

# Simplified Plastic Analysis for Reinforced Web Holes

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A PREVIOUS PUBLICATION<sup>1</sup> has presented a method of analysis for the ultimate strength of beams in the region of web holes reinforced in the manner shown in Fig. 1. Experimental verification of the method has also been established.<sup>1,2</sup> The analysis described in Ref. 1 is quite cumbersome, and it is the purpose of this paper to describe and substantiate an approximate, but much simpler, method of analysis. Attention is directed to plastic collapse of beams with both hole and reinforcement located symmetrically about the neutral axis. The analysis provides a method for construction of interaction diagrams relating values of the moment and shear which, acting together at the center line of the hole, will cause failure.

The structural behavior of a beam in the region of a reinforced hole has been described elsewhere.<sup>1</sup> The zone of the beam (web, flange, and reinforcement) above the hole in a region of positive moment carries compressive stress due to the primary moment, and additional normal stresses due to local bending resulting from Vierendeel action. In addition, the web carries the shear force. In Ref. 1 the shear and normal stresses in the web at failure of the beam are assumed to be related through von Mises' yield criterion, and this considerably complicates the algebra of the solution. When the shear force being carried is significant, that is, when the shear-to-moment ratio at the hole is high, the normal stresses on a cross section through the web are small, and in the following discussion these stresses are ignored under such loading conditions. However, when the shear-to-moment ratio at the hole is low, the web contribution to the bending strength is not neglected.

Further simplification is made herein by assuming that the thickness of the flange and reinforcement are small compared with the distance from the edge of the hole to the outer edge of the flange. Also, it is assumed that the reinforcement is located close to the top and bottom of the hole.

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## NOMENCLATURE

$a$	= half length of hole
$A_f$	= area of one flange
$A_r$	= area of reinforcement above the hole, also equal to the area of reinforcement below the hole
$A_w$	= nominal web area = (web thickness) $\times$ (depth of beam)
$d$	= overall beam depth
$F_y$	= yield stress
$h$	= half height of hole
$k_1, k_2$	= coefficients defining positions of stress reversal
$L$	= distance from center of hole to point of contraflexure
$M$	= moment at center line of hole
$M_0, M_1$	= moments at specific points on the interaction diagram
$M_p$	= plastic moment of basic beam section
$V$	= shear force at center line of hole
$V_{\max}$	= maximum shear force the perforated beam can carry
$V_0$	= shear force at a corner of the interaction diagram
$V_p$	= plastic shear force of basic beam section

## SOLUTION FOR HIGH SHEAR

A diagram of a beam containing a reinforced hole is shown in Fig. 1, and the part of the beam above the hole is shown in Fig. 2a for the case of a high shear-to-moment ratio. Due to the local bending moments, reversal of stress will take place on cross sections at each end of the hole. At the low moment end of the hole this always occurs in the flange, and at the high moment end it occurs in the reinforcement in most practical cases. In certain cases, stress reversal at the high moment end occurs in the flange. However, when this is so, it will always be very close to the underside of the flange because of the large normal forces carried by the flange. It is, therefore, sufficient to consider stress reversal only in the reinforcement because, since the normal stresses

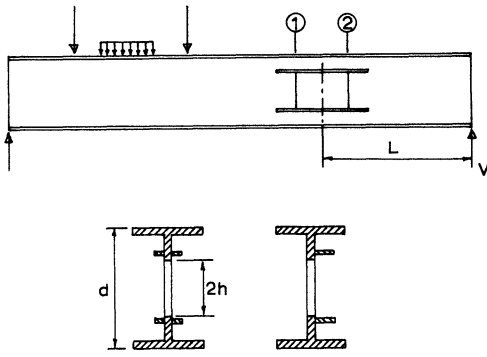


Fig. 1. Beam containing reinforced opening

in the web are being neglected, the solution for stress reversal at the top of the reinforcement is identical to the solution for stress reversal at the underside of the flange.

Assuming that complete yielding of flange and reinforcement occur at both ends of the hole, thus permitting plastic hinges (and so collapse) to develop, the resultant forces on the part of the beam above the hole are shown in Fig. 2a. It is assumed that stress reversal at the high moment end takes place at a fraction  $k_1$  of the reinforcement thickness measured from its lower side, and at the low moment end a fraction  $k_2$  of the flange thickness measured from its upper side.

Equilibrium of the beam to the right of Sects. 1 and 2, as shown in Fig. 2a, requires that:

$$V(L + a) = A_f F_y d + (1 - 2k_1) A_r F_y 2h \quad (1)$$

$$V(L - a) = (1 - 2k_2) A_r F_y d + A_r F_y 2h \quad (2)$$

$$Va = k_2 A_r F_y (d - 2h) \quad (3)$$

Eliminating  $k_2$ ,

$$Va = k_1 A_r F_y (d - 2h) \quad (4)$$

$$M = VL = A_f F_y d + A_r F_y 2h - \frac{Va(d + 2h)}{(d - 2h)} \quad (5)$$

Equations (4) and (5) provide a means of checking the moment ( $M$ ) and shear ( $V$ ) capacities at the center line of a reinforced hole. However, Eqs. (4) and (5) are used only where the shear force is high.

Dividing Eq. (3) by Eq. (4),

$$\frac{k_2}{k_1} = \frac{A_r}{A_f} \quad (6)$$

Since  $k_1$  and  $k_2$  have, by definition, maximum values of unity, it can be seen that if  $A_r < A_f$ , which is the usual case,  $k_1$  reaches 1.0 before  $k_2$  and a limiting value of  $V$  is given by Eq. (4). If, in a special case,  $A_r > A_f$ , the limiting value of  $V$  is given by Eq. (3).

For the usual case, therefore, the maximum value of the shear force  $V_0$  is given by Eq. (4) when  $k_1 = 1$ , i.e.,

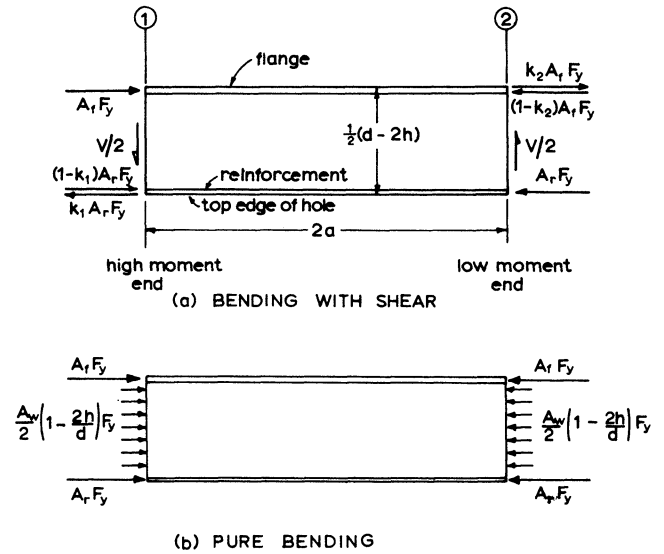


Fig. 2. Forces on part of the beam above the hole

$$V_0 a = A_r F_y (d - 2h) \quad (7)$$

and in Eq. (5), the corresponding moment is:

$$M_0 = (A_f - A_r) F_y d \quad (8)$$

These results are non-dimensionalized by dividing by the full plastic shear capacity  $V_p$  and the plastic moment  $M_p$ , both of which are for the unperforated, unreinforced section:

$$\frac{V_0}{V_p} = \sqrt{3} \left( \frac{V_{max}}{V_p} \right) \left( \frac{d}{a} \right) \left( \frac{A_r}{A_w} \right) \quad (9)$$

$$\frac{M_0}{M_p} = \frac{1 - (A_r/A_f)}{1 + (A_w/4A_f)} \quad (10)$$

in which, with the same approximations,

$$V_p = A_w F_y / \sqrt{3} \quad (11)$$

$$M_p = \left( A_f + \frac{1}{4} A_w \right) d F_y \quad (12)$$

$$\frac{V_{max}}{V_p} = \left( 1 - \frac{2h}{d} \right) \quad (13)$$

and  $A_w$  = the gross web area, and  $A_f$  = the flange area. In Eqs. (9) and (13),  $V_{max}$  is the maximum shear force which can be carried by the unperforated web, assuming it is fully yielded in shear, and the shear capacity of the beam is given by the lesser of  $V_0$  or  $V_{max}$ . It can be noted from Eq. (9) that the full web capacity can be developed if

$$A_r \geq \frac{A_w}{\sqrt{3}} \left( \frac{a}{d} \right) \quad (14)$$

For the rare case when  $A_r > A_f$ , for which the limiting value of  $V$  is given by Eq. (3), Eqs. (9) and (10) are

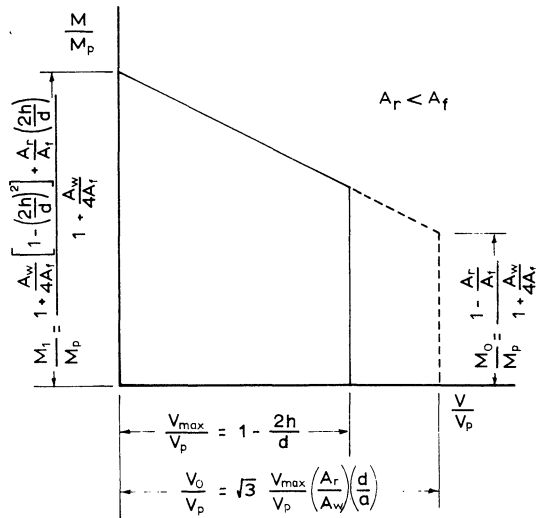


Fig. 3. Moment-shear interaction diagram for reinforced holes

replaced by:

$$\frac{V_0}{V_p} = \sqrt{3} \left( \frac{V_{max}}{V_p} \right) \left( \frac{d}{a} \right) \left( \frac{A_f}{A_w} \right) \quad (15)$$

and

$$\frac{M_0}{M_p} = \frac{\left( \frac{2h}{d} \right) \left( \frac{A_r}{A_f} - 1 \right)}{1 + (A_w/4A_f)} \quad (16)$$

#### SOLUTION FOR NO SHEAR

Under pure bending, the forces on the section above the hole are as shown in Fig. 2b, instead of those in Fig. 2a. There is no stress reversal in the flange or reinforcement. Furthermore, for this case, the normal stresses in the web are included.

For this condition, the moment  $M_1$  on sections through the hole is given by the equation:

$$M_1 = dA_f F_y + 2hA_r F_y + \frac{d}{4} \left[ 1 - \left( \frac{2h}{d} \right)^2 \right] A_w F_y \quad (17)$$

Division by Eq. (12) gives:

$$\frac{M_1}{M_p} = \frac{1 + \frac{A_w}{4A_f} \left[ 1 - \left( \frac{2h}{d} \right)^2 \right] + \frac{A_r}{A_f} \left( \frac{2h}{d} \right)}{1 + (A_w/4A_f)} \quad (18)$$

The corresponding shear force is, of course, zero.

#### SOLUTION FOR INTERMEDIATE SHEAR

In the region of intermediate shear, the significance of the normal stresses in the web decreases until the condition of high shear is reached. Therefore, in this range, an approximate solution is obtained by assuming a linear reduction in moment capacity from  $M_1$  when there is no shear, to  $M_0$  when the shear reaches a value  $V_0$ .

An interaction diagram as shown in Fig. 3 can now be constructed for the case  $A_r < A_f$ , as follows:

1. Plot on  $(V/V_p)$  as abscissa and  $(M/M_p)$  as ordinate the point given by Eq. (18), and  $V/V_p = 0$ .
2. Plot the point given by Eqs. (9) and (10).
3. Join these two points by a straight line, and drop a perpendicular from the second point to the  $V/V_p$  axis.
4. Inspect the value  $V_{max}/V_p$ , and if it is less than the value given by Eq. (9), construct a vertical line through  $V/V_p = V_{max}/V_p$ . This represents the maximum possible factored shear force.

If the less usual case of  $A_r > A_f$  arises, then Eqs. (15) and (16) should be used instead of Eqs. (9) and (10) in step 2.

#### APPLICATIONS

Interaction diagrams obtained in this way are shown for a number of hole and reinforcement sizes in a W14X38 beam in Figs. 4 and 5. Also shown are results for which full account is taken of the web capacity, and for which no assumptions are made regarding the relative size of flange thickness to beam depth, etc.<sup>1</sup> Agreement between these results is generally satisfactory. The approximate theory is most conservative when the hole depth-to-beam depth ratio is low, and the hole is long, so that  $V_{max}$  cannot be attained. This is to be expected, since in this case the web will make a greater contribution to the bending strength. The 7 × 14-in. hole result shown predicts strengths which are a maximum of 12 percent low, and this represents the largest difference of a number of solutions obtained for a variety of hole and beam sizes.

The approximate theory is slightly unconservative when  $V_{max}/V_p$  governs, but this can easily be improved

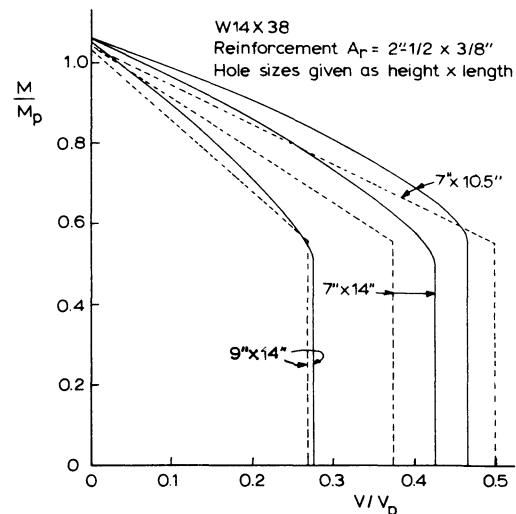


Fig. 4. Comparison of interaction diagrams

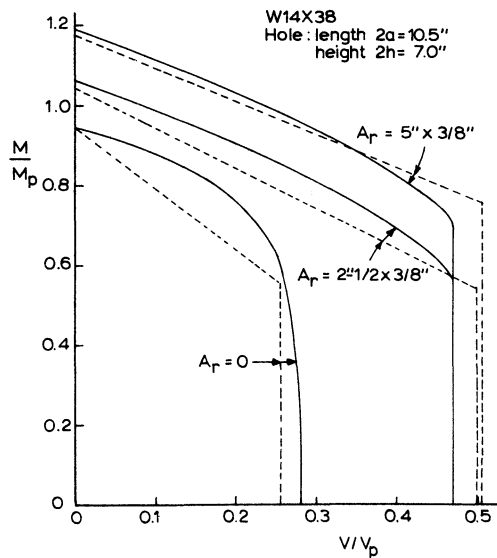


Fig. 5. Comparison of interaction diagrams

by replacing  $d$  by a reduced value (say  $0.95d$ ) to account for the flange thicknesses. However, it has been demonstrated<sup>1</sup> that the theory which includes the web capacity is conservative in this region, and this refinement therefore does not appear to be necessary.

#### NO REINFORCEMENT

It should be noted that, with the exception of Eq. (18), the equations presented so far do not apply to the case when the reinforcement area is zero. This is because, in this case, the strength of the beam is almost entirely dependent upon the capacity of the web above and below the hole to carry normal stresses. Interaction diagrams for this case can be constructed, according to the method given in Ref. 3, by letting  $A_r = 0$  in Eq. (18), and assigning the following values to  $V_0$  and  $M_0$ :

$$\frac{V_0}{V_p} = \left(1 - \frac{2h}{d}\right) \sqrt{\frac{\alpha}{1 + \alpha}} \quad (19)$$

$$\frac{M_0}{M_p} = \frac{1 - \frac{A_w}{4A_f} \left(1 - \frac{2h}{d}\right) \frac{2}{\sqrt{1 + \alpha}}}{1 + (A_w/4A_f)} \quad (20)$$

in which

$$\alpha = \frac{3}{16} \left(\frac{d}{a}\right)^2 \left(1 - \frac{2h}{d}\right)^2$$

An interaction of this type is shown in Fig. 5, where it is compared with results based on exact representation of the flange thickness relative to beam depth.

#### CONCLUSIONS

A method has been given for construction of moment-shear interaction diagrams for beams with holes. The advantage of this method over more exact available methods lies not only in the simple explicit forms for the critical points on the interaction curves, but also in the fact that all the expressions are non-dimensional, and thus provide a basis for development of design charts.

#### ACKNOWLEDGMENTS

The work described herein is based on research sponsored by the Canadian Steel Industries Construction Council. The author wishes to express his appreciation to Dr. John E. Bower of the United States Steel Corporation, for his detailed review of the manuscript, and for a number of helpful suggestions.

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