Torsional Properties of Composite Girders

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THE DESIGN and analysis[4] of curved highway bridges requires the evaluation of the torsional properties of the bridge girder members. The application of "thin walled theory"[1-3] can be applied to this problem with certain modifications.[5] These modifications require consideration of the material property variations and the composite action of the deck slab and girders. Inclusion of these modifications into the general theory has resulted in a series of equations,[5] which were then used to evaluate the torsional properties of typical composite highway girders.[5,6]

The relationship between the warping torsional property and the bending property of the various girders was then determined, yielding a series of approximate equations. These equations will be presented herein and will permit the designer to obtain preliminary maximum normal stresses.

Approximate equations for evaluation of the torsional properties will also be given herein.

The experimental testing of a series of composite girders[7] has indicated that the application of thin walled theory is justified.

NOMENCLATURE

a = $\left[\frac{E_s I_w}{G_s K_T}\right]^{1/2}$ = constant, in.
b_f = d_3 = width of bottom steel flange, in.
b_s = d_1 = width of concrete slab, in.
d_g = total depth of girder, in.
d_2 = $d_g + t_c/2 - t_f/2$ = equivalent depth of idealized section, in.
m = G_s/G_c = modular ratio
m_z = uniformly distributed torque, kip-in./in.
n = E_s/E_c = modular ratio
t_c = thickness of concrete slab, in.
t_f = t_3 = thickness of bottom steel flange, in.
t_1 = t_c/n = equivalent slab thickness, in.
w = thickness of web of steel girder, in.
z = distance from left end of member to any section, in.
E_c = modulus of elasticity of concrete
A, B = polynomial coefficients
E_s = modulus of elasticity of steel
G_c = shear modulus of elasticity of concrete
G_s = shear modulus of elasticity of steel
I_w = warping constant, in.^6
K_T = torsional constant, in.^4
L = length of girder, in.
M_b = bending moment, kip-in.
M_w = warping moment (bimoment), kip-in.^2
M_z = concentrated torque, kip-in.
S_{wc} = warping statical moment at concrete slab, in.^4
S_{ws} = warping statical moment at steel beam flange, in.^4
W_{nc} = normalized warping function at concrete slab, in.^2
W_{ns} = normalized warping function at steel beam flange, in.^2
Z = composite beam section modulus, in.^3
α = distance from center line of concrete slab to shear center of composite section, in.
ϕ = rotation
ϕ' = first derivative of ϕ with respect to length
ϕ'' = second derivative of ϕ with respect to length
ϕ''' = third derivative of ϕ with respect to length
σ_b = normal bending stress, ksi
σ_t = $(\sigma_w + \sigma_b)$ = total normal stress, ksi
σ_w = normal warping stress, ksi
τ_{st} = St. Venant shearing stress, ksi
τ_w = warping shearing stress, ksi

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THEORY

The entire development of the theory and the necessary modifications is given in Refs. 1, 2, 3, and 5. In essence, the torsional properties of composite girders are obtained by converting the concrete slab to an equivalent area of

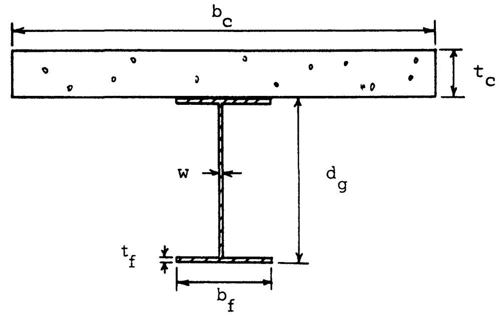


Fig. 1. Typical composite section

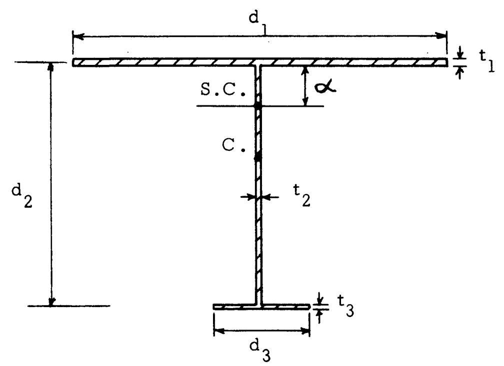


Fig. 2. Idealized composite section

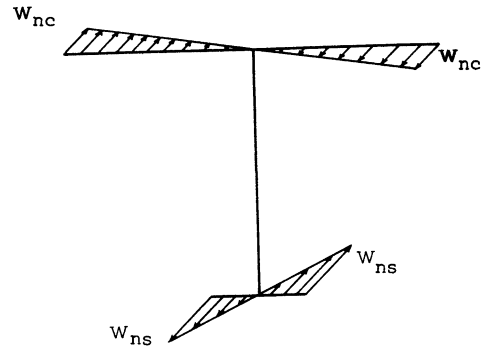


Fig. 3. Normalized warping functions

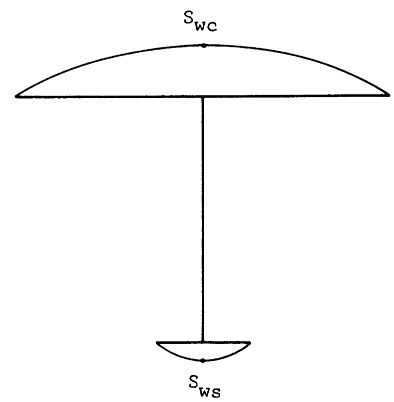


Fig. 4. Warping statical moment

steel, and varying the general equations. The modular ratio $n = E_s/E_c$ is applied to the slab, and the subsequent computations are determined for α, the distance to the shear center; W_n, the normalized warping functions; S_w the warping statical moments; and I_w, the warping constant. The shear modular ratio, $m = G_s/G_c$, is applied in the evaluation of the torsional constant, K_T.

The composite cross section in Fig. 1 is transformed and idealized as shown in Fig. 2. The idealization involved is the neglect of the steel top flange. Application of the thin walled theory,[1,2] with the following assumptions,

$$\begin{aligned} t_1 &= t_c/n \\ t_2 &= w \\ d_2 &= d_g + t_c/2 - t_f/2 \\ d_3 &= b_f \\ t_3 &= t_f \\ n &= E_s/E_c \end{aligned}$$

will result in the following equations:

Shear Center: See Fig. 2.

$$\alpha = \frac{d_3^3 t_3}{(d_1^3 t_1 + d_3^3 t_3)} \cdot d_2 \tag{1}$$

Normalized Warping Functions: See Fig. 3.

$$\text{Slab:} \quad W_{nc} = \frac{\alpha d_1}{2} \tag{2}$$

$$\text{Beam:} \quad W_{ns} = \frac{(d_2 - \alpha)}{2} \cdot d_3 \tag{3}$$

Warping Statical Moments: See Fig. 4.

$$\text{Slab:} \quad S_{wc} = \frac{\alpha d_1^2 t_1}{8} \tag{4}$$

$$\text{Beam:} \quad S_{ws} = \frac{(d_2 - \alpha) d_3^2 t_3}{8} \tag{5}$$

Warping Stiffness:

$$I_w = \frac{\alpha^2}{12} \cdot t_1 d_1^3 + (d_2 - \alpha)^2 \cdot \frac{t_3 d_3^3}{12} \tag{6}$$

Torsional Constant:

$$K_T = \frac{1}{3}\left[d_3 t_3^3 + d_2 t_2^3 + \frac{d_1 t_c^3}{m}\right] \tag{7}$$

where $m = G_s/G_c$

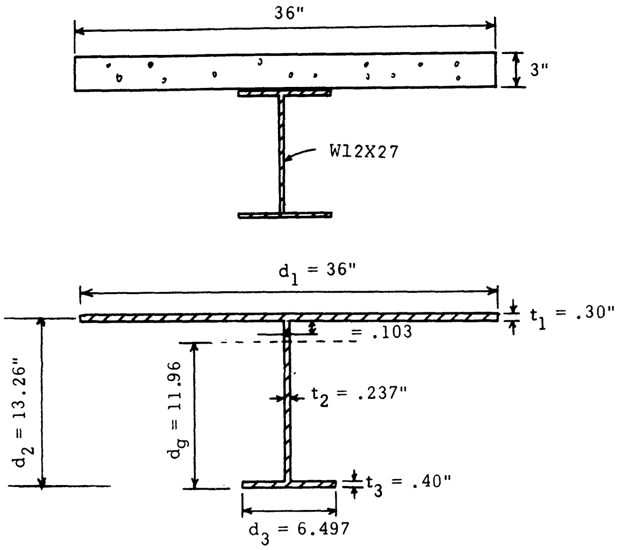


Fig. 5. Example problem

With the evaluation of these torsional parameters, the resulting stresses in the composite section can be evaluated. These stress equations are as follows:

Pure Torsional Shearing Stress:

$$\text{Slab:}\quad \tau_{st-c} = G_c t\phi' \quad \text{or} \tag{8a}$$

$$\tau_{st-c} = t\frac{M_z}{K_T m} \tag{8b}$$

$$\text{Steel:}\quad \tau_{st-s} = G_s t\phi' \quad \text{or} \tag{9a}$$

$$\tau_{st-s} = t\frac{M_z}{K_T} \tag{9b}$$

Warping Shearing Stress:

$$\text{Slab:}\quad \tau_{wc} = -\frac{E_c S_w}{t/n}\phi''' \tag{10}$$

$$\text{Steel:}\quad \tau_{ws} = -\frac{E_s S_w}{t}\phi''' \tag{11}$$

Warping Nominal Stress:

$$\text{Slab:}\quad \sigma_{wc} = E_c W_n \phi'' \tag{12}$$

$$\text{Steel:}\quad \sigma_{ws} = E_s W_n \phi'' \tag{13}$$

Constant a:

$$a = \left[\frac{E_s I_w}{G_s K_T}\right]^{1/2} \tag{14}$$

where I_w and K_T are according to Eqs. (6) and (7).

EXAMPLE

The following will demonstrate the application of Eqs. (1) through (7) in evaluating the torsional properties of a composite section. The properties will then be compared to those values which were obtained by the more exact equations of Ref. 5.

Figure 5a describes a W12×27 in composite action with a 3 in. x 36 in. concrete slab. The idealized configuration of this section is as shown in Fig. 5b, where the dimensions of the section are computed as follows, using center line distances:

$$d_1 = 36.0 \text{ in.}$$

$$t_1 = t_c/n = 3 \times 1/10 = 0.30 \text{ in.}$$

$$d_2 = d_g - t_f/2 + t_c/2 = 11.96 - \frac{0.40}{2} + \frac{3.0}{2}$$

$$= 13.26 \text{ in.}$$

Shear Center:

$$\alpha = \frac{d_3{}^3 t_3}{d_1{}^3 t_1 + d_3{}^3 t_3}\cdot d_2 \tag{1}$$

$$= \frac{6.497^3 \times 0.40}{(36^3 \times 0.30) + (6.497^3 \times 0.40)} \times 13.26$$

$$= 0.103 \text{ in. from top slab}$$

Normalized Warping Functions:

$$W_{nc} = \frac{\alpha d_1}{2} \tag{2}$$

$$= \frac{0.103 \times 36}{2} = 1.86 \text{ in.}^2$$

$$W_{ns} = \frac{(d_2 - \alpha)}{2}\cdot d_3 \tag{3}$$

$$= (13.26 - 0.103) \times \frac{6.497}{2} = 42.5 \text{ in.}^2$$

Table 1. Comparison of Torsional Parameters

Function	Approximate Equations	Considering All Girder Elements Ref. 5)	% Diff.
α (in.)	0.103	0.120	14.2
K_T (in.^{4})	37.20	37.00	0.55
I_w (in.^{6})	1632.00	1620.20	0.74
a (in.)	10.50	10.50	0.0
W_{nc} (in.^{2})	1.86	2.10	11.4
W_{ns} (in.^{2})	42.50	42.70	0.47
S_{wc} (in.^{4})	5.01	5.60	10.50
S_{ws} (in.^{4})	27.80	27.80	0.0

Table 2. Simple Span Bridges—Girder Sizes

W Member	Plate Width (in.)	Plate Thickness (in.)	Plate Width (in.)	Plate Thickness (in.)	Span Length (ft)
W16×36	5	1/4, 1/2, 5/8	11	1/4, 3/8, 1/2, 5/8	16–30
W18×45	5 1/2	1/4, 1/2, 3/4	11 1/2	1/4, 3/8, 1/2, 5/8	18–38
W21×55	6	1/4, 1/2, 3/4	12 1/2	1/4, 3/8, 1/2, 5/8	20–42
W24×68	7	1/4, 1/2, 3/4, 7/8	13	3/8, 1/2, 5/8, 3/4, 7/8	24–50
W27×84	8	1/4, 1/2, 3/4, 1	14	1/2, 5/8, 3/4, 7/8, 1	28–60
W30×99	8 1/2	1/4, 1/2, 3/4, 1	14 1/2	1/2, 5/8, 3/4, 7/8, 1	32–66
W33×118	9 1/2	1/4, 1/2, 3/4, 1, 1 1/8	15 1/2	5/8, 3/4, 7/8, 1, 1 1/8	38–72
W36×35	10 1/2	1/4, 1/2, 3/4, 1, 1 1/8	16	5/8, 3/4, 7/8, 1, 1 1/8	44–78
W36×150	10	1/4, 1/2, 3/4, 1, 1 1/4, 1 3/8	14	3/4, 1, 1 1/4, 1 3/8	46–82
W36×160	10	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2	14	1, 1 1/4, 1 1/2	50–84
W36×170	10	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 5/8	14	1, 1 1/4, 1 1/2	50–86
W36×182	10	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4	14	1, 1 1/4, 1 1/2	52–88
W36×194	10	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4, 1 7/8	14	1, 1 1/4, 1 1/2	54–90
W36×230	14 1/2	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4		. . .	60–96
W36×245	14 1/2	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4		. . .	62–100
W36×260	14 1/2	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4		. . .	64–102
W36×280	14 1/2	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4		. . .	68–104
W36×300	14 1/2	1/4, 1/2, 3/4, 1, 1 1/4, 1 1/2, 1 3/4		. . .	70–106

Warping Statical Moments:

$$S_{wc} = \frac{\alpha d_1^2 t_1}{8} \tag{4}$$

$$= \frac{0.103 \times 36^2 \times 0.30}{8} = 5.01 \text{ in.}^4$$

$$S_{ws} = \frac{(d_2 - \alpha) d_3^2 t_3}{8} \tag{5}$$

$$= \frac{(13.26 - 1.03) \times 6.497^2 \times 0.40}{8} = 27.8 \text{ in.}^4$$

Warping Stiffness:

$$I_w = \frac{\alpha^2}{12} t_1 d_1^3 + (d_2 - \alpha)^2 \frac{t_3 d_3^3}{12} \tag{6}$$

$$= \frac{0.103^2}{12} \times 0.30 \times 36^3 + (13.26 - 0.103)^2 \times \left[\frac{0.40 \times 6.497^3}{12}\right]$$

$$= 1632.0 \text{ in.}^6$$

Torsional Constant:

$$K_T = \frac{1}{3}\left[d_3 t_3^3 + d_2 t_2^3 + \frac{d_1 t^3}{m}\right] \tag{7}$$

$$= \frac{1}{3}\left\{(6.497 \times 0.40^3) + (13.26 \times 0.237^3) + \left[\frac{36.0}{8.8} \times 3^3\right]\right\}$$

$$= 37.2 \text{ in.}^4$$

Constant a:

$$a = \left[\frac{E_s I_w}{G_s K_T}\right]^{1/2} \tag{14}$$

$$= \left[\frac{30 \times 10^3 \times 1632.0}{12 \times 10^3 \times 37.2}\right]^{1/2} = 10.5 \text{ in.}$$

Application of the more exact equations, which include all elements, to calculate the torsional properties of the section shown in Fig. 5 yields the results given in Table 1. Also listed in the table are the values computed by the approximate equations. As can be seen, generally good correlation between the theories occurs.

Table 3. Continuous Span Bridges—Girder Sizes

Web Depth	Web Thickness	Flange Thickness	Flange Width
42	5/16	1 1/2	12
42	5/16	2	12
42	5/16	1 1/2	14
42	5/16	2	14
48	5/16	1 1/2	12
48	5/16	2	12
48	5/16	1 1/2	14
48	5/16	2	14
54	3/8	1 1/2	12, 14, 16, 18
54	3/8	2	12, 14, 16, 18
60	3/8	1 1/2	14, 16, 18, 20, 22, 24
60	3/8	2	14, 16, 18, 20, 22, 24
72	3/8	1 1/2	14, 16, 18, 20, 22, 24
72	3/8	2	14, 16, 18, 20, 22, 24
84	1/2	1 1/2	14, 16, 18, 20, 22, 24
84	1/2	2	14, 16, 18, 20, 22, 24

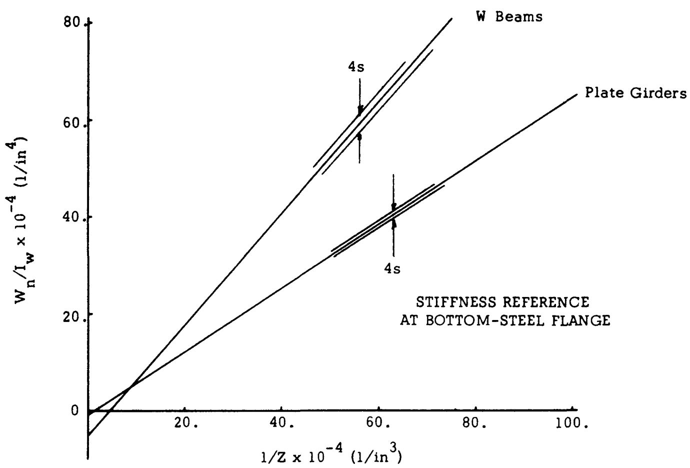


Fig. 6. *W_n/I_w vs. $1/Z$ For* W *and plate girder members—reference bottom flange*

TYPICAL BRIDGE MEMBERS

The evaluation and listing of the torsional properties of simple span bridge members and continuous bridge member are presented in Ref. 5. These data are for those members given in Tables 2 and 3, as recommended in a previous study[6] and present design practices. The concrete slab was assumed equal to 8.5 in., and the effective slab width is assumed equal to 6, 7, 8 and 9 ft. The material properties were assumed equal to:

$$E_s = 30 \times 10^3 \text{ ksi}; \quad E_c = 3 \times 10^3 \text{ ksi}$$
$$G_s = 12 \times 10^3 \text{ ksi}; \quad G_c = 1.36 \times 10^3 \text{ ksi}$$

To present all of this data would require numerous pages; therefore, an examination of some of the trends of the data was conducted. By applying some of the previously presented approximate equations, and the following equations, an evaluation of the maximum normal stress can be obtained.

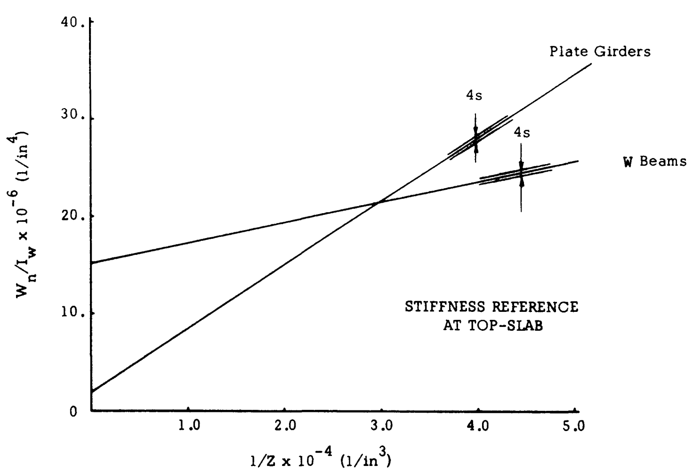


Fig. 7. *W_n/I_w vs. $1/Z$ For* W *and plate girder members—reference top slab*

Maximum Normal Stresses—The maximum normal stresses that may be developed in a section will be the combined effects of bending stresses and warping normal stresses. The warping stresses can be computed by Eq. (12) or (13), or, in general,

$$\sigma_w = EW_n\phi'' \tag{15}$$

or defining the warping moment (bimoment) as

$$M_w = EI_w\phi'' \tag{16}$$

Eq. (15) becomes

$$\sigma_w = \frac{W_n}{I_w} M_w \tag{17a}$$

or

$$\frac{\sigma_w}{M_w} = \frac{W_n}{I_w} \tag{17b}$$

The bending stresses are computed by the conventional equation:

$$\sigma_b = Mb/Z \tag{18a}$$

or

$$\frac{\sigma_b}{M_b} = \frac{1}{Z} \tag{18b}$$

The term (W_n/I_w) in Eq. (17b) is a known quantity for all of the composite sections given in Tables 2 and 3. Similarly the section modulus term $(1/Z)$ in Eq. (18b) is a known quantity. A plot of these data will then give the relationship between (σ_w/M_w) and (σ_b/M_b).

Plotting these data and performing a linear regression analysis results in a series of curves, as given in Figs. 6 and 7. Figure 6 describes the functional relationships on the bottom steel flange. Figure 7 describes the relationships on the concrete slab, as shown in Fig. 3. The resulting equation is of the form:

$$\left(\frac{W_n}{I_w}\right) = A + B\left(\frac{1}{Z}\right) \tag{19}$$

The respective coefficients A and B are given in Table 4, as are the variances and standard deviations, s. A value of $4s$, which indicates that 95 percent of the data is about the mean regression line, is shown in Figs. 6 and 7.

These data indicate the following trends:

W *Beam:*

Steel flange: $$\left(\frac{\sigma_w}{M_w}\right)_s = 1.15\frac{\sigma_b}{M_b} \tag{20}$$

Concrete slab: $$\left(\frac{\sigma_w}{M_w}\right)_c = 0.002\frac{\sigma_b}{M_b} \tag{21}$$

Table 4. Coefficients of Polynomial Fit to Stiffness Data

	Location	A	B	Variance	Std. Deviation (s)
W beam	Steel flange beam	-5.217×10^{-4}	1.149	0.481×10^{-7}	0.219×10^{-3}
	Concrete slab	1.532×10^{-5}	1.995×10^{-2}	0.599×10^{-10}	0.774×10^{-5}
Plate Girder	Steel flange beam	-8.001×10^{-5}	6.532×10^{-1}	0.115×10^{-8}	0.339×10^{-4}
	Concrete slab	1.985×10^{-6}	6.195×10^{-2}	0.111×10^{-10}	0.333×10^{-5}

Plate Girder:

$$\text{Steel flange:} \quad \left(\frac{\sigma_w}{M_w}\right)_s = 0.652\frac{\sigma_b}{M_b} \tag{22}$$

$$\text{Concrete slab:} \quad \left(\frac{\sigma_w}{M_w}\right)_c = 0.062\frac{\sigma_b}{M_b} \tag{23}$$

These expressions could be applied in evaluating the warping stresses if the warping moment (bimoment) was known, as the bending stress and moments are readily computed. However, the warping moment, M_w, is a function of the torsional stiffness EI_w and torsional function ϕ''. An estimate of the warping moment for the composite properties and extreme cases of loadings would thus permit evaluation of the warping stress σ_w. Considering the following four case loadings (see Fig. 8), the maximum ϕ'' function was evaluated, and thus the warping moment. These results are given in Fig. 9.

For example, considering Case I loading, and applying Eqs. (20) through (23) would result in the following, where the constant a can be evaluated by Eq. (14):

W *Beam:*

$$\sigma_{ws} = 1.15\left(\frac{\sigma_b}{M_b}\right)\cdot\frac{mLa}{2} \tag{24}$$

$$\sigma_{wc} = 0.002\left(\frac{\sigma_b}{M_b}\right)\cdot\frac{MLa}{2} \tag{25}$$

Plate Girder:

$$\sigma_{ws} = 0.652\left(\frac{\sigma_b}{M_b}\right)\cdot\frac{mLa}{2} \tag{26}$$

$$\sigma_{wc} = 0.062\left(\frac{\sigma_b}{M_n}\right)\cdot\frac{MLa}{2} \tag{27}$$

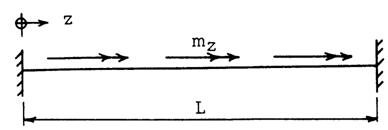
Thus if a composite beam is subjected to a combination of vertical and torsional loadings, the maximum normal stresses may be evaluated by applying the following general equation, Eq. (28):

$$\sigma_T = \sigma_b + B\left(\frac{\sigma_b}{M_b}\right)M_w \tag{28}$$

$$= \sigma_b\left(1 + B\cdot\frac{M_w}{M_b}\right)$$

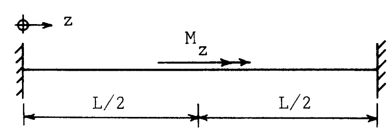
where B is obtained from Table 4 for the respective girder types and M_w is as shown in Fig. 9, relative to the girder property L/a.

CASE I



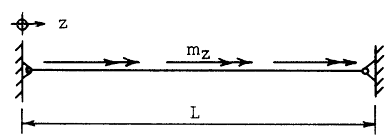
$$M_w = \frac{m_z La}{2}\left[\frac{1 - \cosh L/a}{\sinh L/a}\cdot\cosh z/a - \sinh z/a - \frac{2}{L/a}\right]$$

CASE II



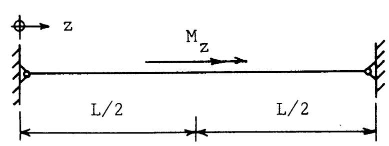
$$M_w = \frac{M_z a}{2}\left[\tanh L/4a \cdot \cosh z/a - \sinh z/a\right]$$

CASE III



$$M_w = m_z a^2\left[-1 + \cosh z/a - \tanh L/2a \cdot \sinh z/a\right]$$

CASE IV



$$M_w = M_z a\left[\left(\frac{\sinh L/2a}{\tanh L/a} - \cosh L/2a\right)\sinh z/a\right]$$

Fig. 8. Warping moments (bimoments) for various torsional conditions

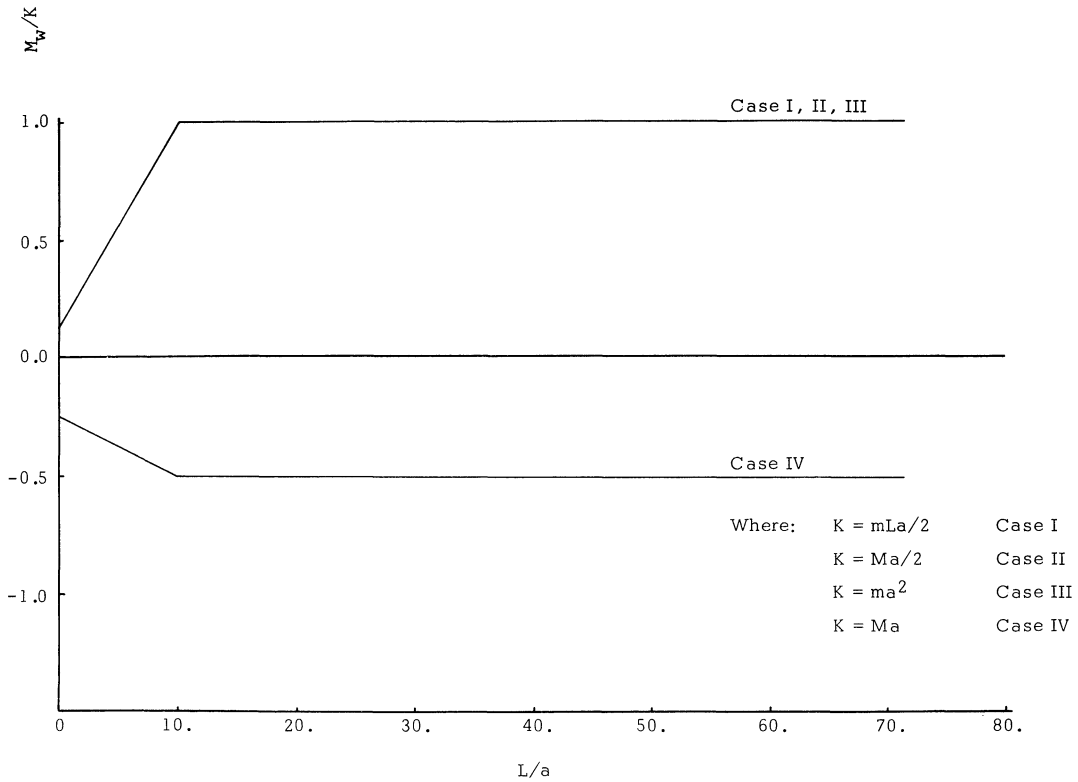


Fig. 9. Maximum M_w vs. L/a

If $L/a \leq 10.0$, it may be desirable to use the exact torsional equations of Ref. 3 to evaluate the warping moment. However, for most composite bridge members, $L/a \gg 10.0$.

CONCLUSIONS

The application of thin walled beam theory has resulted in a series of equations which permit the evaluation of the torsional properties of composite girders.

The evaluation of the properties of typical single span and continuous composite girders, and their trends, has resulted in several empirical equations relating the warping stresses to the bending stresses. Therefore, the total normal stress can readily be computed.

ACKNOWLEDGMENTS

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