Revisiting the P-δ Magnification Factor for Members Subject to End Moments

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ABSTRACT

The 2022 edition of the AISC *Specification for Structural Steel Buildings* allows member and system instability effects to be accounted for by using advanced analysis methods that model these effects directly or through the use of the B_1 (P-δ) and B_2 (P-Δ) multipliers. For compression members subject to end moments only, the B_1 multiplier is expressed as a function of the smaller to larger end moment ratio. This ratio is taken as positive when the member bends in reverse curvature and negative when it bends in single curvature. This code equation for B_1 often overestimates the P-δ effect for double curvature bending under a high axial force but underestimates the P-δ effect for single curvature bending under almost all values of axial force. The reason for the overestimation is because the derivation of the B_1 equation uses the equivalent moment concept and ignores the actual location where the maximum moment occurs in the member. When this condition is taken into consideration, it is shown herein that the equation is no longer applicable for members bent in reverse curvature under any value of axial force. The reason for the underestimation is because the effect of axial force on the moment ratio is ignored. By incorporating this effect in a new equation for B_1, it is shown that this underestimation is drastically reduced. The validity of this proposed B_1 multiplier is established by comparison with the corresponding theoretical values.

Keywords: P-δ effect, B_1 factor, design equation.

INTRODUCTION

When a structural member is subject to the combined actions of compressive axial force and bending moments as shown in Figure 1, the lateral deflection and moment in the member will be magnified from their respective first-order values. The increase is the result of the axial force acting through the deflection of the member away from its chord (i.e., P-δ effect) and the same axial force acting on the relative deflection of one end of the member with respect to the other end (i.e., P-Δ effect).

To account for these two instability effects, the 2022 edition of the AISC *Specification for Structural Steel Buildings* (AISC, 2022), hereafter referred to as the AISC *Specification*, allows designers to use either a second-order elastic analysis that explicitly models these effects or an approximate second-order analysis that makes use of results obtained from a first-order elastic analysis in conjunction with two multipliers, B_1 and B_2. For moment and axial force, the equations that can be used are given by AISC *Specifications* Equations A-8-1 and A-8-2, and reproduced here as Equations 1a and 1b:

$$M_r = B_1 M_{nt} + B_2 M_{lt} \tag{1a}$$

$$P_r = P_{nt} + B_2 P_{lt} \tag{1b}$$

where

M_r = required second-order flexural strength using LRFD or ASD load combinations, kip-in. (N-mm)

P_r = required second-order axial strength using LRFD or ASD load combinations, kips (N)

M_{nt} = first-order moment obtained using the LRFD or ASD load combinations, when the structure is restrained against lateral translation, kip-in. (N-mm)

M_{lt} = first-order moment obtained using the LRFD or ASD load combinations, due to lateral translation, kip-in. (N-mm)

P_{nt} = first-order axial force obtained using the LRFD or ASD load combinations, when the structure is restrained against lateral translation, kips (N)

P_{lt} = first-order axial force obtained using the LRFD or ASD load combinations, due to lateral translation, kips (N)

B_1 = multiplier to account for the P-δ effect

B_2 = multiplier to account for the P-Δ effect

B_1 and B_2 are given by AISC *Specification* Equations A-8-3 and A-8-6, and are reproduced as Equations 2a and 2b:

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$$B_1 = \frac{C_m}{1 - \frac{\alpha P_r}{P_{e1}}} \geq 1 \quad \text{(2a)}$$

$$B_2 = \frac{1}{1 - \frac{\alpha P_{story}}{P_{e\,story}}} \geq 1 \quad \text{(2b)}$$

where α is taken as 1 for LRFD and 1.6 for ASD; P_{e1} is the elastic critical buckling strength of the member in the plane of bending, assuming the member ends do not undergo any lateral translation, kips (N); P_{e1} is given by AISC *Specification* Equation A-8-5 and is equal to $\pi^2 EI^*/(L_{c1})^2$, where EI^* is the flexural rigidity in kip-in.^{2} (N-mm^2), taken as $0.8\tau_b EI$ when used in the direct analysis method of design (AISC *Specification* Section C1.1) or as EI when used in the effective length and first-order analysis method of design (AISC *Specification* Section C1.2, Appendix 7); τ_b is the stiffness reduction parameter discussed in AISC *Specification* Section C2.3; L_{c1} is taken as the laterally unbraced length of the member in the plane of bending, in, (mm); P_{story} is the total vertical load acting on the story under consideration using LRFD or ASD load combinations, kips (N); and $P_{e\,story}$ is the elastic critical bucking strength of the story in the direction of lateral translation being considered, kips (N), determined by sidesway buckling analysis, or by AISC *Specification* Equation A-8-7.

C_m is the equivalent uniform moment factor, assuming the member ends undergo no lateral translation relative to each other. For members not subjected to transverse loading between the two member ends, the equation for C_m is given by AISC *Specification* Equation A-8-4, and reproduced below as

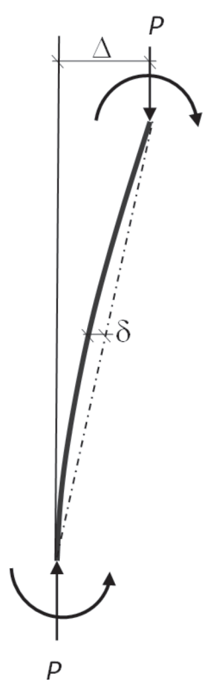

Fig. 1. P-delta effects.

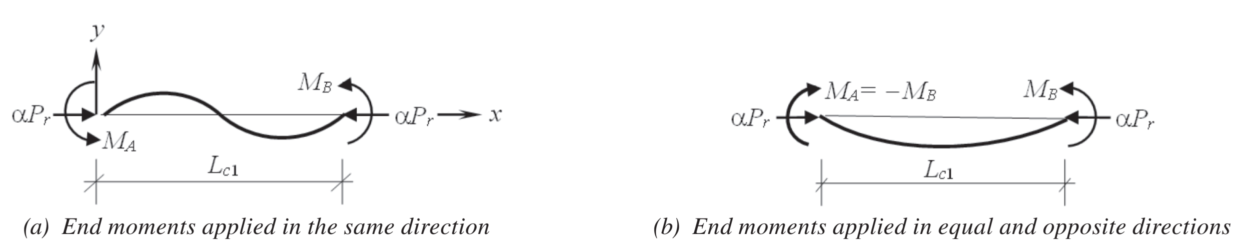

Fig. 2. Prismatic member subject to end moments and a compressive axial force.

$$C_m = 0.6 - 0.4\left(\frac{M_A}{M_B}\right) \quad (3)$$

in which M_A/M_B is the ratio of the smaller to larger end moments calculated using a first-order elastic analysis. M_A/M_B is taken as positive if the member bends in reverse curvature and negative if the member bends in single curvature. This equivalent moment factor allows the maximum moment in a member under end moments M_A and M_B shown in Figure 2(a) to be calculated from the maximum moment of a member subject to a pair of equal and opposite end moments shown in Figure 2(b).

Equation 3 was proposed by Austin (1961). The derivation of this equation can be found in Chen and Lui (1987). The Austin equation is a simplification of the equation

$$C_m = \sqrt{\frac{\left(\frac{M_A}{M_B}\right)^2 + 2\left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right) + 1}{2\left(1 - \cos\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)}} \quad (4)$$

Note that Equation 4 is a function of both the moment ratio M_A/M_B and $\alpha P_r/P_{e1}$. Not accounting for this axial force effect makes Equation 3 less accurate (Ballio and Mazzolani, 1983; Duan et al., 1989; Chen and Shen, 2015). Over the years, researchers have studied this C_m factor (Massonnet, 1959; Chen and Zhou, 1987; Sohal and Syed, 1992; Chen and Wang, 1999), and simplifying equations have been proposed for use in design. However, one aspect of Equation 3, as well as with Equation 4 from which Equation 3 was derived, that many researchers have overlooked is that they do not take into consideration the actual location of the maximum bending moment (when acting in conjunction with an axial force) within in the member. If the location of this maximum bending moment does not fall within the physical length of the member (as will be shown mathematically in a later section), both equations become invalid and should not be used. Addressing these two issues (axial force effect on C_m and applying Equations 3 and 4 when they are not applicable) are the primary goals of this paper.

THEORETICAL BACKGROUND

In Figure 2(a), a prismatic structural member laterally braced at the ends without experiencing any relative end translations in the y-direction is subject to end moments M_A and M_B ($\geq M_A$) and a compressive axial force αP_r. The differential equation governing its behavior can be written as

$$\frac{d^2y}{dx^2} + \left(\frac{\alpha P_r}{EI^*}\right)y = \left(\frac{M_A + M_B}{EI^*}\right)\left(\frac{x}{L_{c1}}\right) - \left(\frac{M_A}{EI^*}\right) \quad (5)$$

where EI^* and L_{c1} are defined as before in Equation 2a.

The solution to this differential equation after enforcing the boundary conditions $y(0) = y(L) = 0$ is

$$y = -\left\{\frac{\left[\left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right) + 1\right]}{\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)}\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{x}{L_{c1}}\right) - \left(\frac{M_A}{M_B}\right)\left[\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{x}{L_{c1}}\right) - 1\right] - \left(\frac{M_A}{M_B} + 1\right)\left(\frac{x}{L_{c1}}\right)\right\}\left(\frac{M_B}{\alpha P_r}\right) \quad (6)$$

And upon taking derivatives, one obtains

$$\frac{d^2y}{dx^2} = \left\{\frac{\left[\left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right) + 1\right]}{\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)}\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{x}{L_{c1}}\right) - \left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{x}{L_{c1}}\right)\right\}\left(\frac{M_B}{EI^*}\right) \quad (7)$$

$$\frac{d^3y}{dx^3} = \left\{\frac{\left[\left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right) + 1\right]}{\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)}\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{x}{L_{c1}}\right) + \left(\frac{M_A}{M_B}\right)\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{x}{L_{c1}}\right)\right\}\left(\frac{\pi}{L_{c1}}\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)\left(\frac{M_B}{EI^*}\right) \quad (8)$$

The equation to determine the location where maximum moment occurs, denoted herein as $\tilde{x}$, can be obtained by setting Equation 8 equal to zero; that is,

$$\tan\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\frac{\tilde{x}}{L_{c1}}\right) = \frac{-\left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right) - 1}{\left(\frac{M_A}{M_B}\right)\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)} \quad (9)$$

It is important to note that the location for maximum moment $\tilde{x}$ determined from Equation 9 must fall in the range $0 \leq \tilde{x} \leq L_{c1}$, or $0 \leq (\tilde{x}/L_{c1}) \leq 1$, to ensure that the maximum moment occurs within the physical length of the member.

Table 1. Values of $\tilde{x}/L_{c1}$						
		$\alpha P_r/P_{e1}$				
		0.1	**0.3**	**0.5**	**0.7**	**0.9**
M_A/M_B	–1.0	0.500	0.500	0.500	0.500	0.500
	–0.8	0.704	0.555	0.525	0.511	0.503
	–0.6	0.935	0.623	0.555	0.525	0.507
	–0.4	1.17	0.705	0.594	0.543	0.512
	–0.2	1.39	0.803	0.644	0.566	0.518
	0	1.58	0.913	0.707	0.598	0.527
	0.2	–1.43	–0.796	–0.626	–0.553	–0.514
	0.4	–1.31	–0.681	–0.528	–0.486	–0.492
	0.6	–1.22	–0.577	–0.417	–0.387	–0.449
	0.8	–1.14	–0.488	–0.306	–0.250	–0.343
	1.0	–1.08	–0.413	–0.207	–0.098	–0.027

Table 2. Theoretical Moment Magnification Factor, $B_{1,theory}$						
		$\alpha P_r/P_{e1}$				
		0.1	**0.3**	**0.5**	**0.7**	**0.9**
M_A/M_B	–1.0	1.14	1.53	2.25	3.94	12.42
	–0.8	1.05	1.39	2.03	3.55	11.18
	–0.6	1.00	1.26	1.82	3.16	9.94
	–0.4	1	1.14	1.61	2.78	8.70
	–0.2	1	1.06	1.42	2.40	7.46
	0	1	1.01	1.26	2.04	6.23
	0.2	1	1	1	1	1
	0.4	1	1	1	1	1
	0.6	1	1	1	1	1
	0.8	1	1	1	1	1
	1.0	1	1	1	1	1

Once $\tilde{x}$ is determined, the maximum moment M_{max} can be obtained by substituting $(\tilde{x}/L_{c1})$ into Equation 7 and multiplying the resulting equation by the flexural rigidity EI^* of the member. M_{max} so obtained is given by

$$M_{max} = \underbrace{\left[\frac{\sqrt{\left(\frac{M_A}{M_B}\right)^2 + 2\left(\frac{M_A}{M_B}\right)\cos\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right) + 1}}{\sin\left(\pi\sqrt{\frac{\alpha P_r}{P_{e1}}}\right)} \right]}_{B_{1,theory}} M_B = B_{1,theory}\, M_B \quad (10)$$

The term in brackets denoted as $B_{1,theory}$ is the theoretical elastic moment magnification factor. This factor accounts for the member instability (P-δ) effect that magnifies the primary moment acting on the member due to the presence of an axial force.

The equivalent moment factor C_m shown in Equation 4 is obtained by equating Equation 10 with the equation for M_{max} when $M_A = -M_B$ (i.e., when the member is subject to a pair of equal and opposite end moments).

Table 3. AISC *Specification* Moment Magnification Factor, $B_{1,AISC}$

		$\alpha P_r/P_{e1}$				
		0.1	**0.3**	**0.5**	**0.7**	**0.9**
M_A/M_B	−1.0	1.11	1.43	2.00	3.33	10.00
	−0.8	1.02	1.31	1.84	3.07	9.20
	−0.6	1	1.20	1.68	2.80	8.40
	−0.4	1	1.09	1.52	2.53	7.60
	−0.2	1	1	1.36	2.27	6.80
	0	1	1	1.20	2.00	6.00
	0.2	1	1	1.04	1.73	5.20
	0.4	1	1	1	1.47	4.40
	0.6	1	1	1	1.20	3.60
	0.8	1	1	1	1	2.80
	1.0	1	1	1	1	2.00

P-δ MOMENT MAGNIFICATION

The presence of a compressive axial force αP_r magnifies the larger of the two end moments (i.e., M_B). The magnification factor $B_{1,theory}$ given in Equation 10 can be used to magnify M_B only if $\tilde{x}$ satisfies the condition $0 \le (\tilde{x}/L_{c1}) \le 1$ to ensure that the maximum moment created by the combined actions of end moments and axial force falls within the physical length of the member. Values of $(\tilde{x}/L_{c1})$ for a range of M_A/M_B and $\alpha P_r/P_{e1}$ computed from Equation 9 are given in Table 1. The gray highlighted numbers mean the condition $0 \le (\tilde{x}/L_{c1}) \le 1$ is violated. Note that $\tilde{x}/L_{c1}$ does not fall within the physical length of the member when $M_A/M_B > 0$ (i.e., when the member bends in reverse curvature) and when $\alpha P/P_{e1}$ is small for certain values of $M_A/M_B < 0$ (i.e., for certain values of moment ratio when the member bends in single curvature).

By enforcing the condition $0 \le (\tilde{x}/L_{c1}) \le 1$, the theoretical values of the moment magnification factor $B_{1,theory}$ calculated from Equation 10 are given in Table 2. Because the condition $0 \le (\tilde{x}/L_{c1}) \le 1$ is violated in the gray-shaded area, the maximum moment acting on the member is therefore taken as the larger of the two end moments (i.e., M_B), so the moment magnification factor is set equal to 1.

When Equation 3 is substituted into Equation 2a, the AISC *Specification* moment magnification factor ($B_{1,AISC}$) for members subject to end moments M_A and M_B and an axial compressive force αP_r can be written as

$$B_{1,AISC} = \frac{0.6 - 0.4\left(\dfrac{M_A}{M_B}\right)}{1 - \dfrac{\alpha P_r}{P_{e1}}} \ge 1 \qquad (11)$$

Values of $B_{1,AISC}$ computed using this equation are given in Table 3. Upon comparison with Table 2, it can be seen that Equation 11 underestimates the moment magnification effect for almost all $\alpha P_r/P_{e1}$ when $M_A/M_B < 0$ (single curvature bending) but overestimates its effect for high $\alpha P_r/P_{e1}$ when $M_A/M_B > 0$ (double curvature bending) in the yellow-shaded area. More importantly, the numbers shown in the yellow-shaded area should not even be used since the condition $0 \le (\tilde{x}/L_{c1}) \le 1$ is violated as depicted in Table 1.

The moment magnification effect is underestimated because the term $\alpha P_r/P_{e1}$ is ignored in the numerator of Equation 11. To remedy this, a nonlinear regression analysis was performed on Equation 4, and a new equation for C_m is proposed as follows.

$$C_m = 0.6 - \left(0.4 + 0.25\frac{\alpha P_r}{P_{e1}}\right)\left(\frac{M_A}{M_B}\right) \qquad (12)$$

Using this equation for C_m, the improved *P*-δ moment magnification factor is proposed as

$$B_{1,proposed} = \begin{cases} \dfrac{0.6 - \left(0.4 + 0.25\dfrac{\alpha P_r}{P_{e1}}\right)\left(\dfrac{M_A}{M_B}\right)}{1 - \dfrac{\alpha P_r}{P_{e1}}} \ge 1 & -1 \le \dfrac{M_A}{M_B} \le 0 \\ 1 & \text{otherwise} \end{cases} \qquad (13)$$

Values of the moment magnification factor computed using Equation 13 are shown in Table 4. When compared to the theoretical values given in Table 2 and plotted together in Figure 3, good agreement is observed. The maximum error is less than 5%, which occurs at $\alpha P_r/P_{e1} = 0.9$.

Table 4. Proposed Moment Magnification Factor, $B_{1,proposed}$						
		$\alpha P_r/P_{e1}$				
		0.1	**0.3**	**0.5**	**0.7**	**0.9**
M_A/M_B	–1.0	1.14	1.54	2.25	3.92	12.25
	–0.8	1.04	1.40	2.04	3.53	11.00
	–0.6	1	1.26	1.83	3.15	9.75
	–0.4	1	1.13	1.62	2.77	8.50
	–0.2	1	1	1.41	2.38	7.25
	0	1	1	1.20	2.00	6.00
	0.2	1	1	1	1	1
	0.4	1	1	1	1	1
	0.6	1	1	1	1	1
	0.8	1	1	1	1	1
	1.0	1	1	1	1	1

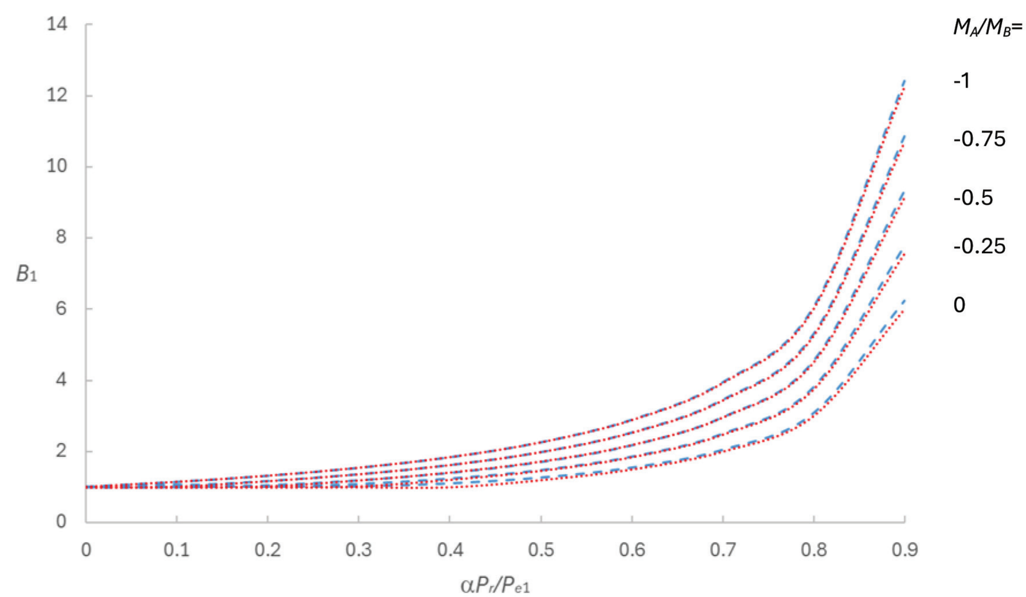

Fig. 3. Comparison of theoretical (blue dashed lines) and proposed (red dotted lines) moment B_1 *factor.*

CONCLUSIONS

When a member is subject to a pair of end moments and an axial compressive force, the end moments cause the member to deflect from its chord, and the axial force acts through this lateral deflection to produce additional moment in the member. This phenomenon is referred to as the P-δ effect. To account for this effect, the 2022 AISC *Specification* permits the use of a second-order analysis or the use of the P-δ moment magnification factor B_1. The current equation for B_1 has two fallacies: It does not take into account that the magnified moment may not occur within the physical length of the member, and it ignores the axial force effect in the numerator of the equation. In this paper, it is shown that the moment magnification effect does not apply to cases where the end moments cause the member to bend in reverse curvature; by introducing an axial force term in the numerator of the moment magnification equation, the P-δ effect can be more accurately represented. Comparison of B_1 values calculated using the proposed equation and their theoretical values show that the errors are well within 5%.

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