Buckling of Hyperbolic Paraboloids

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LARGE SPHERICAL DOME roofs are being constructed regularly because of their beauty, adaptability, and low cost. A decade ago this type of roof was rather rare because of the high cost of construction and the significant unknowns in the design. Considerable research and development in design and construction has resulted in analysis procedures that have a high degree of reliability and are carried out at minimum cost (often the dome analysis is much more economical than the analysis of a similar size conventional type structure.)[1,2] Ingenious fabrication and erection methods have paralleled the analysis development with resulting relatively low final costs. Designs have been carried to completion for roofs up to 850 ft in base diameter (the Astrodome is about 620 ft in base diameter). Preliminary feasibility designs have been made for roofs over one mile in diameter.

The buckling analysis is an important part of dome design. Several large domes have failed in the last decade and present methods of buckling analysis have been used to explain the failures. Experience in the field and laboratory has allowed reductions in factors of safety from about 10 to as low as 1.5 with satisfactory results.

The hyperbolic paraboloid has long been a favorite of owners and architects. In spite of this, many times the hypar has not been built because of the seemingly high cost and difficulty of analysis. The large hypar seems to be about a decade behind the dome in its development. Besides its aesthetic value, the hypar is ideal for structures that must span a square or rectangular area. The writer is of the opinion that the hypar can be used to span large open areas, and that in the next few years it will be more economical than the two-way truss and similar systems and will approach the economy of the domed structure.

As is the case for a practical economical domed structure, stability will be a critical item in economical hypars. The purpose of this paper is to present theoretical equations that will assist the professional in designing large hypars. The writer would like to emphasize that load-deflection analyses to predict buckling loads based upon standard computer programs have not been very successful in dome designs. Non-conservative results are usually obtained using these procedures. Similar unreliable results could be expected for hypars. Therefore, the following is given as a start towards a reliable hypar buckling theory.

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STABILITY THEORY

A typical hypar that is made of structural shapes is shown in Fig. 1. The covering of the structural shapes can be designed to be structurally effective if desired. The following results can also be used should the covering have structural rigidity.

A number of solutions have been published for homogeneous hypars. Reissner[3] used a traditional approach in which the equilibrium and compatibility equations for an element were solved to arrive at a differential equation which was solved for the buckling load. Leet[4] ran experiments and found that initial imperfections and deflections had an effect on the critical load. Many of the results were similar to those for domes. Dayaratnam and Gerstle[5] used an energy approach to arrive at the critical load. Banavalkar and Gergelz[6] used a finite element approach combined with the energy method to determine the buckling load for a hypar built of cold-formed light-gage panels.

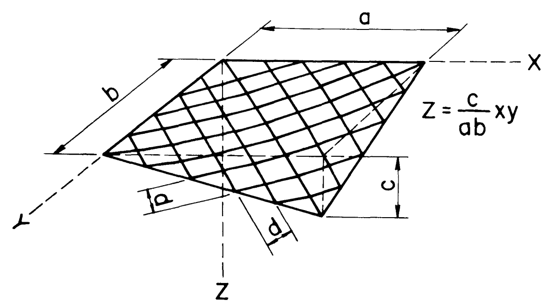

Figure 1.

Reissner[3] arrived at the following equation:

$$p_{cr} = \frac{\pi^2 D}{l_{cr}^2}\frac{c}{ab} + \left(\frac{c}{ab}\right)^3 \frac{4Et}{\pi^2} l_{cr}^2 \qquad (1)$$

where:

p_{cr} is the critical vertical buckling load (load per unit area)
a, b, and c are as shown in Fig. 1
t is the thickness of a homogeneous hypar
E is the modulus of elasticity
D is the flexural rigidity given by $D = \dfrac{Et^3}{12(1-v)^2}$
v is Poisson's ratio
l_{cr} is the critical buckling length given by

$$l_{cr}^4 = \frac{\pi^4 D}{4Et}\left(\frac{ab}{c}\right)^2 \qquad (2)$$

Equation (1) suggests a way to arrive at the buckling load for a steel framed or latticed or stiffened hypar. The procedure is similar to that already shown to be useful for domes, plates and other shell structures. The first term contains the flexural rigidity D and the second term contains the extensional rigidity Et. The buckling load for framed and other hypars can be solved provided equivalent flexural and extensional rigidities can be found for nonhomogeneous hypars. This so called "split rigidity concept" has been used by the writer to analyze many domes in the last several years. It is often convenient to find effective bending and membrane or extensional thicknesses to replace the homogeneous thicknesses in Eq. (1). From Fig. 1, the effective bending thickness is given by (assuming Poisson's ratio is zero):

$$D = E\frac{1}{12}t_B^3 = \frac{EI}{d}$$

or

$$t_B = \left(\frac{12I}{d}\right)^{1/3} \qquad (3)$$

where:

t_B is the effective bending thickness
I is the moment of inertia of the member
d is the spacing of the member

The effective membrane or extensional thickness is given by

$$Et_m = E\frac{A}{d}$$

or

$$t_m = \frac{A}{d} \qquad (4)$$

where:
t_m is the effective membrane thickness
A is the area of the member

For a framed, latticed or stiffened hypar, Eqs. (1) and (2) become

$$p_{cr} = \frac{2}{\sqrt{3}} Et_m^2\left(\frac{t_B}{t_m}\right)^{3/2}\frac{c^2}{a^2b^2} \qquad (5)$$

and the critical buckling length or length of buckle is

$$l_{cr} = \left[\frac{\pi^4}{48}\frac{t_B^3}{t_m}\left(\frac{ab}{c}\right)^2\right]^{1/4} \qquad (6)$$

It is not at all unusual for the effective bending thickness to be 20 to 30 times the effective membrane thickness for a practical framed shell. As a result, the buckling load or the structural efficiency of a framed shell is very high. This is a major reason why domes have become more economical, and thus more popular, in recent years. Equation (5), with the term $(t_B/t_m)^{3/2}$, indicates the same should be true for hypars.

Design Example—Assume that the following data is given (see Fig. 1):

$a = b = 150$ ft
$c = 20$ ft
$l = 4$ ft
Members are C10×25
Area = 7.35 in.^{2}
$I = 144$ in.^{4}

Then,

$$t_m = \frac{7.35}{48} = 0.16 \text{ in.}$$

$$t_B = \left[\frac{12 \times 144}{48}\right]^{1/3} = 3.3 \text{ in.}$$

$$p_{cr} = \frac{2}{\sqrt{3}} \times 29 \times 10^6 \times (0.16)^2\left(\frac{3.3}{0.16}\right)^{3/2} \times \left(\frac{20}{150 \times 150 \times 12}\right)^2$$

$$= 0.45 \text{ lb/in.}^2 = 65 \text{ lbs/ft}^2$$

$$l_{cr} = \left[\frac{\pi^4}{48}\frac{(3.3)^3}{0.16}\left(\frac{150 \times 150 \times 12}{20}\right)^2\right]^{1/4}$$

$$= 540 \text{ in.} = 45 \text{ ft}$$

DISCUSSION AND SUMMARY

The previous example illustrates one of the serious problems a designer would face in a typical problem. If the design live plus dead load were 40 lbs/ft^2, the previous calculation would indicate a factor of safety of about 1.6. It is questionable if the factor of safety is actually this high. Deflections (of the order of several inches) due to dead and live loads and fabrication and erection tolerances could reduce the critical load seriously. Therefore, the designer must, in this case, estimate the effects of other factors. Methods for doing this are available for domes. However, very little information is available for hypars.

In addition to the deflections from all causes, plasticity, residual stresses, and edge beams will have an effect on the critical buckling load. They can be approximated by an experienced designer.

General techniques are available for a detailed engineering evaluation of the points outlined above.[7] These could be applied to hypars and will probably be carried out in the near future.

REFERENCES

1. *Buchert, K. P.* Space Frame Buckling *AISC Engineering Journal, Vol. 5, No. 4, October, 1968.*
2. *Buchert, K. P. and Crooker, J. O.* Reticulated Space Structures *Journal of the Structural Division, ASCE, Vol. 96, No. ST3, March, 1970.*
3. *Reissner, E.* Some Aspects of the Theory of Thin Elastic Shells *Journal, Boston Society of Civil Engineers, Boston, Mass., Vol. XLII, No. 2, April, 1956.*
4. *Leet, K. M.* Study of Stability in the Hyperbolic Paraboloid *Journal of the Mechanics Division, ASCE, Vol. 92, No. EMI, February, 1966.*
5. *Dayaratnam, P. and Gerstle, K. H.* Buckling of Hyperbolic Paraboloids *World Conference on Shell Structures, October, 1962, San Francisco, California, National Academy of Sciences.*
6. *Banavalkar, P. V. and Gergelz, P.* Analysis of Thin Steel Hyperbolic Paraboloid Shells *ASCE, Engineering Meeting, Phoenix, Arizona, January 1971, Preprint 1320.*
7. *Buchert, K. P.* Zur Stabilitat grosser doppelt gekrummter und versteifter Schalen *Der Stahlbau, February, 1965.*