Discussion

The Effective Length of Columns in Unbraced Frames

Paper presented by JOSEPH A. YURA (April, 1971, issue)

Discussion by **PETER F. ADAMS**

The solutions proposed by Yura could be of great significance in the design of tall steel structures; the purpose of this discussion is to point out implications of the solutions which may not have been intended, and also to question certain of the techniques suggested in the paper.

Yura has discussed two different aspects of column design. In the first situation, the column yields before buckling; in the second, stiffer portions of the structure are used to support more flexible portions. Although the examples used in the paper relate to the design of axially loaded columns, it must be emphasized that the K-factor determined for a member is used more often in the interaction equations relating to beam-column design.[1]

In the 1969 AISC Specification, for example, the following interaction equation must be satisfied:[9]

$$\frac{f_a}{F_a} + \frac{C_m f_b}{\left(1 - \frac{f_a}{F_e'}\right) F_b} \leq 1.0 \tag{6}$$

It has been clearly shown that Eq. (6) correlates well with the computed strengths of columns in "sway permitted" frames and this equation has been recommended for the design of such members.[10] In Eq. (6) the K-factor (used in the terms F_a and F_e') is computed on the basis of elastic action, even though the actual member may yield significantly before reaching its ultimate strength.[10] The adjustment of the K-factor as now suggested by Yura could lead to a significant overestimate of the strength of the member, resulting in an inadequate design. For example, the difference in strength for a member having the proportions of the column in Design Example 1 can be seen in Fig. 5.22 of Ref. 11. If the capacity of a member were based on $KL/r = 23$ ($K = 1.0$), as implied by Yura, it would appear to be much greater than the actual capacity based on $KL/r = 54$ ($K = 2.35$ as determined from the nomograph.)

In the less frequent case of axially loaded columns, it is again doubtful whether an attempt should be made to reduce the effective length factor by considering the inelastic action of the column. For the column of Design Example 1, significant yielding will certainly occur (due to residual stresses) before the column buckles. The beams may also yield before the column buckles, however, and at the instant of buckling the end restraint may actually be decreased instead of increased.

If the designer can be assured that column buckling will occur before the restraining girders yield, then the effective length factor could be reduced. The bending stiffness of the partially yielded column (the effective EI value) has been determined as[12]

$$E_T I = EI \sqrt{\frac{\sigma_y}{\sigma_{rc}} (1 - P/P_y)} \tag{7}$$

where P represents the axial load on the column at the instant of buckling and P_y represents the yield load ($P_y = A\sigma_y$). The yield stress is denoted by σ_y and the maximum compressive residual stress by σ_{rc}.

The effective column stiffness predicted by Eq. (7) is substantially higher than that implied by Eq. (5) and thus the K-factor obtained will be greater than that obtained by Yura's procedure. For the member used in Design Example 1, $K = 1.72$ using Eq. (7), instead of 1.0 as determined by Yura. In the above calculation, it was assumed that $\sigma_{rc} = 0.3\, \sigma_y$ and that[9]

$$\frac{P}{P_y} = \frac{A F_a}{A F_y}\left[\frac{5}{3} + \frac{3}{8}\left(\frac{KL/r}{C_c}\right) - \frac{1}{8}\left(\frac{KL/r}{C_c}\right)^3\right] \tag{8}$$

Yura has correctly shown that for mixed framing systems such as that shown in Fig. 8, the buckling

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strength is relatively independent of the distribution of axial loads on the various columns. This leads to his suggested design procedure, that is, to increase the axial loads on the rigidly framed columns so that these columns are subjected to the full story loads (the pinned columns are designed to resist their own axial loads). The process (if the pinned columns carry most of the vertical load) leads to the columns being designed for almost twice the vertical load on the story. For example, in Fig. 8 the columns have been designed to resist a total vertical load of 500 kips, rather than the actual total load of 290 kips. This is not particularly desirable, since the rigidly framed columns (which must be designed for the increased load) have a K-factor of 2.0, while the pinned columns use $K = 1.0$. In other buildings where the K-factor for the rigidly framed columns may be larger, the additional material required by Yura's solution would be correspondingly increased.

In addition, the proposed procedure fails to account for the actual internal forces for which the structure must be designed.[13] Columns B in Fig. 8 would be designed using interaction equations, such as Eq. (6). When the frame is deformed in a sidesway mode, the rigidly framed portions of the structure would be subjected to the additional moments and shears produced by the $P\Delta/L$ forces from the pinned columns.[13] These forces would increase the bending stresses throughout the rigid portions of the structure and thus influence the bending term, f_b, in the interaction equations. The proposed solution influences only the axial force term, f_a.

To replace the procedure proposed by Yura, a second order analysis could be performed using standard computer programs.[14] A more convenient (approximate) technique might be to modify the results of the normal first order elastic analysis by introducing additional loads at each floor level equal to the sum of the story shears produced by the $P\Delta$ effect. In a given story the shear is equal to $\Sigma P\Delta/L$, where ΣP represents the sum of the axial loads in the columns of the story.[13] This technique introduces additional moments into the structure (beams as well as columns) which are resisted in proportion to the flexural stiffnesses of the various portions of the bent or bents. Since $P\Delta/L$ forces for all columns have then been included in the analysis, the effective length factors may be computed as for the braced frame[11] (or conservatively taken as unity).

In summary, the use of the inelastic buckling solution to determine the K-factor is dangerous for the case of beam columns and is of doubtful validity even in the case of axially loaded columns. Yura's approach to the design of systems containing columns of widely varying stiffness (including pinned columns) does not account for the actual internal forces developed by the supported columns; it is suggested that an approximate second order solution be used in its place.

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Response by **JOSEPH A. YURA**

Professor Adams basically disagrees with all major points in the paper. While much of the discussion relates specifically to the use of the interaction equations for beam-columns, which was beyond the scope of the original paper, Adams also contends that the methods have doubtful validity for axially loaded members. This discussion will answer those points which question the validity of the methods as related to structures loaded primarily by gravity loads and also show how the procedures in the original paper can be extended to beam-columns.

Concerning the effect of inelastic column behavior on the evaluation of K, Adams states that the beams may also yield before the column buckles, which could increase K. The possibility of beam yielding prior to buckling is real, but it is no greater for inelastic columns than for elastic ones. In fact the effect of beam yielding would be much more important if the columns buckled in the elastic range. But usually this is ignored in design.

There are valid reasons why column inelasticity should be considered while ignoring yielding in the beams. Beam yielding is generally confined to small regions (plastic hinge locations). If plastic hinges form at the ends of beams due to gravity load, and then buckling occurs in a sway mode, the restraint at the "leeward" end will theoretically be zero (like a pinned end), while the hinge at the "windward" end will unload elastically. Consequently the beam stiffness will be only one-half that for a purely elastic sway situation ($3EI/L$ vs. the $6EI/L$ used in the nomographs). On the other hand the inelastic effects in columns affect the rigidity along the entire length of the column. For a column with an L/r of 20, the stiffness is approximately $\frac{1}{20}$th of the full elastic value. In addition,

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numerous tests have shown that the plastic hinges are actual idealizations at the ends of the beams where the moment gradient is large. When the plastic moment is reached, there is no sudden reduction in stiffness as implied by simple plastic theory. Rather, the end restraint remains reasonably close to that for a rigid connection for moments up to 15 percent greater than the plastic moment. So the actual beam stiffness will generally be close to its elastic value up to the theoretical ultimate load of the structure.

Adams indicates that the inelastic bending stiffness of a column is given by Eq. (7) and that this stiffness is substantially higher than that used by AISC in the establishment of the allowable column stresses. A comparison of the AISC column strength (factor of safety removed) with Eq. (7) is shown in Fig. 1. The writer would hesitate to use the phrase "substantially" higher to describe the difference between the two curves, but that is unimportant. Equation (7), which was derived almost twenty years ago[15], is based on an *assumed* distribution and magnitude of residual stress. The implication that Eq. (7) gives the correct stiffness is unjustified.

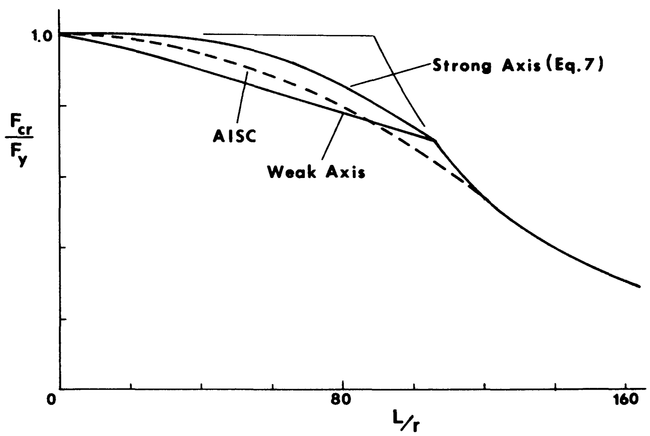

Fig. 1. Basic column strength

It is well known that the magnitude and distribution of residual stress can vary substantially. The AISC equation is based on the recommendations of the Column Research Council. For most residual stress patterns, the AISC equation provides a reasonable lower limit to the theoretical column strength[1]. For the assumed distribution and magnitude used by Adams, buckling about the weak axis plots below the AISC equation, as shown in Fig. 1. As stated in Ref. 1, the AISC equation "provides a suitable compromise between weak and strong axis buckling of wide-flange sections having an average maximum compressive residual stress of $0.3F_y$".

The writer followed the Column Research Council and AISC recommendations as a practical estimate of stiffness. However, this does not mean that the writer's approach is invalid if expressions other than the AISC equation are used for column stiffness. For example, if Eq. (7) is specified or used for the column stiffness, the procedure for determining $G_{inelastic}$ would be to adjust $G_{elastic}$ by the ratio of the inelastic column strength derived from Eq. (7) to the elastic Euler stress at the particular KL/r under consideration, as stated in the paper. Adams claims that if Eq. (7) is used for the inelastic column strength in Design Example 1, $K = 1.72$ will result. This is not true. The way Adams applied the method using his Eq. (8) is incorrect and he does not follow the procedure described in the paper. His approach is shown graphically in Fig. 2. Adams starts with elastic $G = 5.72$ and elastic $KL/r = 54$, but instead of getting F_{cr}/F_e at this KL/r, so that $G_{inelastic} = (0.96/2.73) \times G_{elastic} = 2.02$, he actually follows the path shown by ①. At elastic $KL/r = 54$, he moves vertically until the AISC column strength curve is reached and then moves horizontally to the right at this stress level until the strength curve from Eq. (7) is reached at a $KL/r = 69$. The F_{cr}/F_e at $KL/r = 69$ is obviously incorrect. The inelastic G after the first cycle is determined by Adams to be 3.12, whereas the correct value is 2.02. After two cycles of this incorrect procedure, a K of 1.72 is established. However the correct procedure gives $K = 1.3$ after two cycles, which is exactly the same as that derived in the paper. A few more cycles show that K approaches 1.0 when Eq. (7) is used for the inelastic column stiffness. Adams' error resulted from the use of the AISC column formula for strength and separately using Eq. (7) for stiffness. It is well known that stiffness and strength are directly related for column buckling.

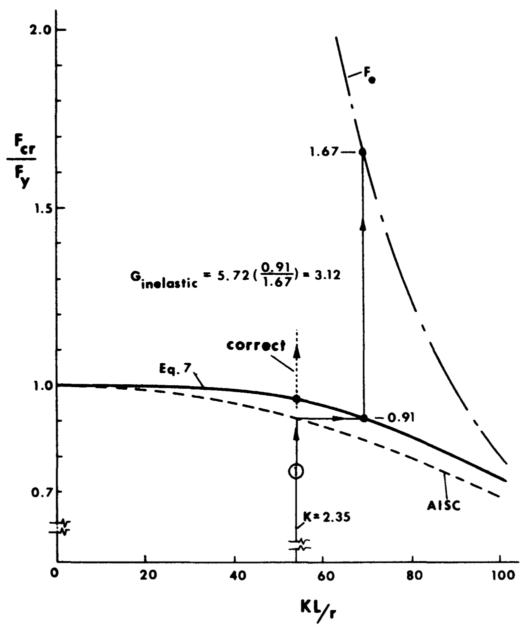

Fig. 2. Adams inelastic K solution

For beam-column design, Adams feels that the design will be inadequate if the K which is adjusted for inelastic action is used in the interaction equation. His conclusion is based on the statement that Ref. 10 shows that K should be based on elastic action, even though the column may yield significantly before reaching ultimate strength. Since the writer was the principal author of Ref. 10, he has complete knowledge of the calculations involved in the development of the paper. In all cases except those involving infinitely stiff beams, the axial load was in the *elastic* range of column action, where the elastic K is valid. For the cases where infinitely stiff beams were used, $G = 0$ and no amount of inelastic column action can change K from its elastic value. Near ultimate load, yielding occurred at the tops of the columns due to the bending moments, but this yielding was confined to very small regions of the columns, so the column length was primarily elastic. Contrary to Adams' statement, the structures were not highly yielded at all and the inelastic K concept would not make a single change in any of the comparisons in Ref. 10.

In Eq. (6) the first term relates to the strength of the column under axial load alone. Since the inelastic K has been developed for this situation, it is reasonable to use the inelastic K in calculating F_a. On the other hand, F_e' is related to the bending moment, so the writer suggests that the elastic KL be used in evaluating F_e', at least until further studies can be completed to determine the effect of inelastic column action on the bending term.

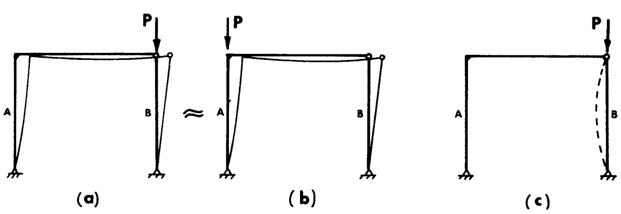

Fig. 3. Equivalent frame concept

Salem has shown that the buckling strength of the two frames shown in Fig. 3 is almost identical. It should be obvious that for Case (a) the stability of the system in a sway mode is controlled by the stiffness of column **A** and the beam. The size of column **B** has no effect on the sidesway stability load because both ends are pinned, but column **B** must at least have the size necessary to prevent buckling as an Euler column as shown in Fig. 3c. Thus, for Case (a) column **B** must be designed as an Euler column in order to support the applied load P. In addition, column **A** must be at the size required to produce a sidesway buckling strength of P or greater. To get the required size of column **A**, an engineer could use the charts developed by Salem, which solve Case (a) exactly. The size of column **A** required to support the load in a sidesway mode would be almost identical to the size required for the loading condition shown in Fig. 3b. A comparison of the size required for Cases (a) and (b), as determined by the exact Salem solutions, is given in Fig. 4. The comparison shows that when the beam is infinitely stiff ($G = 0$), the replacement of Case (a) by Case (b) will result in a required moment of inertia that is 20 percent conservative. However, for G greater than 2, the difference between the two solutions is less than 5 percent. For large G, the two conditions produce identical results. The advantage of this observation is that Case (a), which *apparently* is beyond the scope of the alignment chart solution, can be replaced by Case (b), which can be solved by the charts. Of course the engineer might just use the exact Salem solutions if they are available.

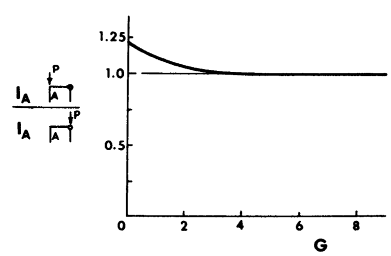

Fig. 4. Required Moment of Inertia for Column A

Adams' comments related to the procedures for the design of columns with different buckling loads overlooks the fact that the structure must be able to safely support both types of instability. In the sway mode, pinned end columns have no stability. K is *not* 1.0 in the sway mode; it is ∞. So the stability in a sway mode must depend on other portions of the structure. The stiffer columns are those with rigid connections. The method given in the paper places the loads on the stiffest columns to check the sway mode of buckling. For structures loaded primarily by gravity loads, where column stability affects the size of the members, the procedures given in the paper are valid. The implication by Adams that the design requires larger member sizes than necessary is untrue. The size of the members in a structure must be sufficient to prevent buckling. Adams' contention that the proposed procedure fails to account for the actual internal forces is incorrect for determining the buckling load. The procedure is based on Salem's exact solutions, which consider the shears produced by the $P\Delta/L$ forces from the pinned column. The design procedure assesses the stability of the system.

When the columns are subjected to bending moments in addition to axial loads, the interaction equations must be used to check the member. It is well known that bending moments have little influence on the buckling of the structure, and that the interaction equation checks the individual strength and stability of the member. Because the interaction equation as generally used for unbraced frames assumes that all columns buckle simultaneously, some adjustment is necessary when this condition is not satisfied. The stability of such a system should be assessed separately from the interaction equations using the procedure outlined in the paper and the strength of an individual member in the story checked by the following interaction equation:

$$\frac{f_a}{F_a} + \frac{C_m f_b}{[1 - (\Sigma P/\Sigma P_e')]F_b} \leq 1.0 \qquad (9)$$

where f_a is calculated for the actual axial load on the member (not the equivalent load), F_a is based on KL from the nomograph, ΣP is the sum of the loads on the story, and $\Sigma P_e' = \Sigma F_e' \times A$ for all the members in the story, using KL for calculating F_e'. The only difference between Eq. (9) and Eq. (6) is that the summation of the story loads is used in the bending amplifier rather than the individual load. If the columns of the frame are in the elastic range, the term $[1 - (\Sigma P/\Sigma P_e')]$ automatically checks the stability of the system as designated in the paper. If all the columns in a story buckle simultaneously, Eq. (9) reduces to the standard Eq. (6). The procedure just suggested may at first appear to be complicated, but actually it is rather straightforward. Note that the term $[1 - (\Sigma P/\Sigma P_e')]$ is a constant for all the columns in a story. The use of Eq. (9) will be illustrated in the following example.

Design Example 4.—The frame shown in Fig. 5 is similar to that used in Design Example 2 of the original paper. But in this case, a wind load of 10 kips is also acting on the structure. Design the columns using F_y = 36 ksi. Assume out-of-plane buckling of column **B** is prevented by girts.

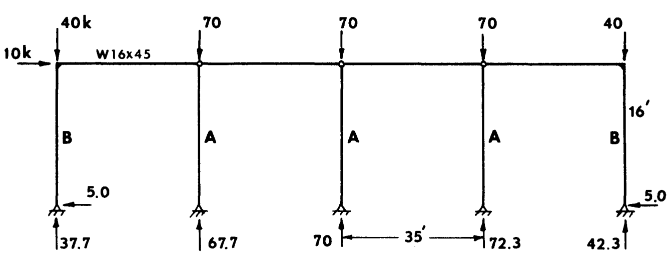

Fig. 5. Design Example 4

Gravity load: Same as design used in the original paper.

Columns A: W8×24; P_{allow} = 73 kips

Columns B: W12×31; P_{allow} = 145.2 kips, KL/r = 75

Note: Some advantage of inelastic column action could have been considered in the evaluation of K, but the procedure was not used for clarity of presentation.

This design was governed by stability:

Applied load = 290 kips

Sidesway buckling strength:

3 Cols. **A** (W8×24): 0

2 Cols. **B** (W12×31):

$2 \times 145.2 = \underline{290.4}$

$290.4 > 290$ kips **ok**

Gravity plus wind: The AISC Specification permits a 33% increase in allowable stress. This is more easily accomplished by designing for ¾ of the applied loads:

Columns **A**: $P_{max} = 72.3 \times 0.75 = 54.3$ kips; $M = 0$ ∴ gravity load governs

Columns **B**: $P_{max} = 42.3 \times 0.75 = 31.8$ kips; $M = 80 \times 0.75 = 60$ kip-ft

Try W12×31:

$\frac{KL}{r_x} = 75$; $F_a = 15.90$ ksi; $F_e' = 26.55$ ksi;

$A = 9.13$ in.^{2}; $S_x = 39.5$ in.^{3}

$f_a = 31.8/9.13 = 3.48$ ksi

$f_b = 60(12)/39.5 = 18.25$ ksi

$\Sigma P = 290 \times 0.75 = 218$ kips

$\Sigma P_e'$ (use sidesway K):

3 Cols. **A** (W8×24, pinned ends, $K = \infty$): $P_e' = 0$ kip

2 Cols. **B** (W12×31, $K = 2.0$): $2 \times 26.55 \times 9.13 = \underline{485}$

$\Sigma P_e' = 485$ kip

Check Eq. (9):

$$\frac{3.48}{15.90} + \frac{0.85(18.25)}{24\,[1 - (218/485)]} = 1.394 > 1.0 \textbf{ ng}$$

Try W14×34:

$K = 2.3$; $\frac{KL}{r_x} = 75$; $F_a = 15.90$ ksi;

$F_e' = 26.55$ ksi; $A = 11.2$ in^2; $S_x = 54.7$ in^3

$f_a = 31.8/11.2 = 2.84$ ksi

$f_b = 60(12)/54.7 = 13.16$ ksi

$\Sigma P = 218$ kips

$\Sigma P_e' = 2 \times 26.55 \times 11.2 = 595$ kips

Check Eq. (9):

$$\frac{2.84}{15.90} + \frac{0.85(13.16)}{24[1 - (218/595)]} = 0.914 < 1.0 \textbf{ ok}$$

Check stability: Since ΣP on a story is unchanged by the wind load, a check for buckling under gravity load only is sufficient.

Columns **A**: Use W8×24

Columns **B**: Use W14×34

Adams suggests an approximate second order analysis for systems containing columns with various stiffnesses. Using the moment from the approximate analysis, the interaction equations are used with the effective length factors taken as 1.0. Using this procedure for the frame shown in Fig. 5, the moment from the approximate analysis after 5 cycles was 84.6 kip-ft. The first order moment was 60 kip-ft. Using this moment and an axial load of 30 kips, a W14×34 was required. The sum of the terms of the interaction equation was 0.899, which is similar to that obtained using Eq. (9), with a first order analysis and KL.

The writer would like to point out some limitations for the procedure suggested by Adams. These limitations will be illustrated by designing the frame shown in Fig. 6. The structure is loaded primarily by axial load. The very small lateral load is placed on the structure to permit an approximate second order analysis as suggested by Adams.

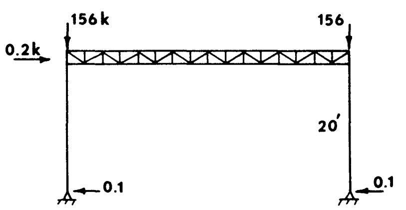

Fig. 6. Design Example 5

Design Example 5—Design the columns using F_y = 36 ksi and a W8 section. The column webs are in the plane of the frame. Neglect out-of-plane behavior.

AISC approach:

Neglect the very small moment (2 kip-ft). Treat as axial load only. $K = 2.0$ for pinned end and very stiff beams.

$KL = 2 \times 20 = 40$ ft.

From the column tables in the AISC Manual:

Use W8×67 ($P_{allow} = 178 > 156$ kips)

Note: A W8×67 would also be required if the interaction equations were used with $M = 2$ kip-ft. The bending term is only 3 percent.

Adams approach:

Try W8×40:

$L/r = 68$; $F_a = 16.64$ ksi; $F_e' = 32.3$ ksi;

$S_x = 35.5$ in.^{3}; $F_b = 24$ ksi; $A = 11.8$ in.^{2}

$M_{1st\ order} = 2$ kip-ft

$M_{2nd\ order}$ (after 10 cycles of analysis) $= 6.7$ kip-ft

$f_b = 6.7(12)/35.5 = 2.26$ ksi

$f_a = 156/11.8 = 13.2$ ksi

Check the interaction equations as for a braced frame ($K = 1.0$, $C_m = 0.6$), but use the moment from the second order analysis:

$$\frac{13.2}{16.64} + \frac{2.26}{24[1 - (13.2/32.3)]} = 0.889$$

Use W8×40

The Adams approach gives a section considerably lighter than the buckling solution (W8×47), yet this is not possible since the buckling solution, $K = 2.0$, is exact for the problem shown in Fig. 6. There are a number of reasons why the Adams solution gives poor results for this case. First, an elastic structural analysis is used, yet the column is loaded into its inelastic range. (For $F_y = 36$ ksi, the AISC Specification assumes inelastic action starts when $f_a = 9.5$ ksi.) Consequently the actual deflections will be larger. Second, the Adams approach does not really check the stability of the structure if the structural analysis is performed at the working load. The structure must be able to support at least a 67 percent overload (or closer to a 92 percent overload if buckling governs in the elastic range) before material failure. A second order elastic analysis shows that for the W8×40, the cross section will reach its ultimate strength at an overload of only 35 percent. At this overload level, the maximum moment at the top of the column is 61.4 kips, which is more than nine times the value at working load.

Additional calculations show that a W8×48 could carry an overload of 70 percent based on the approximate analysis. A W8×58 would be satisfactory up to a 110 percent overload, which appears very safe. But the approximate structural analysis is about 20 percent unconservative when beams are rigid and axial loads dominate. When G is greater than 2.0, the approximate analysis will be satisfactory for elastic structures.

Thus the following limitations should be imposed on this second order analysis approach:

1. The analysis can not be performed at just working load. The structure must also be checked at an overload level of at least 67 percent, so that there is an added factor of safety against instability. If buckling occurs in the elastic range, the overload should be closer to 92 percent in order to correspond to the factor of safety implied by the specifications.
2. When checking at the overload level, the analysis must consider inelastic column action. An elastic analysis may be incorrect. The inelastic structural analysis becomes quite difficult.
3. When axial loads are significant, numerous cycles of structural analysis are necessary to converge on the proper deflection.

The points listed above are illustrated in Design Example 5. The limitations significantly curtail the usefulness of the approximate second order elastic analysis for multistory frames where axial loads are usually high. Generally, if bending predominates and the axial loads are less than 80 percent of the buckling load, the second order approach will produce satisfactory results.

In summary the writer has indicated how the methods developed for axially loaded members can be extended to beam-columns. In addition, Adams' statements concerning the validity of the inelastic buckling solution were shown to be unjustified. The inelastic K can be used for evaluating the strength of axially loaded members and used in the axial load term of Eq. (6) for beam-columns. The suggested design procedure for columns and beam-columns in structures with various column stiffnesses does not produce designs that are unduly conservative, and the internal forces are correctly considered. On the contrary, the method suggested by Adams may produce unsafe designs in structures with significant axial loads if the analysis is conducted at working load only and/or inelastic column action is not considered in the structural analysis.

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Discussion by **BRUCE G. JOHNSTON**

In the April, 1971 issue of *Engineering Journal*, Professor Yura discusses two important matters related to the Effective Length of Columns in Unbraced Frames: (1) The effect of column inelasticity in reducing effective length factors as computed by the use of the Lawrence alignment charts, and (2) the bracing effect of adjacent columns that are not loaded in as great a proportion of the buckling load as the column being designed. Both of these concepts are valid ones.

Professor Yura goes on to suggest that through an iterative procedure "When the elastic KL/r is reasonably low (about 50 or less), the actual K will usually converge to 1.0. . . ." A numerical design example is given and the reader is referred to Yura's paper for further details.

The writer cautions against the use of the iterative procedure, as the conclusion reached is obviously invalid. In the sidesway buckling of a column in a multistory building under vertical load (assuming simultaneous buckling of all columns in a floor), the buckling load may be determined on the basis that in the laterally deflected position the sum of the moments at the top and bottom of the columns are in equilibrium with the external moments, $P\Delta$, produced by the column loads. The columns *must* develop sufficient bending strength to provide this equilibrium, regardless of whether they are in the inelastic range or the elastic. Thus, in the first increment of buckling, moments are introduced into the beams and there is *some* joint rotation at both the top and bottom. Thus, K is *greater than 1* and no amount of manipulation of the nomographic charts can alter the fact. As the stiffness of the beams unrealistically approaches infinity, K approaches unity, but in tall buildings the columns may be considerably stiffer than the beams and K, even though *properly* corrected for the inelastic reduction effect, will be very appreciably greater than unity.

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The correct procedure for reducing K to some value between the elastic prediction and unity has not, to the writer's knowledge, yet been developed. A single empirical reduction, using the Yura approach, should be acceptable. In the meantime, the more rational study of the complete frame, involving the equilibrium between column and beam end moments in the laterally deflected position, including the effect of lateral wind or earthquake loads, is proceeding at Lehigh and elsewhere. Presumably this will sidestep the need for the calculation of K. But, so long as the AISC interaction formula is used, with one of the limiting situations being the determination of F_a for the axial loaded condition the value of K should not be arbitrarily assumed as unity for any laterally unbraced building frame.

Response by **JOSEPH A. YURA**

Professor Johnston suggests that the iterative procedure outlined in the paper is improper because of the writer's statement that the K-values usually converge to 1.0 when the elastic KL/r is about 50 or less. His conclusion is based on the fact that during buckling moments must develop at the ends of columns. The moments cause some joint rotation, therefore K is greater than 1.0. This is true but the *amount* of joint rotation is related to the ratio of the column stiffness to beam stiffness:

$$G = \frac{\left(\dfrac{E_T I}{L}\right)_{column}}{\left(\dfrac{EI}{L}\right)_{beam}}$$

As G approaches zero, the amount of end rotation approaches zero and K converges to 1.0. Now G can approach zero in two fundamental ways. One way is for the beams to approach infinite stiffness compared to the column I/L, but Johnston states that this is unrealistic in a multistory frame and the writer agrees. However

Johnston ignores the other way in which G can approach zero, that is, E_T can become quite small. The tangent modulus E_T at $KL/r = 20$ is only one-twentieth of the full elastic value E. In fact the concept that the column stiffness $E_T I/L$ is less than the full elastic value for $KL/r < C_c$ was the major point in the paper.

Actually the writer would have been more correct if the statement read ". . . is reasonably low (about 50 or less), the actual K will usually converge to values *close to* 1.0. . . .", but the intent is the same. However, Johnston claims that the "proper" K will be very appreciably greater than unity while simultaneously indicating that he has no knowledge of a correct procedure for determining K for the inelastic cases. On the contrary, the procedure suggested by the writer is well documented. References 1, 2, and 3 contain theory and design examples to calculate the inelastic K.

The results of a study on the stability of columns with equal restraints at both ends, which represents columns in multistory frames, are shown in Fig. 1. Two values of G were chosen, one fairly large to represent large columns relative to the beams. The exact solutions to the buckling equation are shown by the solid lines. $F_y = 36$ ksi was used. When $G = 10$ and L/r is greater than 40, the solution is elastic ($\sigma_{cr} < F_y/2$) and $K = 3.01$. As the column slenderness ratio reduces, the buckling stress is greater than the elastic limit and the column is in the inelastic range. The K-values are less than 3.01. At $L/r = 38$, $K = 2.93$ and the buckling coefficient continues to reduce, down to $K = 1.04$ at $L/r = 10$. The buckling solution using just the story height is shown as a dashed curve. At elastic $KL/r = 50$ ($K = 3.01$), the true solution and the story height ($K = 1.0$) solution are coincident. Consequently, when $KL/r = 50$, $K \approx 1.0$. The effective length is *not* substantially greater than the story height. A similar observation can be made for the case of $G = 2.5$. Studies by Lu[4] have also shown that at low slenderness ratios the sidesway buckling solutions coincide with no sway behavior ($K = 1.0$).

The solid points shown Fig. 1 are buckling solutions based on the iterative procedure outlined in the April 1971 paper. The results are almost exactly the same as those given by the solid curve. Contrary to Johnston's comments, the iterative procedure does not *manipulate* the nomographs. It uses the nomographs to solve the buckling equation. The solutions shown in Fig. 1 provide evidence that the writers observation was correct and no restrictions need be placed on the iterative procedure as suggested by Johnston.

Johnston's final comment concerning current work which will presumably sidestep the need for calculating K is indeed just a stability analysis with a different name. The results given in the nomographs are based on equilibrium between column and beam end moments

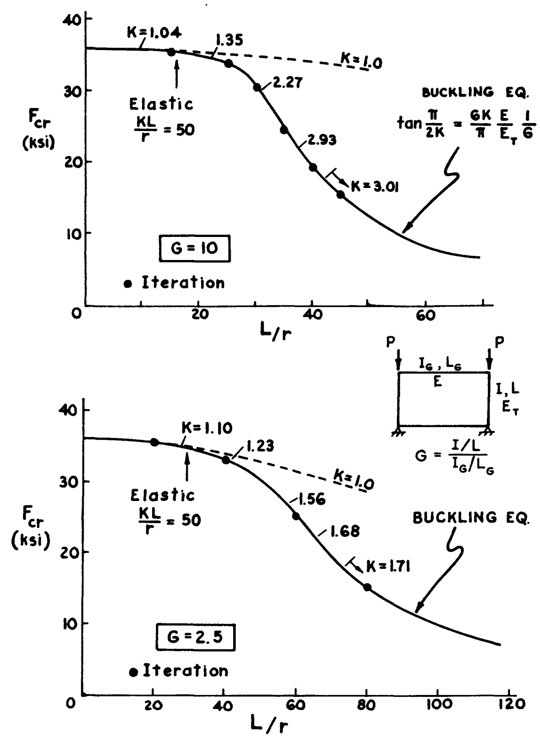

Fig. 1. Inelastic buckling solutions

in the laterally deflected position. Calculating K and performing a buckling analysis is same thing. The writer concurs with Johnston's statement that the value of K should not be *arbitrarily assumed* as unity. Nowhere in the paper does the writer make such a suggestion.

REFERENCES

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2. *Tall, L., Ed.* Structural Steel Design *The Ronald Press, New York, 1964 pp. 336–337, Prob 10.1 on p. 342.*
3. *McQuire, W.,* Steel Structures *Prentice Hall, Englewood Cliffs, N. J., 1968, pp. 509–513.*
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Discussion by **ALFRED ZWEIG**

In Professor Yura's very interesting paper he made reference to a discussion by the writer of a paper by T. R. Higgins which was published in the July 1965 issue of the AISC Engineering Journal. In this discussion the suggestion was made to apply the method developed by T. R. Higgins to the case of a rigid frame with alternately turned columns. To demonstrate the advantage of such an arrangement the writer used an example which was

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also used by Yura as Design Example 3; Yura arrived at the same answer as the writer did in 1965, namely that it is possible to reduce the column size from a W12×65 to a W12×53 section by turning adjacent columns alternately.

Being cognizant that in the method developed by Higgins only the buckling resistance of the strong axis column was utilized, the writer published in September, 1968, in the *Journal of the Structural Division of the ASCE*, a paper entitled "Buckling Analysis of One Story Frames," with a more exact and more generally applicable analysis of the buckling problem in question. Referring to this paper, it can be shown that a W10×49 section will suffice for Yura's Design Example 3.

The reason for this difference lies in the fact that Yura's suggestion is based on Salem's discussion to the writer's above-mentioned paper. In this discussion Salem suggested a simplified approach which is always on the safe side. Using the writer's more accurate analysis, however, it is sometimes possible to obtain a somewhat smaller column size than with Salem's approximation.

Applying the more accurate analysis to Design Example 3, we find for a W 10×49 with $r = \sqrt{I_y/I_x} = \sqrt{93/273} = 0.583$, and with reference to Table 2 of the writer's paper, by interpolations, $K = 1.43$. This coefficient corresponds to a theoretically friction-free pin at the column base.

This ideal condition does not exist in the actual detail of a column base in an industrial building; to account, therefore, for this fact, the Commentary to Sect. 1.8 of the AISC Specification recommends in Fig. C.1.8.2 for the G-value, in lieu of the theoretically required infinity, a value of 10. Applying this G-value to a rigid frame with equal column stiffness results in the reduction of the theoretical K-value from 2 to 1.65. This 82.5 percent reduction in the K-value was, therefore, used by the writer and by Yura in his Example 3.

Applying, for the same reason, the same reduction of 82.5 percent to the above established theoretical K-value of 1.43, we find a K-value of $0.825 \times 1.43 = 1.18$. For $KL_y/r_y = 1.18 \times 240/2.54 = 111$, $F_A = 11.54$ kips per sq in., and the allowable column load is $11.54 \times 14.4 = 166$ kips or 10 kips more than the actual load used in Example 3.

Summarizing, Salem recognized the fact that the total buckling load of a rigid frame with an infinite stiff horizontal member equals approximately the sum of the buckling loads of each frame column. This approximation is on the safe side. Yura's suggestion to utilize this fact for practical design purposes is a significant help to the designer especially since it does not require any additional design aids. The writer's more accurate method, however, as developed in the above quoted ASCE paper, permits frequently further steel economy as demonstrated in the example of this comment.