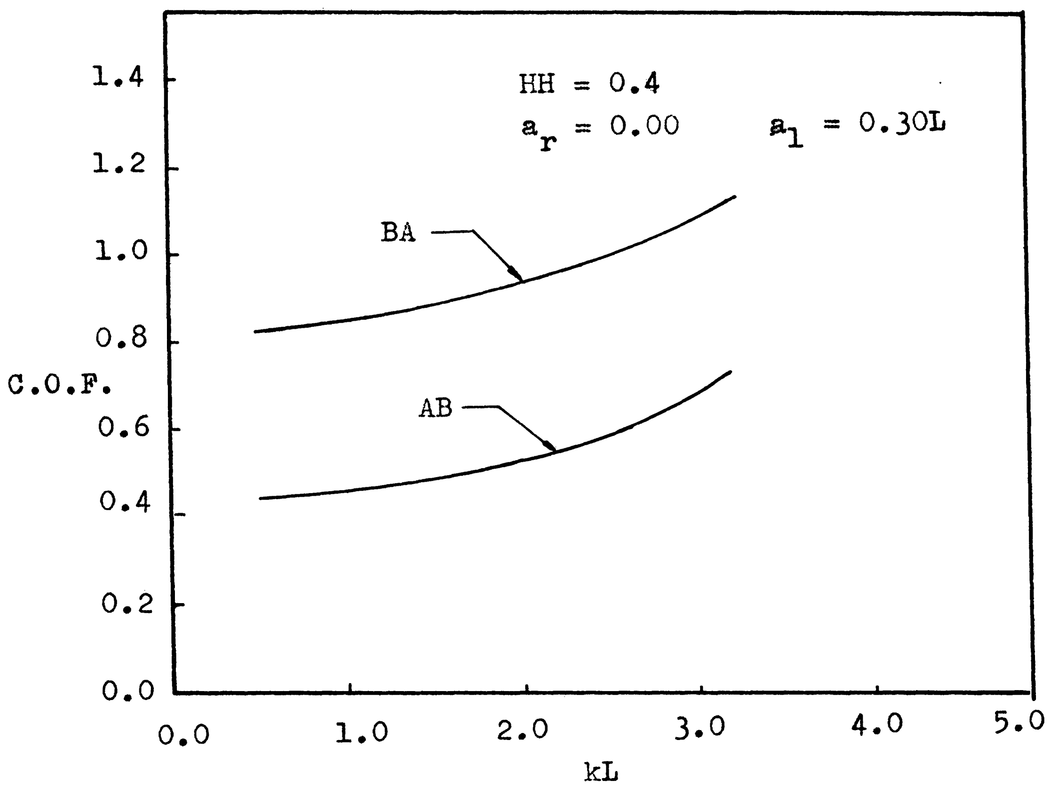
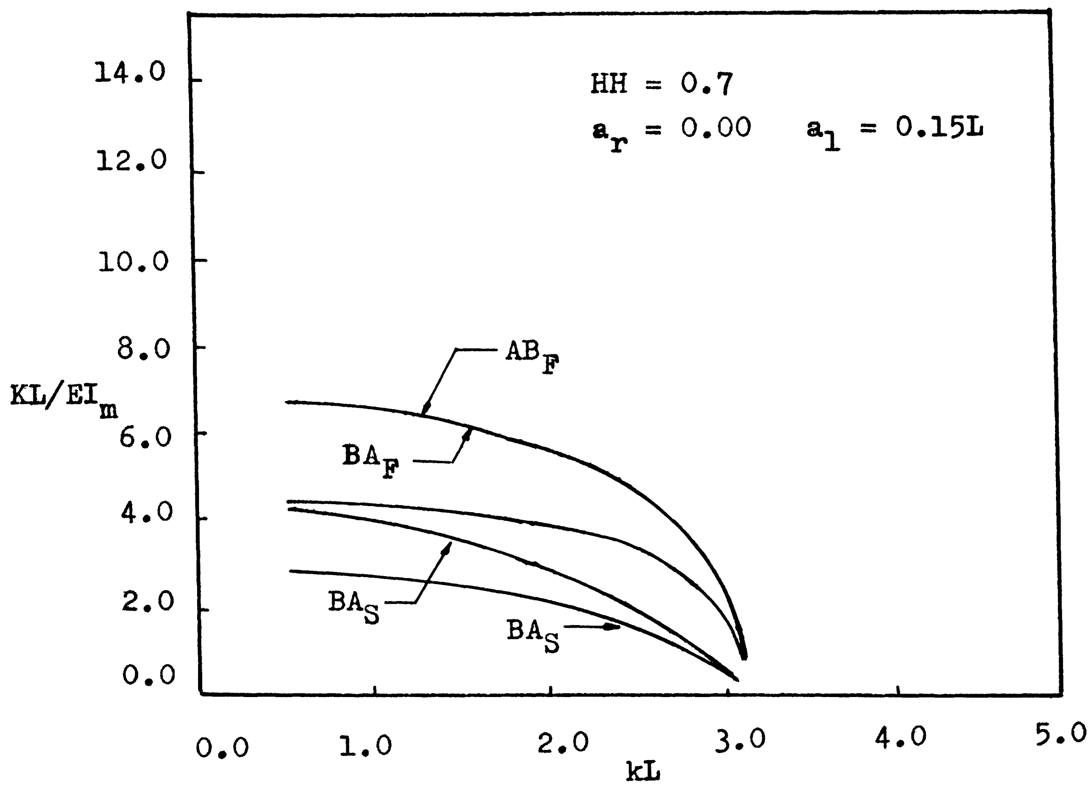


Chart 12. HH = 0.4, a_r = 0.00, a_l = 0.30L



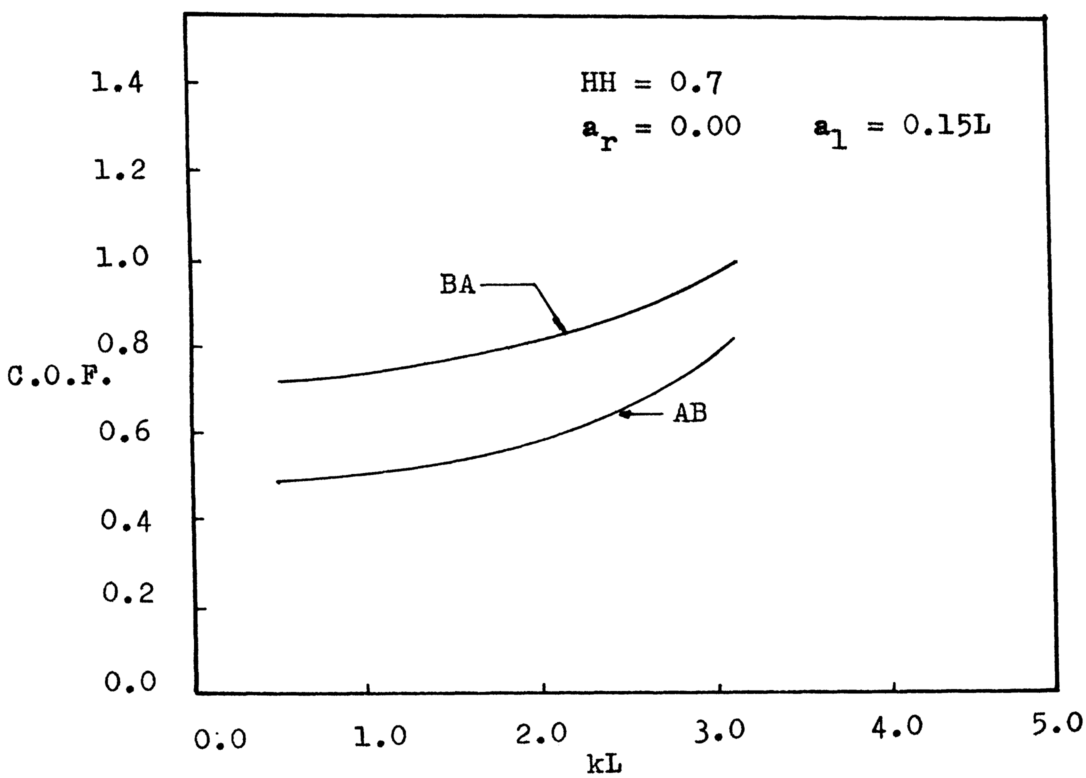
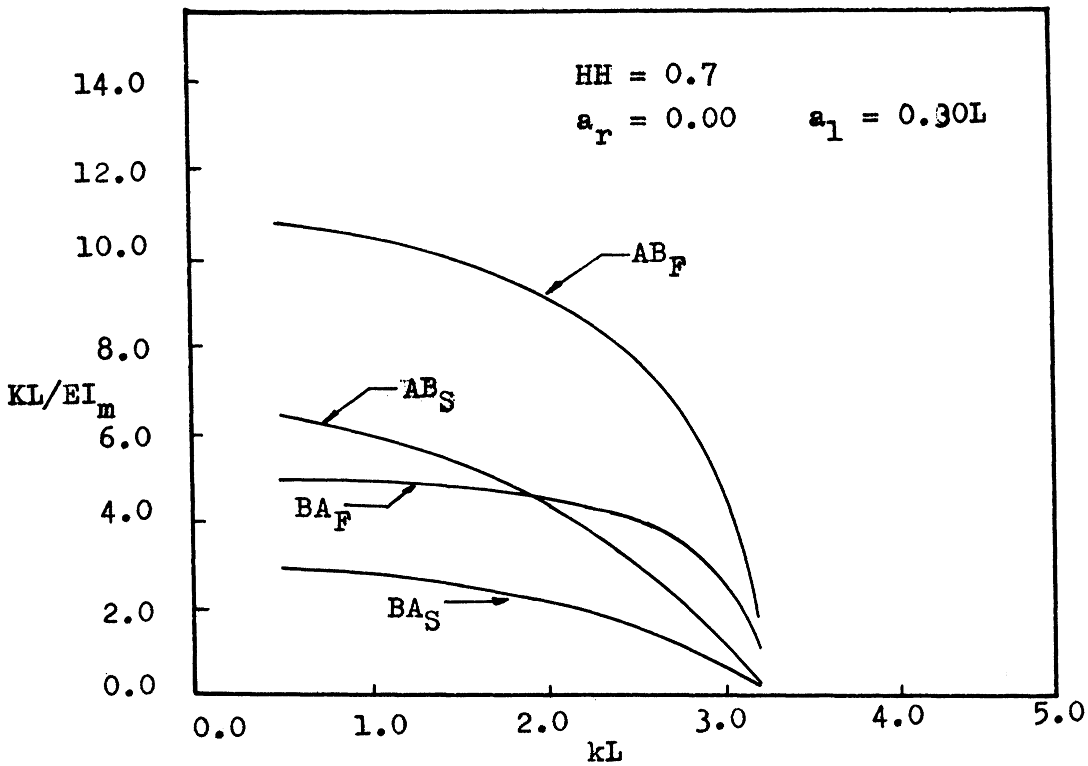


Chart 13. $HH = 0.7$, $a_r = 0.00$, $a_l = 0.15L$



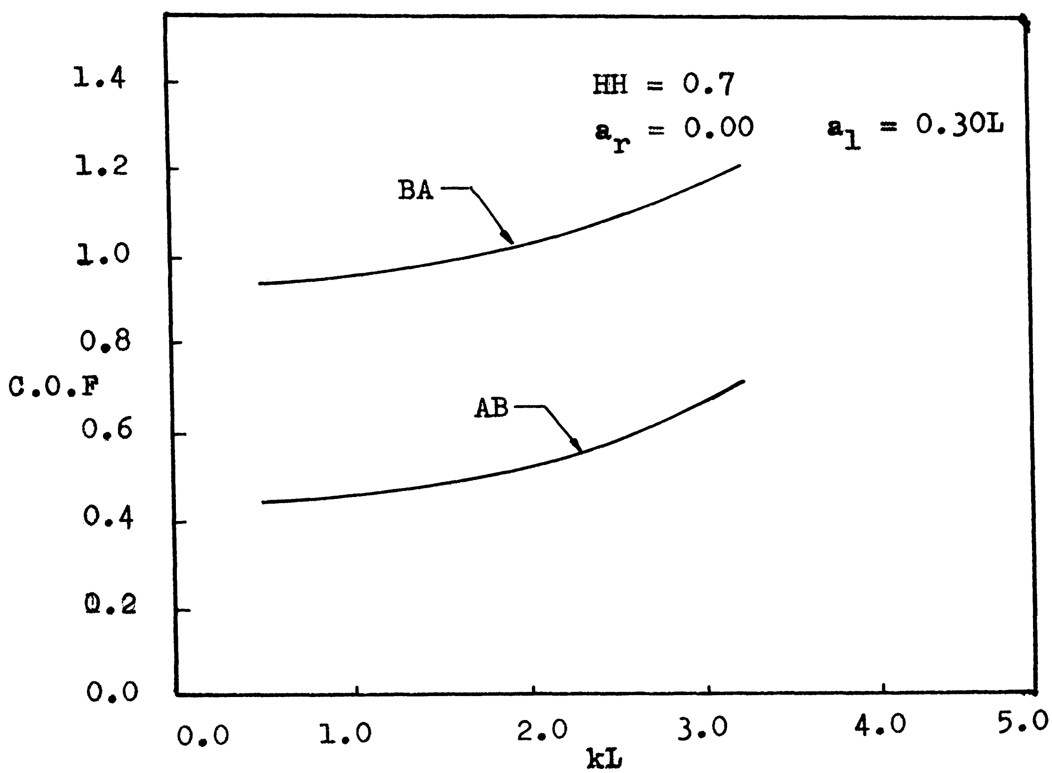
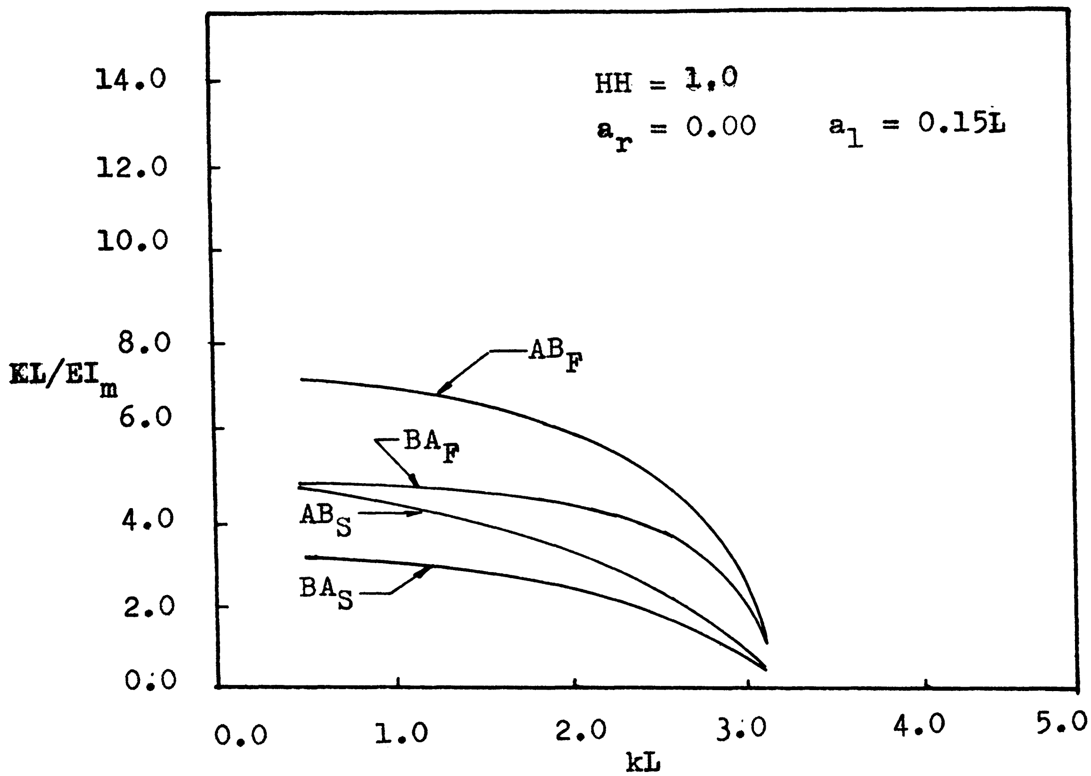


Chart 14. HH = 0.7, a_r = 0.00, a_l = 0.30L



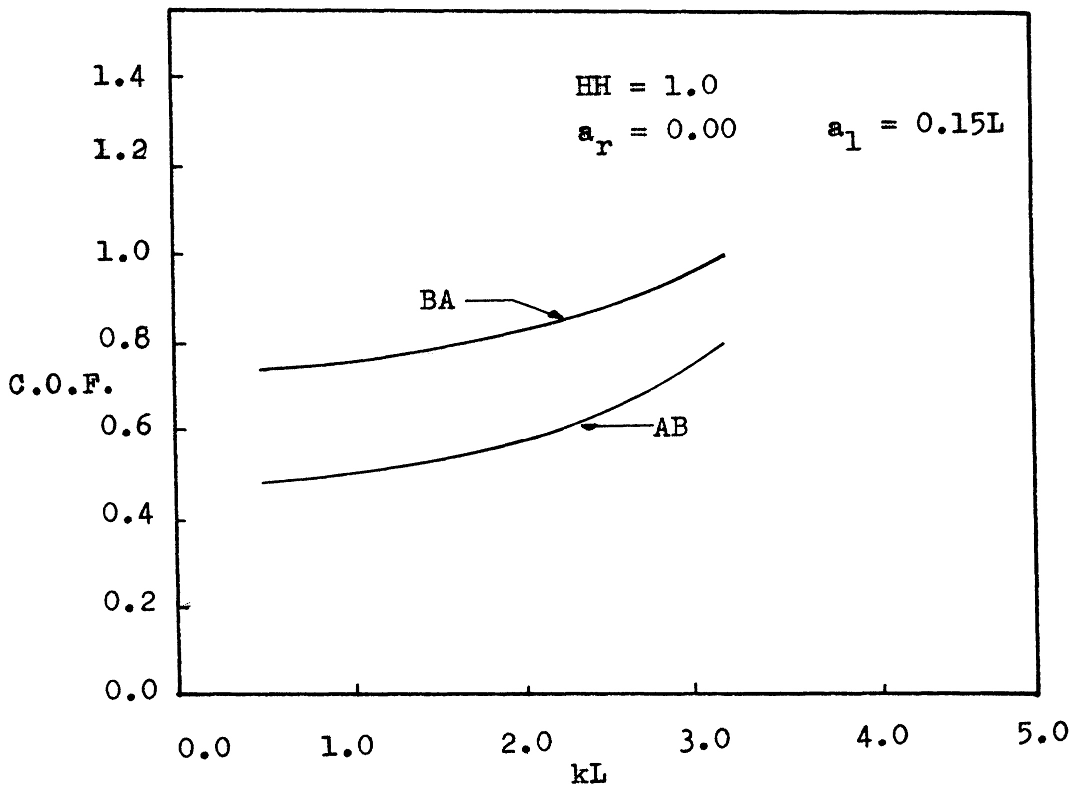
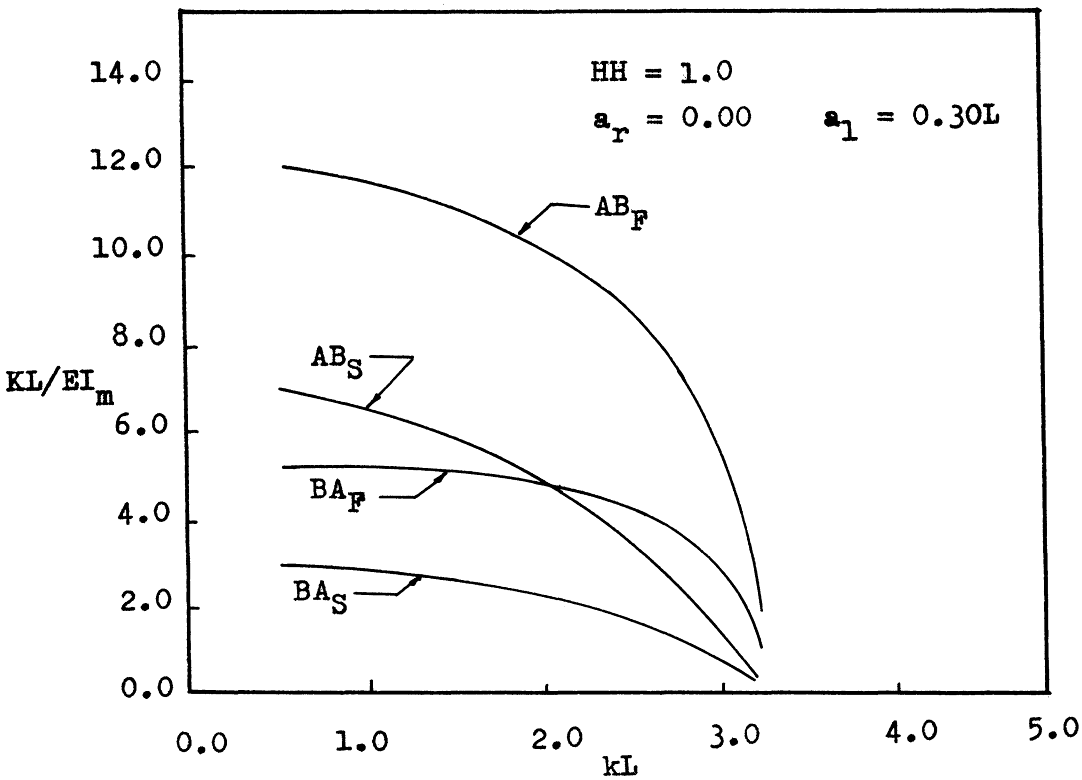


Chart 15. $HH = 1.0,\ a_r = 0.00,\ a_l = 0.15L$



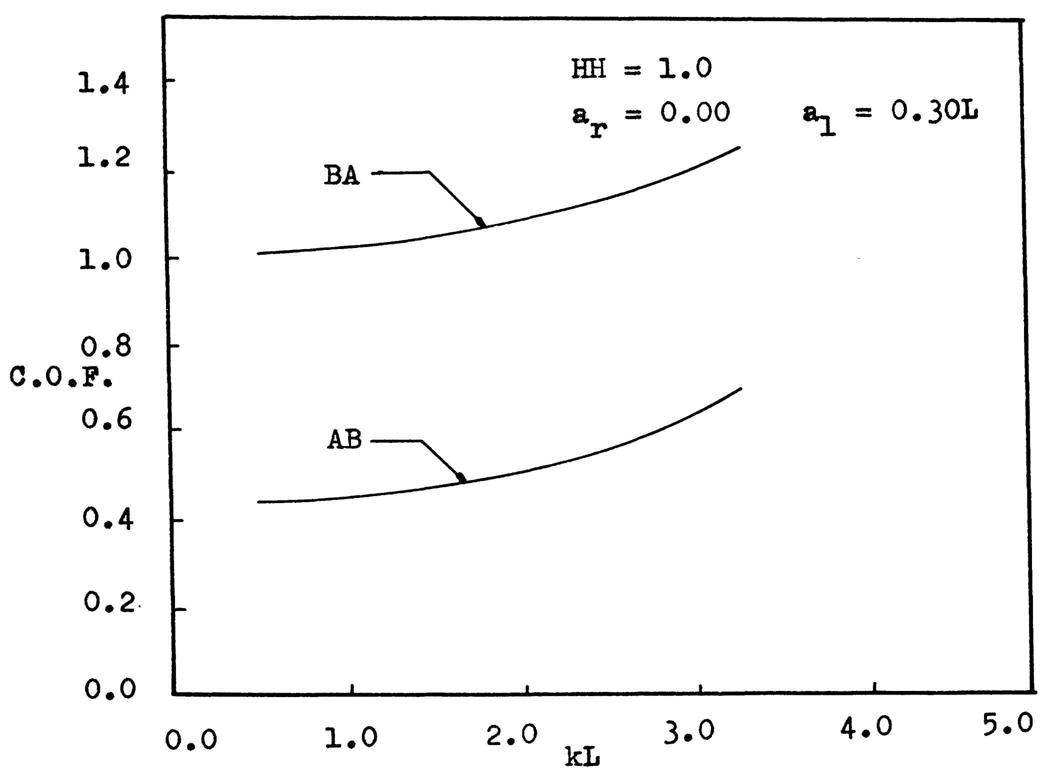


Chart 16. HH = 1.0, a_r = 0.0, a_l = 0.30L