# Approximate Torsional Analysis of Curved Box Girders by the M/R-Method

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CURVED BRIDGES are becoming increasingly prevalent in highway construction because of improved geometric designs and construction techniques. The current trend in this type of structure is to shape the girders so that they follow the curvatures of the horizontal alignment, creating a continuous flow of the major structural elements.[1]

While the appearance and structural efficiency are often enhanced by the use of curved girders, the analysis and design of these members are likely to be more complicated, which, in some cases, may be the only major factor that prevents the adoption of such a system. It is desirable, therefore, to develop approximate methods which may help practicing engineers overcome this hindrance.[1,2,3]

The objective of this paper is to present to design engineers a simplified method for the torsional analysis of single-span or continuous curved box girders, which, by virtue of their excellent strength in resisting torsion, are generally recognized as ideal supporting elements for horizontally curved structures. The accuracy and limitations of the approximate method, as well as the effects of the various parameters inherent in the problem, are discussed herein. The results are then compared with those obtained in exact solutions based on the transfer matrix method.[4]

## ASSUMPTIONS

1. The dimensions and section properties of the girder may vary in the spanwise direction. However, the cross sections are symmetrical with respect to the vertical axis.

2. The curvatures may vary within each span, but are not reversed in direction.

3. The line of bearings at each support is radial.

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4. The thickness of each plate element is small as compared with its width, and in turn, the width is small as compared with the span length.

5. Internal diaphragms are adequately provided, so that distortions of the cross sections will not occur.

6. Secondary stresses due to warping are considered to be negligible, which is justified when assumption 5, above, is satisfied.

## DEVELOPMENT OF THE APPROXIMATE METHOD (M/R-METHOD)

**Internal Forces**—Consider an infinitesimal segment of a curved girder (Fig. 1) for which three equilibrium equations may be formulated as follows:

$$\frac{dV}{Rd\alpha} = \frac{dV}{dx} = -p \tag{1}$$

$$\frac{dM}{Rd\alpha} = \frac{dM}{dx} = \mp \frac{T}{R} + V \tag{2}$$

$$\frac{dT}{Rd\alpha} = \frac{dT}{dx} = \pm \frac{M}{R} - t \tag{3}$$

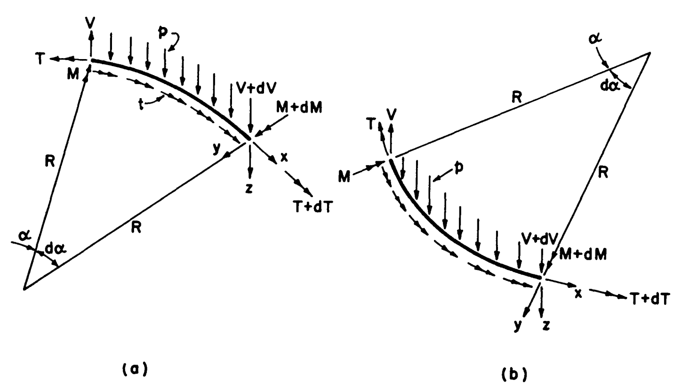


*Fig. 1. Sign convention for external and internal forces*

where

$M$ = bending moment
$R$ = radius of curvature
$T$ = torsional moment
$V$ = shear moment
$p$ = distributed vertical load
$t$ = applied torque
$x$ = independent variable along the longitudinal axis
$\alpha$ = independent angular variable

When a term carries two signs, the upper sign applies to curves defined in Fig. 1a and the lower to curves defined in Fig. 1b.

Integration of Eq. (1) gives the well-known relation (just as in the case of a straight beam) that the change in shear forces between any two points on the girder is equal to the area of the load diagram between the same two points. Equations (2) and (3), however, are coupled and the values of $M$ and $T$ cannot be determined as easily. For an exact analysis, one may first differentiate Eq. (2) or Eq. (3), and after making the substitution into the other, may proceed to solve the resulting second-order differential equation.[5]

When the central angle $\alpha$ is small, and when the bending-torsional stiffness ratio lies below a certain limit in the case of indeterminate structures, the bending moments are not significantly influenced by the torsional moments. One may therefore determine $M$ approximately (but with sufficient accuracy for practical purposes) by dropping the $T/R$ term in Eq. (2), so that

$$\frac{dM}{Rd\alpha} = \frac{dM}{dx} = V \tag{4}$$

In other words, the bending moments may be evaluated closely by considering the curved girder as a straight member with a span equal to its arc length, provided certain requirements are met. In fact, this approach has long been adopted in practice, even though in some cases its limitations are not fully recognized.

The approximate method described herein for the torsional moment analysis of curved girders may be considered as a logical extension of the preceding solution for the bending moments. A similar procedure for the approximate evaluation of torsional moments was also recently suggested independently in Ref. 6.

Integration of Eq. (3) yields the basic relationship that the change in torsional moments between any two points on a curved girder is equal to the area of the $(\pm M/R - t)$-diagram between the same two points. Whereas the torsional moments will be exact if the bending moments have been computed in an exact manner, the former may be determined approximately if approximate $M$-values obtained from Eq. (4) are used in the $(\pm M/R - t)$-diagram. Moreover, one may go one step further and consider, as before, that in the torsional analysis the girder is straight and has a span length equal to the developed length of the actual curved girder. The accuracy of the proposed method depends on the magnitude of the central angle as well as the bending-torsional stiffness ratio $(EI/GJ)$ of the curved beam.

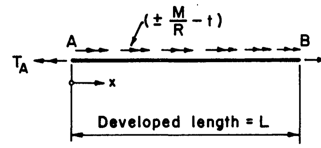


*Fig. 2. Torsional loading on developed girder*

Since only the relative change in torsional moments can be determined from the corresponding $(\pm M/R - t)$ area, the torsional moment at any section can be computed only after that at a reference section is known. For a curved girder having a constant $EI$ and $GJ$ and torsionally-fixed at both ends of a span (but which otherwise may be continuous over several supports), the torsional moment at end **A** of span **AB** (Fig. 2), according to the approximate method, is

$$T_A = \frac{1}{L}\int_0^L \left(\pm\frac{M}{R} - t\right)(L - x)dx \tag{5}$$

which can be readily derived by the force method. The torsional moment at any section located at a distance $s$ from end **A** is therefore

$$T = T_A - \int_0^S \left(\pm\frac{M}{R} - t\right)dx \tag{6}$$

Equations (5) and (6) are analogous to the conjugate-beam method for the determination of deflections of beams. Thus, in the case discussed above, one can consider a corresponding fictitious beam which is simply supported (because the angle of twist $\theta = 0$ at both ends) and subjected to a distributed $(\pm M/R - t)$ loading (Fig. 3). The reaction at end **A** of the conjugate beam will then be equal to the torsional moment of the real beam at **A**, and the shear at any other section will be equal to the torsional moment of the real beam correspondingly. To satisfy Eq. (3), a positive $(\pm M/R - t)$ quantity should be an upward load in the conjugate beam.

In most steel box girder bridges constructed in this country, it has been the practice to provide a bearing under each web of the girder, thereby rendering a torsionally fixed condition at every support. The approximate method will therefore be most useful in these cases. However, as demonstrated later, the method can be equally applied to other situations in which point supports are present.

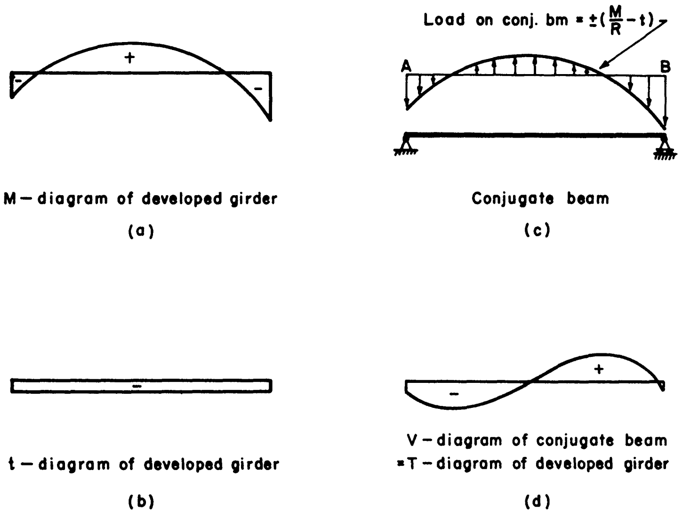


*Fig. 3. Conjugate beam M/R-method*

**Displacements**—The longitudinal slopes and angles of twist of a curved girder have a significant influence upon each other, and the vertical deflections are dependent on both.

Since the equivalent straight-girder concept of bending analysis may be extended readily to include the approximate determination of vertical deflections, no further attempts will be made herein to define the procedure, other than to examine the accuracy of the approximate results.

In order to arrive at an approximate method of evaluating the angles of twist of a curved girder, examine an infinitesimal circular segment as shown in Fig. 4. In the following discussions, downward deflections are considered positive, whereas positive rotations are defined in Fig. 4, in which $\theta$ = angle of twist, and $\phi$ = longitudinal slope of the girder.

The following equations may be written to relate $\theta$ and $\phi$:

$$\frac{d\theta}{Rd\alpha} = \frac{d\theta}{dx} = \pm\frac{\phi}{R} + \frac{T}{GJ} \tag{7}$$

$$\frac{d\phi}{Rd\alpha} = \frac{d\phi}{dx} = \pm\frac{\theta}{R} + \frac{M}{EI} \tag{8}$$

When a term carries two signs, the upper sign refers to the type of curves defined in Fig. 4a, and the lower to that in Fig. 4b.

Differentiating Eq. (7) with respect to $x$, and substituting Eqs. (8) and (3) into the resulting expression,

$$\frac{d^2\theta}{dx^2} = \pm\frac{1}{R}\frac{d\phi}{dx} + \frac{1}{GJ}\frac{dT}{dx}$$
$$= -\frac{\theta}{R^2} \pm \frac{1}{EI}\left(\frac{M}{R}\right) + \frac{1}{GJ}\left(\pm\frac{M}{R} - t\right) \tag{9}$$

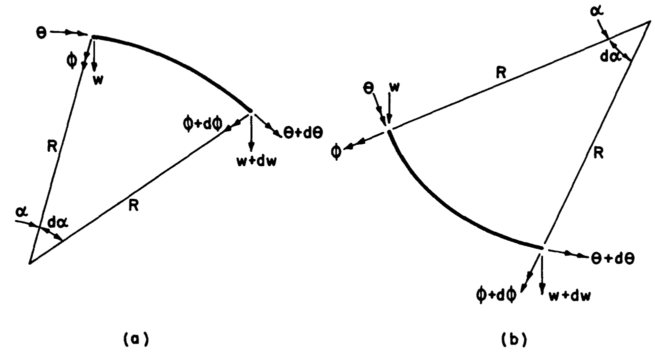


*Fig. 4. Sign convention for rotations*

In practical cases, $\theta/R^2$ is small as compared with the two remaining terms on the right hand side of Eq. (9), and can be neglected without significant effects. Thus, Eq. (9) may be simplified as follows:

$$\frac{d^2\theta}{dx^2} = \frac{1}{EI}\left(\frac{M}{R}\right) + \frac{1}{GJ}\left(\frac{M}{R} - t\right)$$
$$= \frac{1}{EI}\left[\pm\frac{M}{R} + \frac{EI}{GJ}\left(\pm\frac{M}{R} - t\right)\right] \tag{10}$$

Applying the conjugate beam analogy, one may conclude that for a curved girder with torsionally-fixed ends, the quantity $EI\theta$ at any point of the real girder is equal to the corresponding moment in the simply-supported conjugate beam under a distributed load of $\pm M/R + (EI/GJ)(\pm M/R - t)$. One may also note that the moment due to the second part of this load is equal to $EI/GJ$ times the moment caused by the load used previously in the approximate torsional moment analysis.

When $t = 0$, Eq. (10) reduces to

$$\frac{d^2\theta}{dx^2} = \pm\frac{1}{EI}\left(1 + \frac{EI}{GJ}\right)\left(\frac{M}{R}\right) \tag{11}$$

The quantity $EI\theta$ is thus equal to $(1 + EI/GJ)$ times the moment in the conjugate beam under the distributed load of $M/R$. In other words, the numerical value of the angle of twist in such cases is

$$|\theta| = \frac{1}{R}\left(1 + \frac{EI}{GJ}\right)(|w|) \tag{12}$$

where $w$ is the corresponding vertical deflection calculated on the basis of the approximate bending analysis.

**Summary**—In summary, under most conditions encountered in practice (e.g., central angle $\leqslant 30°$ and $EI/GJ \leqslant 2.5$), the bending and torsional analysis of curved box girders may be uncoupled and investigated independently. By straightening the curved girder to its full developed length, the bending moments and vertical shear forces can be readily determined as customarily done in the past. The proposed method suggests that the torsional analysis can be carried out in a similar manner, except that (1) a straight conjugate beam subjected to a distributed load of $(\pm M/R - t)$ in the torsional moment analysis, and (2) a straight conjugate beam subjected to a distributed load of $[\pm M/R + (EI/GJ)(\pm M/R - t)]$ for the determination of the angles of twist are to be considered, where $M$ is the bending moment obtained in the approximate flexural analysis, $R$ is the radius of curvature, and $t$ is the applied distributed torque in the spanwise direction. Since $M/R$ is a parameter in the forcing function in the analysis, the approximate method is called the $M/R$-method to differentiate it from the classical conjugate beam method for the determination of beam deflections.

The following sections will examine the accuracy and limitations of the approximate method as well as several other aspects related to the problem in general.

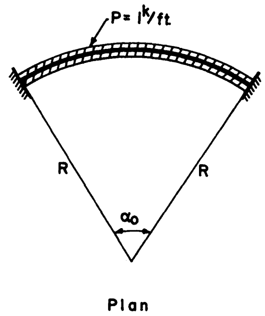


*Fig. 5. Circular girder with fixed-ends*

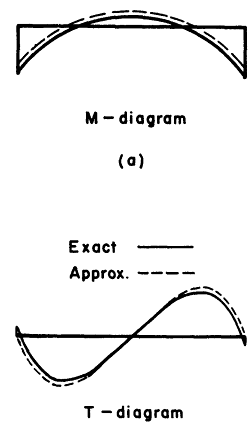


*Fig. 6. Comparison of M- and T-diagrams in exact and approximate analyses*

### EFFECT OF CENTRAL ANGLE $\alpha_0$

The accuracy of the approximate torsional analysis depends, among other factors, on the accuracy of the approximate bending analysis; both are influenced to a great extent by the value of the central angle $\alpha_0$ and the bending-torsional stiffness ratio $EI/GJ$.

As an indication of the effect of $\alpha_0$, consider a single-span fixed-ended (with respect to both bending and torsion) circular beam subjected to a uniform load of 1 kip/ft (Fig. 5). Assume that $R = 300$ ft and $EI/GJ = 2.5$.

Figure 6 shows typical qualitative comparisons between exact and approximate bending and torsional moment diagrams. The approximate torsional moment diagram is defined by third degree curves because the approximate bending moment curve is a second degree parabola. It may be noted that the approximate $M$-diagram is always above the exact, so that the $-M$'s are always smaller and the $+M$'s always larger than the corresponding exact values. Moreover, the points of inflection in a curved girder are located slightly farther away from the supports than those in an equivalent straight girder of its developed length. Further studies show that this is also true in the case of continuous girders. These observations, therefore, might well be taken into account when change of flange areas is contemplated in curved girder designs by the approximate method.

Table 1 shows a comparison of the exact and approximate results for $\alpha_0 = 30°$ and $45°$ respectively, while the percentages of error in $-M_{max}$, $+M_{max}$, and $T$ at $0.2L$ are given in Table 2 for various values of $\alpha_0$. In the latter tabulation, the torsional moments at the

**Table 1. Comparison of Exact and Approximate Solutions for $\alpha_0 = 30°$ and $\alpha_0 = 45°$**

| | | $\alpha_0 = 30°$ | | $\alpha_0 = 45°$ | |
|---|---|---|---|---|---|
| | $x/L$ | Exact | Approx. | Exact | Approx. |
| $M$ (kip-ft) | 0.0 | −2096 | −2056 | −4818 | −4626 |
| | 0.1 | −984 | −946 | −2313 | −2128 |
| | 0.2 | −116 | −82 | −348 | −185 |
| | 0.3 | 506 | 535 | 1065 | 1203 |
| | 0.4 | 880 | 905 | 1916 | 2036 |
| | 0.5 | 1005 | 1028 | 2200 | 2313 |
| $T$ (kip-ft) | 0.0 | 8.3 | 0.0 | 59.5 | 0.0 |
| | 0.1 | −71.3 | −77.5 | −217.0 | −261.6 |
| | 0.2 | −99.1 | −103.4 | −317.9 | −348.8 |
| | 0.3 | −87.8 | −90.4 | −286.1 | −305.2 |
| | 0.4 | −50.4 | −51.7 | −165.3 | −174.4 |
| | 0.5 | 0.0 | 0.0 | 0.0 | 0.0 |

$x$ = distance from support
$L$ = developed length of curved girder

supports are also given. It may be noted that in the conjugate beam analysis, the shear (and hence the torsional moment of the actual beam) vanishes at the supports because the $M/R$-loading is self-equilibrated.

It may be concluded from the above discussion that for $EI/GJ = 2.5$, the approximate method is satisfactory when $\alpha_0 \leqslant 30°$, and that the maximum torsional moments obtained in the approximate analysis are always on the safe side for a fixed-ended curved beam.

## EFFECT OF BENDING-TORSIONAL STIFFNESS RATIO $EI/GJ$

The previous fixed-ended circular beam ($R = 300$ ft) is again considered in the following, except that $\alpha_0$ is now maintained at either 25° or 30°, while the value of $EI/GJ$ is varied in each case.

It may be concluded from Table 3 that the approximate torsional analysis is acceptable for practical purposes, provided that (1) $EI/GJ \leqslant 4.0$ when $\alpha_0 \leqslant 25°$,

**Table 2. Comparison of Exact and Approximate Results Showing Influence of $\alpha_0$**

| | Bending | | | | | | Torsion | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $-M_{max}$ | | | $+M_{max}$ | | | $T$ at $0.2L = T_1$ | | | $T$ at Support $= T_s$ | |
| $\alpha_0$ | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | $\|T_s/T_1\| \cdot (100)$ (%) |
| 10 | 229.0 | 228.5 | −0.2 | 113.9 | 114.2 | 0.3 | −3.81 | −3.83 | 0.5 | 0.04 | 1.0 |
| 20 | 922.1 | 913.9 | −0.9 | 452.1 | 456.9 | 1.1 | −30.0 | −30.6 | 2.0 | 1.12 | 3.7 |
| 30 | 2096 | 2056 | −1.9 | 1005 | 1028 | 2.3 | −99.1 | −103.4 | 4.3 | 8.25 | 8.3 |
| 45 | 4818 | 4626 | −4.0 | 2200 | 2313 | 5.1 | −317.9 | −348.8 | 9.7 | 59.5 | 18.7 |
| 60 | 8785 | 8225 | −6.4 | 3779 | 4112 | 8.8 | −704.9 | −826.8 | 17.3 | 234.5 | 33.3 |

Notes: 1. $M$ and $T$ in kip-ft
2. In all cases, $T_s = 0$ at the support in the approximate analysis
3. $L$ = developed length of curved girder

**Table 3. Comparison of Exact and Approximate Results Showing Effect of $EI/GJ$**

(a) $\alpha_0 = 30°$

| | Bending | | | | | | Torsion | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $-M_{max}$ | | | $+M_{max}$ | | | $T$ at 0.2$L$ — $T_1$ | | | $T$ at Support — $T_s$ | |
| $\frac{EI}{GJ}$ | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | $\|T_s/T_1\| \cdot (100)$ (%) |
| 2.5 | 2096 | 2056 | −1.9 | 1005 | 1028 | 2.3 | −99.1 | −103.4 | 4.3 | 8.3 | 8.3 |
| 3 | 2100 | 2056 | −2.1 | 1000 | 1028 | 2.8 | −98.4 | −103.4 | 5.1 | 9.3 | 9.5 |
| 5 | 2116 | 2056 | −2.8 | 984.5 | 1028 | 4.4 | −95.9 | −103.4 | 7.8 | 13.4 | 14.0 |
| 10 | 2149 | 2056 | −4.3 | 950.3 | 1028 | 8.2 | −90.6 | −103.4 | 14.1 | 22.3 | 24.6 |
| 50 | 2286 | 2056 | −10.1 | 808.2 | 1028 | 27.2 | −68.4 | −103.4 | 51.2 | 59.1 | 86.5 |
| 100 | 2350 | 2056 | −12.5 | 742.1 | 1028 | 38.5 | −58.0 | −103.4 | 78.3 | 76.2 | 131.4 |

(b) $\alpha_0 = 25°$

| | Bending | | | | | | Torsion | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $-M_{max}$ | | | $+M_{max}$ | | | $T$ at $0.2L = T_1$ | | | $T$ at Support $= T_s$ | |
| $\frac{EI}{GJ}$ | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | $\|T_s/T_1\| \cdot (100)$ (%) |
| 1 | 1441 | 1428 | −0.9 | 708.8 | 714.0 | 0.7 | −58.9 | −59.8 | 1.5 | 2.0 | 3.4 |
| 2 | 1446 | 1428 | −1.2 | 704.5 | 714.0 | 1.3 | −58.3 | −59.8 | 2.6 | 2.9 | 5.0 |
| 3 | 1450 | 1428 | −1.5 | 700.3 | 714.0 | 2.0 | −57.8 | −59.8 | 3.5 | 3.8 | 6.6 |
| 5 | 1457 | 1428 | −2.0 | 692.3 | 714.0 | 3.1 | −56.8 | −59.8 | 5.3 | 5.5 | 9.7 |
| 10 | 1475 | 1428 | −3.2 | 674.2 | 714.0 | 5.9 | −54.4 | −59.8 | 9.9 | 9.5 | 17.5 |

See notes in Table 2 for supplementary information or (2) $EI/GJ \leqslant 2.5$ when $\alpha_0 \leqslant 30°$. Such limits, although arbitrary and subject to personal judgment, are necessary because the approximate method will give the same answer for a given $\alpha_0$, irrespective of the value of $EI/GJ$. As a consequence, spanwise variation of $GJ$ can be ignored entirely in the approximate analysis if, for instance, the weighted average of $EI/GJ$ is within the above limits.

It may also be noted that in the case of $\alpha_0 = 30°$, the approximate bending moments will be of questionable value when $EI/GJ$ exceeds 10. For $EI/GJ = 50$, the error is $-10.1\%$ in $-M_{max}$ and $+27.2\%$ in $+M_{max}$.

### EFFECT OF END RESTRAINTS AND LOADING CONDITIONS

Five additional cases will be investigated to examine (1) the effect of flexural end restraints, and (2) the effect of two types of loading conditions. All of the beams considered are single-span circular beams, with $R = 300$ ft, $\alpha_0 = 30°$, and $EI/GJ = 2.5$. The supports are assumed to be torsionally-fixed.

The cases investigated are:

1. A uniform load of 1 kip/ft over the entire span:
   a. Both ends are simply supported with respect to bending.
   b. The left end is simply supported and the right end is fixed with respect to bending.
2. One concentrated load of 100 kips acting at the mid-span:
   a. Both ends are fixed with respect to bending.
   b. Same as 1 (a).
   c. Same as 1 (b).

While the torsional moment diagrams are defined by third-degree curves in the case of uniformly-loaded beams (because the approximate moment diagrams are second-degree parabolas), and by second-degree curves in the case of concentrated-loaded beams (because the approximate moment diagrams are linear), the general shapes of both groups of diagrams nevertheless look quite similar when the support conditions are identical. For this reason, and for simplicity, the results are shown qualitatively in Fig. 7 without specifying the type of loading.

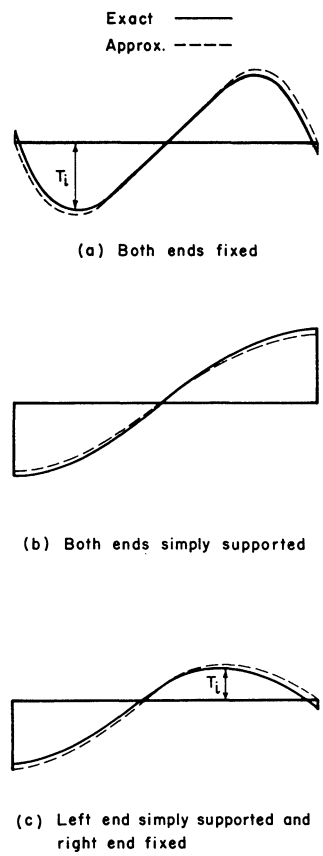


*Fig. 7. Torsional moment diagrams showing effect of various end restraints in bending*

**Table 4. Comparison of Exact and Approximate Results Showing Influence of Flexural End Restraints**

| | | Bending | | | | | | Torsion | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $-M_{max}$ (kip-ft) | | | $+M_{max}$ (kip-ft) | | | $T_i$ (see Fig. 7) (kip-ft) | | | $T$ at **S**-support (kip-ft) | | |
| Load | Supports | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) | Exact | Approx. | Error (%) |
| Uniform load = 1 kip/ft | **F-F** | 2096 | 2056 | −1.9 | 1005 | 1028 | 2.3 | −99.1 | −103.4 | 4.3 | 8.3 | 0 | (—) |
| | **S-S** | 0 | 0 | 0 | 3175 | 3084 | −2.9 | — | — | — | −553.5 | −538.3 | −2.7 |
| | **S-F** | 3208 | 3084 | −3.9 | 1720 | 1735 | 0.9 | 181.6 | 185.0 | 1.9 | −264.3 | −269.2 | 1.9 |
| 100 kips at mid-span | **F-F** | 2012 | 1964 | −2.4 | 1937 | 1964 | 1.4 | −1245 | −1285 | 3.2 | 9.9 | 0 | (—) |
| | **S-S** | 0 | 0 | 0 | 4019 | 3927 | −2.3 | — | — | — | −529.1 | −514.0 | −2.9 |
| | **S-F** | 3078 | 2945 | −4.3 | 2426 | 2454 | 1.2 | 206.5 | 208.2 | 0.8 | −251.7 | −257.0 | 2.1 |

**F** = fixed with respect to bending
**S** = simply supported with respect to bending

Table 5. Comparison of Exact and Approximate Vertical Deflections and Angles of Twist

| Load | Supports | $(EIw)_{max} \times (10)^{-3}$ Exact | $(EIw)_{max} \times (10)^{-3}$ Approx. | Error (%) | $(EI\theta)_{max} \times (10)^{-3}$ Exact | $(EI\theta)_{max} \times (10)^{-3}$ Approx. | Error (%) |
|---|---|---|---|---|---|---|---|
| Uniform | **F-F** | $0.203\ R^4$ | $0.196\ R^4$ | $+3.4$ | $-0.665\ R^3$ | $-0.687\ R^3$ | 3.3 |
| load = | **S-S** | $1.109\ R^4$ | $0.982\ R^4$ | $-11.5$ | $-3.625\ R^3$ | $-3.437\ R^3$ | $-5.2$ |
| 1 kip/ft | **S-F** | $0.432\ R^4$ | $0.408\ R^4$ | $-5.6$ | $-1.421\ R^3$ | $-1.426\ R^3$ | 0.4 |
| 100 kips | **F-F** | $0.770\ R^3$ | $0.749\ R^3$ | $-2.7$ | $-2.544\ R^2$ | $-2.622\ R^2$ | 3.1 |
| at mid- | **S-S** | $3.379\ R^3$ | $2.998\ R^3$ | $-11.3$ | $-11.06\ R^2$ | $-10.49\ R^2$ | $-5.2$ |
| span | **S-F** | $1.415\ R^3$ | $1.311\ R^3$ | $-7.4$ | $-4.66\ R^2$ | $-4.59\ R^2$ | $-1.5$ |

**F** = fixed with respect to bending
**S** = simply supported with respect to bending

Table 4 gives an indication of the errors of the approximate solutions. In general, the discrepancies of the torsional moments are largest at the supports and reduce progressively toward the mid-span. Except for the case of simply-supported beams (with respect to bending), the errors in torsional moments are on the safe side. Moreover, all the errors appear to be acceptable in practice.

### VERTICAL DEFLECTIONS AND ANGLES OF TWIST

Table 5 gives a comparison of the exact and approximate solutions for the circular girders investigated in the preceding section.

### CONTINUOUS CURVED GIRDERS

The approximate method is applicable to the analysis of continuous curved girders. However, aside from the influence of the $EI/GJ$ ratios, the accuracy depends to a great extent on (1) the total central angle of the entire girder from one end to the other (or the sum of the central angles of all the spans when the curvature varies), (2) the central angle of each individual span, and (3) the torsional restraint provided at the supports. The following recommendations, based on the findings of a number of numerical investigations (but again subject to personal judgment), are suggested in order to maintain the same degree of accuracy in the approximate analysis as previously determined for single-span girders. The simplified method will be valid even if there are reversed curvatures in the continuous girders, provided that they do not occur between any two adjacent torsionally-fixed supports.

The limitations are prescribed as follows:

1. The central angle of each span should not exceed 30° (25°) and the weighted average of $EI/GJ$ in each span should not exceed 2.5 (4.0).
2. If all the supports are torsionally-fixed, the central angle of the entire girder, or the sum of all the central angles in case of variable curvatures, should not exceed 90°.
3. If one or more of the supports are not torsionally-fixed, it is further recommended that:
   a. There should be at least one torsionally-fixed support in the entire span.
   b. The central angle (or sum of central angles) should not exceed 40° (32°) between two adjacent torsionally-fixed supports, nor 25° (20°) between a torsionally-free end support and the first torsionally-fixed support, and the weighted average of $EI/GJ$ should not be larger than 2.5 (4.0).
   c. The central angle of the entire girder, or the sum of all the central angles in case of variable curvatures, should not exceed 90°.

When all the supports are torsionally-fixed, the procedure of analysis will be similar to what has been described previously, i.e.:

1. Straighten the entire curved girder to its full developed length and provide corresponding supports. Determine the bending moments $M$ by any method of indeterminate analysis.
2. Considering one span at a time, apply the distributed $(\pm M/R - t)$ loads on the simply-supported straight conjugate beam and determine the torsional moments in that span.
4. The algebraic difference of the two end torsional moments adjacent to a support is the torsional moment reaction at that support.

Sometimes it may be necessary or desirable to provide point supports for a continuous curved girder. (A point support, for instance, may be in the form of a single bearing placed directly under an internal transverse diagram connecting the webs of a box girder.) In such cases, the span of the conjugate beams described in step 2, above, should be between two adjacent torsionally-fixed supports.

The procedure outlined above will yield the same bending moments in the girder whether the supports are torsionally-fixed or torsionally-free. The bending moments, however, are not significantly affected by these conditions.

Further, consider the case in which torsion is due to concentric loads alone (i.e., $t$ does not enter into consideration). When $R$ and $EI$ are constant in the continuous girder, the approximate method will give only one set of torsional moments, irrespective of the degrees of torsional restraints provided at the intermediate supports, in that the torsional moment reactions will always be equal to zero at these points. This is necessarily so because the bending analysis must satisfy the condition that the longitudinal slope of the girder must be continuous at an intermediate support, which requires that the end shear of two neighboring conjugate beams adjacent to a support be equal under the $M/EI$ loading. Moreover, recall that the end torsional moments are the end shears of the conjugate beams under the $M/R$ loading. Since both $EI$ and $R$ are constant, it may be concluded that the end shear in these latter conjugate

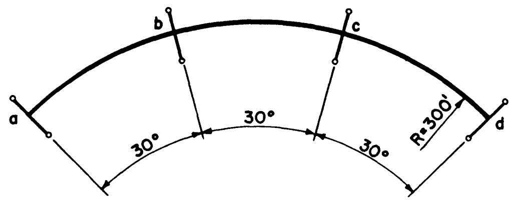


*Fig. 8. Three-span continuous curved girder*

beams must also be equal, so that there cannot be any torsional moment reaction at the intermediate supports.

The results of two investigations are examined below.

*1. Three-span continuous girder (Fig. 8):* Data: $R = 300$ ft; $EI/GJ = 2.5$; central angle of each span $= 30°$; total central angle of girder $= 90°$; all supports are torsionally-fixed, but permit free rotations with respect to bending; movable uniform load $= 1$ kip/ft.

Table 6 gives a comparison of the results obtained by the exact and approximate solutions.

**Table 6. Comparison of Exact and Approximate Results of a Three-span Continuous Curved Girder**

| Case | Loading Condition | $M$ or $T$ (kip-ft) | Exact | Approx. | Error (%) |
|---|---|---|---|---|---|
| | | Bending | | | |
| 1 | 1 kip/ft over **ab** and **cd** | $+M$ at $0.4L$ from **a** | 2519.9 | 2467.4 | −2.1 |
| 2 | 1 kip/ft over **ab** and **bc** | $-M$ at **b** | 2992.9 | 2878.6 | −3.8 |
| 3 | 1 kip/ft over **bc** | $+M$ at mid-span of **bc** | 1862.4 | 1850.6 | −0.6 |
| 4 | 1 kip/ft over **ad** | $+M$ at $0.4L$ from **a** | 1992.8 | 1973.9 | −0.9 |
| | | $-M$ at **b** | 2535.7 | 2467.4 | −2.7 |
| | | $+M$ at mid-span of **bc** | 549.8 | 548.6 | −0.2 |
| | | Torsion | | | |
| 1 | 1 kip/ft over **ab** and **cd** | $-T$ at **a** | 439.2 | 430.7 | −1.9 |
| 2 | 1 kip/ft over **ab** and **bc** | $+T$ at left of **b** | 21.3 | 35.9 | (—) |
| | | $+T$ at right of **b** | 51.8 | 35.9 | (—) |
| 3 | 1 kip/ft over **ab** | $+T$ at left of **b** | 246.7 | 251.2 | 1.8 |
| | | $+T$ at right of **b** | 265.5 | 251.2 | −5.4 |
| 4 | 1 kip/ft over **bc** | $-T$ at left of **b** | 225.4 | 215.3 | −4.5 |
| | | $-T$ at right of **b** | 213.7 | 215.3 | 0.8 |
| 5 | 1 kip/ft over **ad** | $-T$ at **a** | 324.9 | 323.0 | −0.6 |
| | | $+T$ at $0.8L$ from **a** | 228.7 | 228.2 | −0.2 |
| | | $+T$ at left of **b** | 102.6 | 107.7 | 5.0 |
| | | $+T$ at right of **b** | 115.9 | 107.7 | 1.6 |
| | | $-T$ at $0.3L$ right of **b** | 40.3 | 47.4 | (—) |

Four loading cases are considered for bending, the first three of which are critical loading conditions required to yield, respectively, a maximum moment at each of the sections under consideration. The last case deals with a uniform load extending over the entire length, corresponding to the dead load condition.

It can be observed that the maximum error in bending moments occurs at the interior supports and is in the same order of magnitude as that of a single-span **S-F** beam shown in Table 5. Similarly, the region of negative moment extends farther from an interior support than that in a straight girder of its developed length, and should be taken into consideration in the design.

Five loading cases are considered for torsion. The first case yields both the maximum torsional moment and the maximum vertical shear force at the end support. The second case corresponds to the loading condition for maximum vertical shear at the two girder sections adjoining support b. It may be noted that the torsional moments are insignificant in this case. Moreover, the errors, while considerably large in terms of percentage, have no practical importance and therefore are not indicated in the table. As a comparison, the maximum torsional moments at these two sections are given in cases 3 and 4. The fifth case again represents that of a uniform dead load.

The results summarized in Table 6 appear to be acceptable for practical purposes.

*2. Two-span continuous girder (Fig. 9):* Data: $R = 300$ ft; $EI/GJ = 3.0$; central angle of left span $= 30°$; central angle of right span $= 20°$; total central angle of girder $= 50°$; all supports permit free rotations with respect to bending; uniform load $= 1$ kip/ft over the left span only.

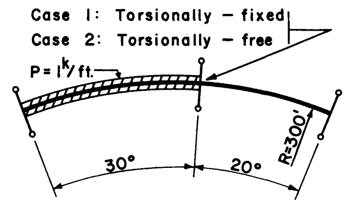


*Fig. 9. Two-span continuous curved girders*

Two cases are investigated:

a. Case 1: All supports are torsionally-fixed.
b. Case 2: The two end supports are torsionally-fixed, but the intermediate support is torsionally-free.

The results of the study are summarized in Table 7. While the total central angle of the girder is 50° and $EI/GJ$ is 3.0, so that this combination falls outside the limits specified for good approximate results, it is intentionally adopted herein to demonstrate the degree of error to be expected in such cases, and to examine the effects of the two extreme torsional-restraint conditions provided at the interior support.

The results show that: (1) the bending moments are not significantly affected by the mode of torsional restraint encountered at the interior support, and (2) the approximate torsional moments are closer to the exact values when the interior support is torsionally-fixed. In the latter case, however, the torsional moment at the two girder sections immediately adjacent to the interior

**Table 7. Comparison of Exact and Approximate Results of Two-span Continuous Curved Girders**

| | | Bending ($M$ in kip-ft) | | | | | Torsion ($T$ in kip-ft) | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Case 1 | | Case 2 | | | Case 1 | | Case 2 | |
| | $x/L$ | Approx. | Exact | Error (%) | Exact | Error (%) | Approx. | Exact | Error (%) | Exact | Error (%) |
| Left Span | 0.0 | 0 | 0 | 0 | 0 | 0 | −376.8 | −375.8 | 0.3 | −367.4 | 2.6 |
| | 0.2 | 1604 | 1616 | −0.7 | 1619 | −0.9 | −284.2 | −282.5 | 0.6 | −274.0 | 3.7 |
| | 0.4 | 2221 | 2227 | −0.3 | 2233 | −0.5 | −75.4 | −72.5 | 3.9 | −63.5 | 18.7 |
| | 0.6 | 1851 | 1829 | 1.2 | 1838 | 0.7 | 146.4 | 148.7 | −1.6 | 158.5 | −7.6 |
| | 0.8 | 458 | 424 | 8.0 | 436 | 5.0 | 277.8 | 275.4 | 0.9 | 286.2 | −2.9 |
| | 1.0 | −1851 | −1972 | −6.1 | −1957 | −5.4 | 215.3 | 202.9 | 6.1 | 215.1 | 0.1 |
| Right Span | 0.0 | −1851 | −1972 | −6.1 | −1957 | −5.4 | 215.3 | 231.3 | −6.9 | 215.1 | 0.1 |
| | 0.2 | −1480 | −1589 | −6.9 | −1577 | −6.2 | 99.0 | 107.0 | −7.5 | 91.7 | 8.0 |
| | 0.4 | −1110 | −1198 | −7.4 | −1190 | −6.7 | 8.6 | 9.6 | −10.4 | −4.9 | (—) |
| | 0.6 | −740 | −802 | −7.7 | −796 | −7.0 | −56.0 | −60.3 | −7.1 | −74.3 | −24.6 |
| | 0.8 | −370 | −402 | −8.0 | −399 | −7.3 | −94.7 | −102.3 | −7.4 | −116.0 | −18.4 |
| | 1.0 | 0 | 0 | 0 | 0 | 0 | −107.7 | −116.3 | −7.4 | −130.0 | −17.2 |

Case 1: Interior Support is torsionally-fixed.
Case 2: Interior Support is torsionally-free.

support is more or less the average of the two exact values. It may also be noted that the errors in maximum torsional moments in the loaded span are small and therefore acceptable. The torsional moments in the unloaded span, while exhibiting larger errors, are of no practical interest because different loading conditions will be required for maximum vertical shear forces in that span.


## SUMMARY

Curved box girders have been recognized recently as excellent supporting elements in bridge structures which must follow a horizontal curved alignment. The exact analysis of these members, however, is often complicated and tedious, unless a computer solution is readily accessible.

This paper presents a simplified method for the approximate solution of torsional moments and angles of twist in curved box girders of single or multiple spans. The box girders are assumed to be adequately stiffened by internal transverse diaphragms so that both warping and distortion stresses may be considered as negligible.

The proposed method adopts an approach which is quite similar to the classical conjugate beam method for the determination of beam deflections. The corresponding fictitious beam in this case, however, is straight and subjected to distributed loads expressed in terms of $M/R$. For this reason, the proposed method is denoted as the $M/R$-method to differentiate it from the classical.

It is found that within the limits specified herein, the method will give results with sufficient accuracy for practical purposes.



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